Algebraic D-groups
and non-linear differential Galois theory

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I. Ax-Schanuel : how and why ?

Ax (1970)

$$x = (x_1, ..., x_n) \in (\mathbb{C}\{\{z_1, .., z_t\}\})^n$$
$$\forall i, y_i(z) := exp(x_i(z)) \in \mathbb{C}\{\{z_1, .., z_t\}\}$$
$$\rightsquigarrow y = exp(x) \in (\mathbb{C}\{\{z_1, ..., z_t\}\}^*)^n$$

Assume x *non-degenerate*, i.e.

$$\forall m \in \mathbb{Z}^n \setminus 0, m_1 x_1 + ... + m_n x_n \notin \mathbb{C}.$$

Then, $tr.deg.(\mathbb{C}(x, y)/\mathbb{C}) \geq n + rk(\frac{Dx}{Dz})$.

If $K = \mathbb{C}(z_1, ..., z_t)$, it suffices to show that

$$tr.deg.(K(x, y)/K) \geq n.$$

Choosing a sufficiently general curve in $\mathbb{C}^t$, it suffices to check it when $t = 1$. So :

$$K = \mathbb{C}(z)^{alg}, \partial = \frac{d}{dz}$$

$(\mathcal{K}, \partial)$ a differential extension, with $\mathcal{K}^\partial = \mathbb{C}$

$(x, y) \in (\mathcal{K} \times \mathcal{K}^*)^n$, x non-degenerate. Then

$$\partial y/y = \partial x \Rightarrow deg.tr.K(x, y)/K \geq n$$

. More generally (cf. Ax (1972))

• $K = \mathbb{C}(z)^{alg}, \partial = \frac{d}{dz}, (\mathcal{K}, \partial), \mathcal{K}^{\partial} = \mathbb{C}$

• G: commutative algebraic group *defined over* $\mathbb{C}$, with no additive quotient; typically,

$G = \mathbf{G}_m^n$, or $G = A$, an ab. var.

• so, Lie algebra $LG/\mathbb{C}$. Let $x \in LG(\mathcal{K})$ s.t. *for any proper algebraic subgroup* H/K *of* G

$$x \notin LH(\mathcal{K}) + LG(\mathbb{C}).$$

• $y = exp_G(x) \in G(\mathcal{K}) \Leftrightarrow \partial \ell n_G(y) = \partial_{LG}(x)$;

$$\Rightarrow tr.deg.(K(x, y)/K) \geq dimG.$$

Here

We assume :

$x \in K$ (exponential case, Lindemann-Weierstrass)

or

$y \in K$ (logarithmic case, Grothendieck conjecture)

but we allow

G **non-constant**, i.e. G/K.

Why ?

Bombieri-Pila-Wilkie-Pila-Zannier-Masser-Pila-..., for instance :

Manin-Mumford (easier version): *Let $A/\mathbb{Q}^{alg}$ be an abelian variety. An algebraic subvariety $X/\mathbb{Q}^{alg}$ of A passes through finitely many torsion points of A, unless X contains a translate of a non-zero abelian subvariety of A.*

Strategy of the proof: : write $A = \mathbb{R}^{2g}/\mathbb{Z}^{2g}$, so the torsion points become rational points in $[0,1]^{2g}$ while X pulls back to a real analytic variety $\mathcal{X}$. By o-minimality, $\mathcal{X}$ meets $<< T^{\epsilon}$ rational points of denominator $\leq T$, *outside of the semi-algebraic subvarieties $\mathcal{Z}$ it contains*. But back to A, one point generates many by Galois action (Kummer for Mordell-Lang), so their orders are bounded.

Requires a control of the $\mathcal{Z}$ in $\mathcal{X}$.

$exp_A(\mathcal{Z}) \subset \mathcal{X}$ provides an algebraic dependence relation over $\mathbb{C}$ for the restriction of the $exp(x_i)$ to $\mathcal{Z}$ and then, Ax forces the $x_{i|\mathcal{Z}}$ to be $\mathbb{Z}$-lin dep., so $\mathcal{Z}$ must be "linear" ("geodesic"), yelding a translate of abelian subvariety of A in X.

