**PROGRAMME DU COURS "*GKZ*-HYPERGEOMETRIC FUNCTIONS" PAR FRITS BEUKERS**

THÉORIES GALOISIENNES ET ARITHMÉTIQUES DES ÉQUATIONS DIFFÉRENTIELLES
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The concept of hypergeometric function was introduced as a concept to encompass many of the existing classical functions such as $\arcsin, \arctan, \log$ etc. However, they also extend the classical functions in a natural way. For example, Riemann discovered their analytic continuation and monodromy group. Equally important, hypergeometric functions occur at many places in mathematics and mathematical physics (both classical and modern). They have been generalised in many senses : the order of the differential equation, the number of variables and $q$-analogues have been introduced.

One important line of generalisation was introduced around 1988 by Gel'fand, Kapranov and Zelevinski. They called their functions A-hypergeometric functions and nowadays they are referred to as *GKZ-hypergeometric functions*. We summarise their definition here to emphasize that the underlying data are very combinatorial in nature.

Start with a finite subset $A \subset \mathbb{Z}^r \subset \mathbb{R}^r$. We assume

– The $\mathbb{Z}$-span of $A$ is $\mathbb{Z}^r$
– There is a linear form $h$ such that $h(\mathbf{a}) = 1$ for all $\mathbf{a} \in A$.

Define a vector of parameters

$$\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{R}^r$$

Write $A = \{\mathbf{a}_1, \ldots, \mathbf{a}_N\}$. The lattice of relations $L \subset \mathbb{Z}^N$ is formed by all $\mathbf{l} = (l_1, \ldots, l_N) \in \mathbb{Z}^N$ such that

$$l_1\mathbf{a}_1 + l_2\mathbf{a}_2 + \cdots + l_N\mathbf{a}_N = \mathbf{0}$$

We introduce the variables $v_1, v_2, \ldots, v_N$. For every $\mathbf{l} \in L$ we define the operator

$$\Box_{\mathbf{l}} = \prod_{l_i>0} \left(\frac{\partial}{\partial v_i}\right)^{l_i} - \prod_{l_i<0} \left(\frac{\partial}{\partial v_i}\right)^{-l_i}$$

For every linear form $m$ on $\mathbb{R}^r$ we define the operator

$$Z_m = m(\mathbf{a}_1)v_1\frac{\partial}{\partial v_1} + \cdots + m(\mathbf{a}_n)v_N\frac{\partial}{\partial v_N} + m(\alpha)$$

The GKZ-differential equations read

– $\Box_{\mathbf{l}}\Phi = 0$ for all $\mathbf{l} \in L$
– $Z_m\Phi = 0$ for all forms $m$.

A formal explicit solution can be given quite easily. Choose $\gamma = (\gamma_1, \ldots, \gamma_N)$ such that $\gamma_1\mathbf{a}_1 + \cdots + \gamma_N\mathbf{a}_N = \alpha$. Then

$$\Phi = \sum_{\mathbf{l}\in L} \frac{v_1^{l_1+\gamma_1} \cdots v_N^{l_N+\gamma_N}}{\Gamma(l_1+\gamma_1+1)\cdots\Gamma(l_N+\gamma_n+1)}$$

is a formal solution of the GKZ-system.

Note that $\gamma$ is determined modulo $L \otimes \mathbb{R}$.

The function $\Phi$ is a template for all classical hypergeometric functions (Gaussian, Appell, Lauricella, Horn). The study of convergent power series solutions is associated with triangulations of the polytope $A$ and the so-called secondary polytope.

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Another important feature are the monodromy group of the GKZ-system and the arithmetic properties of the solutions. Remarkably enough the latter question has been addressed by B. Dwork in his book *Generalised hypergeometric Functions* from about 1990. There he independently developed a general theory for hypergeometric functions which is completely parallel to the development of GKZ-functions. Dwork's arithmetic study is of importance for transcendence and irrationality since all known Siegel G-functions are in fact restrictions of GKZ-functions.

The question of monodromy for GKZ-functions has not been solved yet. The case of finite monodromy has been the subject of a recent study by F. Beukers.

In the lectures we give an introduction to the basic theory of GKZ-functions and cover as many of the above aspects as possible.

## Références

[1] B. Dwork, *Generalised hypergeometric Functions*, Oxford University Press (1990).

[2] J. Stienstra, *GKZ-Hypergeometric Structures*, Proceedings of the Summer School "Algebraic Geometry and Hypergeometric Functions", Istanbul, June 2005, preprint 2005 ; available at `http ://ar`$\chi$`iv.org/abs/0511.5351`.