Relative Manin-Mumford (Pink-type conj.): *a proper subscheme* $\mathbf{X}/S$ *of an abelian scheme* $\mathbf{A}/S$ *meets finitely many of the torsion points of the various fibers, unless it contains a translate of an abelian subscheme.*

Requires studying a non constant G/K.

Refs. :

D. Masser, U. Zannier : Torsion anomalous points and families of elliptic curves; CRAS Paris 346, 2008, 491-494.

J. Pila : Rational points of definable sets and finiteness results for special subvarieties, Prep. 20090721

. Connexions with other talks

Legendre-Gauss: $(E) : \eta^2 = \xi(\xi-1)(\xi-z)$ over $S = \mathbf{P}_1 \setminus \{0,1,\infty\}$. Then, $\omega = \frac{d\xi}{\eta} \in H^0(E, \Omega^1_{E/S}) \subset H^1_{dR}(E/S)$ satisfies

$$L(\omega) = \frac{1}{2} d(\frac{\eta}{(\xi-z)^2}) \equiv 0 \in H^1_{dR}(E/S)$$

where $\quad L := z(1-z)\frac{d^2}{dz^2} + (1-2z)\frac{d}{dz} - \frac{1}{4}.$

For a section $\mathbf{y}(z) := (\xi(z), \eta(z)) \in E(S)$, set:

$$\ell n_E\Big(\mathbf{y}(z)\Big) = \int_\infty^{\mathbf{y}(z)} \omega := \mathbf{x}(z) \in LieE(\tilde{S}^{an}).$$

On torsion sections,

$$\mathbf{y}(z) = (1,0) \mapsto \mathbf{x}(z) = \pi F(\tfrac{1}{2}, \tfrac{1}{2}, 1; z);$$
$$\mathbf{y}(z) = (z,0) \rightsquigarrow i\pi F(\tfrac{1}{2}, \tfrac{1}{2}, 1; 1-z);$$

$\frac{\xi d\xi}{\eta}$ gives $F(-\frac{1}{2}, \frac{1}{2}, 1, z)$, etc ....

. $\rightsquigarrow$ **Frits's talks**.

On a general section,

$L(\mathbf{x}(z)) \in \mathcal{O}(S) \subset \mathbb{C}(S) \subset K$!!! Call this $\mu(\mathbf{y}(z))$.
So, $\mathbf{x}(z)$ satisfies the inhomogeneous LDE:

$$L(\mathbf{x}(z)) = \mu(\mathbf{y}(z))$$

But μ is also related to the non linear Painlevé VI equation

. $\rightsquigarrow$ **Guy's talks**.

P VI

$$\frac{d^2\xi}{dz^2} = \frac{1}{2}(\frac{1}{\xi}+\frac{1}{\xi-1}+\frac{1}{\xi-z})(\frac{d\xi}{dz})^2-(\frac{1}{z}+\frac{1}{z-1}+\frac{1}{\xi-z})\frac{d\xi}{dz}$$

$$+\frac{\xi(\xi-1)(\xi-z)}{z^2(z-1)^2}\Big(\alpha+\beta\frac{z}{\xi^2}+\gamma\frac{z-1}{(\xi-1)^2}+\delta\frac{z(z-1)}{(\xi-z)^2}\Big)$$

For any $(\mathcal{K}, \partial =')$ over $(K, d/dz)$:

$$\mu : E(\mathcal{K}) \to \mathbb{G}_a(\mathcal{K})$$

$$\mu(\mathbf{y}) = \frac{1}{2}\frac{\eta}{(\xi-z)^2}+\Big(z(1-z)\frac{\xi'}{\eta}\Big)'+z(1-z)\frac{\xi'}{\eta}.\frac{\eta'}{\eta},$$

R. Fuchs rewrote PVI as:

$$z(1-z)\mu(\mathbf{y}) = \alpha\eta + \beta t\tfrac{\eta}{\xi^2} + \gamma(z-1)\tfrac{\eta}{(\xi-1)^2}$$
$$+(\delta-\tfrac{1}{2})z(z-1)\tfrac{\eta}{(\xi-z)^2}.$$

Set $z = z(\tau)$ (Legendre's λ), $\xi \sim \wp_{\{1,\tau\}}(\zeta)$.

Then, $\omega \rightsquigarrow d\zeta$, $L \rightsquigarrow \frac{d^2\zeta}{d\tau^2}$, 2-tor. $\rightsquigarrow \epsilon_i, \alpha$'s $\rightsquigarrow \kappa_i$, and PVI reads: find $\zeta(\tau) \in LieE(\tilde{S}^{an})$ s.t.

$$\tfrac{d^2\zeta}{d\tau^2} = \tfrac{1}{4\pi^2}\textstyle\sum_{i=0}^{3}\kappa_i\wp'_{\{1,\tau\}}(\zeta+\varepsilon_i).$$

Exercises : i) rewrite in terms of $q = e^{2\pi i\tau}$

. $\rightsquigarrow$ **Jacques & Lucia's talks**.

. ii) replace $\mathbb{C}$ by $\mathbb{C}_p$ (cf. Coleman)

. $\rightsquigarrow$ **Bruno's talks**.

II. Picard-Vessiot & Kolchin

(K, ∂) : a diff. field with $K^\partial = C$ alg. closed.
$(\hat{K}, \partial)$: a fixed differential closure, $\hat{K}^\partial = C$.
G/C : a connected algebraic group (possibly non commutative), say $G \subset GL_{n/C}$.
Then, ∂ extends to a derivation

$$D_\partial : K[G] \to K[G]$$

respecting the group structure : $D_\partial = 0$ on the coordinates functions of $G/\mathbb{C}$.
For $y \in G(K), T_yG \simeq LG$. Pulling $\partial y \in T_yG$ to 1_G by translation by y, we get the *logarithmic derivative on* G

$$\partial\ell n_G : G \to LG.$$

$$G \subset GL_n \rightsquigarrow \partial\ell n_G y = \partial y.y^{-1}.$$

E.g., $\partial\ell n_{\mathbf{G}_m} y = \frac{\partial y}{y}$, while $\partial\ell n_{\mathbf{G}_a} y = \partial y$.

Given $a \in LG(K)$, P-V. theory studies the differential extension $K(y)/K$, where y is a (any) solution in $G(\hat{K})$ of

$$\partial\ell n_G(y) = a,$$

and its Galois group

$$Aut_\partial(K(y)/K) = J_a(C) \subset G(C).$$

In general, $\partial\ell n_G(uv) = \partial\ell n_G u + u(\partial\ell n_G v)u^{-1}$,

$$\partial\ell n(u^{-1}v) = u^{-1}(-\partial\ell nu + \partial\ell nv)u.$$

So: i) $K(y)$ depends only on the *orbit* of a under the action of $u \in G(K)$. Namely, if $\partial\ell ny = a$, then, $\tilde{y} = uy$ satisfies $\partial\ell n(\tilde{y}) = \tilde{a}$, where $\tilde{a} = \partial\ell nu + uau^{-1}$, and $K(y) = K(\tilde{y})$.

. ii) $\forall\sigma \in Aut_\partial$, $\sigma y = y.\rho(\sigma), \rho(\sigma) \in G^\partial(\widehat{K})$, Here $G^\partial(\widehat{K}) = \{g \in G(\widehat{K}), \partial\ell ng = 0\} = G(C)$.

Thm. : *$Im(\rho) = J_a(C)$, where J_a/C is an algebraic subgroup of G/C; there is a Galois correspondence (e.g., $K(y)^{J_a(C)} = K$); and*

$$tr.deg.(K(y)/K) = dim(J_a).$$

How to *compute* $(J_a)^0$? From the Gospel, Michael I.31.(1) & Marius I.31.(2), we get:

Thm. : *assume K alg. closed. Then, J_a is a minimal algebraic subgroup J/C of G such that $LJ(K)$ meets the orbit of a under $G(K)$.*

Assume a *non degenerate* : *for any proper algebraic subgroup $H \subset G$, the $G(K)$-orbit of a does not meet $LH(K)$.* Then $J_a = G$.

If G is abelian, $a \rightsquigarrow \overline{a} \in LG(K)/\partial\ell n_G G(K)$, hence

. • Kolchin's theorem on $\mathbf{G}_m^n$

. • Ostrowski's theorem on $\mathbf{G}_a^n$.

III. D-groups and Pillay's theory

(K, ∂) :*alg. closed* diff. field with $K^\partial = C$.
$(\hat{K}, \partial)$: a fixed differential closure, $\hat{K}^\partial = C$.
G/K : a connected commutative algebraic group over K. Say G is commutative.

Assume that ∂ extends to a derivation

$$D_\partial : K[G] \to K[G]$$

respecting the group structure (we then say that G *is a D-group*). Equivalently, the twisted tangent bundle

$$0 \to LG \to T_\partial G \to G \to 0$$

admits a regular section $s : G \to T_\partial G$.

For $y \in G(K), \partial y \in (T_\partial G)_y$, so D_∂ yields a *logarithmic derivative on G :*

$$\partial \ell n_{G,s} : G \to LG : y \mapsto \partial y - s(y).$$

Given $a \in LG(K)$, Pillay's theory studies the differential extension $K(y)/K$, where y is a solution in $G(\hat{K})$ of

$$\partial \ell n_G(y) = a,$$

and its Galois group $Aut_\partial(K(y)/K)$. To avoid "new constants", we request that

$$G \text{ is "}K\text{-large : } G^\partial(\hat{K}) = G^\partial(K)$$

where $G^\partial(\hat{K}) = \{g \in G(\hat{K}), \partial \ell n_G g = 0\}$.

. Since $\partial\ell n_G(uv) = \partial\ell n_G u + \partial\ell n_G v$,

. i) $K(y)$ depends only on the class of a in $Coker(\partial\ell n_G, K) := LG(K)/\partial\ell n_G(G(K))$.

. ii) $\forall\sigma \in Aut_\partial$, $\sigma y = y + \xi(\sigma), \xi(\sigma) \in G^\partial(K)$,

Thm.: *$Im(\xi) = N_a^\partial(K)$, where N_a/K is a connected algebraic D-subgroup of $(G/K, s)$; there is a Galois correspondence (in particular, $K(y)^{N_a^\partial(K)} = K$); and*

$$tr.deg.(K(y)/K) = dim(N_a).$$

How to *compute* N_a ? Going to G/N_a, we get :

Thm.: *N_a is a minimal algebraic D-subgroup N of (G, s) such that a lies in $LN(K) + \partial\ell n_G G(K)$.*

Assume a *non degenerate* : *for any proper algebraic D-subgroup $H \subset G$, $a + \partial\ell n_G G(K)$ does not meet $LH(K)$.* Then $N_a = G$.

Example : D-modules

$G = V/K$, with $s(y) = By, B \in Hom(V,V)$. Then, $\partial\ell n_V := \partial_V : V \to LV \simeq V$:

$$y \mapsto \partial_V(y) = \partial y - By.$$

So, (V,s) is "large" only over the alg. closure $\mathbf{K}_{(B)}$ of the P-V field $K_V = K(V^\partial_{(B)})$.
Given $a \in (LV = V)(K)$, we study the DE

$$\partial_V y = a, \ i.e. \ \partial y - By = a.$$

Its Galois group $Gal_\partial(\mathbf{K}(y)/\mathbf{K}) = (N_a)^\partial(\mathbf{K})$ is a $\mathbb{C}$-subspace N^∂ of V^∂, where N is a minimal $\mathbf{K}$-vector subspace of $V \otimes \mathbf{K}$ stable under ∂_V, such that $v \in \partial_V(V(\mathbf{K})) + N(\mathbf{K})$.

Actually, N must then be defined over K (see Part V). Moreover, if V is completely reducible over K, one can speak of *the* minimal D-submodule N satisfying these properties; furthermore, they then automatically descend to K. So, in this "pure" case, N is the smallest D-submodule of V over K such that $v \in \partial_V(V(K)) + N(K)$.

Abelian varieties provide another example of "pure" algebraic D-groups, and we will now restrict to this case.

• $K = \mathbb{C}(S); \partial \in H^0(S, TS)$: a vector field;
• $\pi : \mathbf{G} \to S$: a group scheme; $\mathbf{e} =$ 0-section,
• $L\mathbf{G}$: the pull-back $\mathbf{e}^*(T_{\mathbf{G}|S})$ of the relative tangent bundle of $\mathbf{G}$ over S.
• At the generic point of S, we get G/K, with (relative) tangent bundle $TG \simeq G \times LG$
• The (full) tangent bundle $T\mathbf{G}$ of $\mathbf{G}$ sits in an exact sequence

$$0 \to T_{\mathbf{G}|S} \to T\mathbf{G} \to \pi^*(TS) \to 0$$

of vector bundles over $\mathbf{G}$, and is also a group scheme over TS. When t runs through S, its fibers $(T\mathbf{G})_{(t,\partial_t)}$ yield a group scheme $T_\partial \mathbf{G}$ over S, whose generic fiber is the **twisted tangent bundle** $T_\partial G/K$.
• A section $\mathbf{x}$ of $\mathbf{G}/S$ provides a section $\mathbf{x}_*(\partial)$ of $T_\partial \mathbf{G}/S$, written ∂x at the generic point.
• Viewed over K, $T_\partial G$ is a *group extension of G by LG*;
• Viewed over G, $T_\partial G$ is a *torsor under LG*, whose class in $H^1(G, TG)$ is given, in the proper case, by the **Kodaira-Spencer** map.

IV . Interlude : Kummer theory

• $K =$ number field, $\overline{K} =$ algebraic closure.

• $A =$ an abelian variety over K, $dimA := g$. Set $End(A/K) = End(A/\overline{K}) := \mathcal{O}$.

• $y \in A(K)$. Assume that y *generates* A, i.e. $\mathbb{Z}.y$ is Zariski closed in $A \Leftrightarrow Ann_{\mathcal{O}}(y) = 0$.

Following the elliptic work of Bashmakov and Tate-Coates ($\sim$ 1970), we have :

Theorem K : *there exists* $c = c(A, K, y) > 0$ *such that for all* $n > 0$, $[K(\frac{1}{n}y) : K] \geq cn^{2g}$.

Refs.: K. Ribet : Duke math. J. 46, 1979, 745-761;
D.B. : Proc. Durham Conference 1986, “New advances in transcendence theory”, ed. A. Baker, CUP 1988, 37-55.

- $A_{tor} = \cup_n A[n], \ K_\infty = K(A_{tor})$

- $L_\infty = \cup_n K_\infty(\frac{1}{n}y), \quad L_{(\ell)} = \cup_m K_\infty(\frac{1}{\ell^m}y).$

- $T_\infty(A) := proj.lim_n \ A[n] = \Pi_{\ell \in \mathcal{P}} T_\ell(A)$

We will actually prove that $Gal(L_\infty/K_\infty)$ *is isomorphic to an open subgroup of* $T_\infty(A)$, or equivalently (Nakayama) :

i) for all primes ℓ, $Gal(L_{(\ell)}/K_\infty)$ is an open subgroup of $T_\ell(A) \simeq \mathbb{Z}_\ell^{2g}$;

ii) for almost all ℓ, $Gal(K_\infty(\frac{1}{\ell}y)/K_\infty) \simeq A[\ell]$.

$$\begin{array}{cccc}
\overline{K} & & & \\
| & & & \\
K_\infty(\frac{1}{n}y) & & \xi_y & \\
| & \}N & \hookrightarrow & A[n] \simeq (\mathbb{Z}/n\mathbb{Z})^{2g} \\
K_\infty & & \rho & \\
| & \}J & \hookrightarrow & GL(T_\infty(A)) \\
K & & &
\end{array}$$

$$\xi_y(\sigma) = \sigma(\tfrac{1}{n}y) - \tfrac{1}{n}y, \quad \xi_y(\tau\sigma\tau^{-1}) = \tau(\xi_y(\sigma)).$$

Proof (in the mod ℓ case)

1. Galois theoretic step .
(Of necessity, base extension to $K_\infty \rightsquigarrow A$ becomes "K_∞-large" for the morphism $[\ell]_A$.)

$Im(\xi_y) \simeq N$ is a J-submodule of $A[\ell]$. Assume $N \neq A[\ell]$. Then $\exists \alpha \in \mathcal{O}, \alpha \notin \ell\mathcal{O}$ s.t.

$\alpha.y$ is divisible by ℓ in $A(K_\infty)$.

2. Galois descent

There exists $\ell_0(A, K)$ such that $\forall \ell > \ell_0$, if a point $y' \in A(K)$ is divisible by ℓ in $A(K_\infty)$, then, y' is already divisible by ℓ in $A(K)$, i.e.

$$A(K)/\ell.A(K) \hookrightarrow A(K_\infty)/\ell.A(K_\infty)$$

3. (Diophantine) geometric step

There exists $\ell_1(A, K, y)$ such that $\alpha.y \in \ell.A(K)$ with $\ell > \ell_1$ implies $\alpha \in \ell.\mathcal{O}$.

Proof of 1.

- $A[\ell]$ is a semi-simple J-module (Faltings), so there exists $\alpha_\ell \in End_J(A[\ell])$ killing N.
- $End_J(A[\ell]) \simeq End(A) \otimes \mathbf{F}_\ell$ (Faltings), so α_ℓ yields $\alpha \in \mathcal{O}, \alpha \notin \ell\mathcal{O}$ killing N.
- $\xi_{\alpha.y} = \alpha\xi_y$, so, $\frac{1}{\ell}\alpha.y$ is fixed by N.

Proof of 2.

$$\begin{array}{ccccc} ? & \rightarrow & A(K)/\ell.A(K) & \rightarrow & A(K_\infty)/\ell.A(K_\infty) \\ \downarrow & & \downarrow & & \downarrow \\ H^1(J, A[\ell]) & \rightarrow & H^1(\Gamma_K, A[\ell]) & \rightarrow & H^1(\Gamma_{K_\infty}, A[\ell])^J \end{array}$$

Serre's result on homotheties and Sah's lemma imply $H^1(J, A[\ell]) = 0$ for large ℓ.

Proof of 3.

Mordell-Weil (or a trick of Cassels's), both based on heights.

[Similar arguments *in the ℓ-adic case*.]

V. Logarithms on abelian schemes

• $K = \mathbb{C}(S)$ or $\mathbb{C}(S)^{alg}$, $S/\mathbb{C}$ = smooth affine curve, ∂ = a derivation on K with $K^{\partial} = \mathbb{C}$, $\widehat{K}$ = diff. closure, $\mathcal{U}$ = univ. domain.

• A/K, coming from an abelian scheme $\mathcal{A} \to S$. A_0 = its $K/\mathbb{C}$-trace. Its universal extension $\tilde{A}$ has dimension $2g$:

$$0 \to W_A \to \tilde{A} \to^{\pi} A \to 0$$

Exponential sequence :

$$0 \to T_B\tilde{\mathcal{A}} \to L\tilde{\mathcal{A}}^{an} \to^{exp} \tilde{\mathcal{A}}^{an} \to 0$$

• $y \in \tilde{A}(K)$, generating $\tilde{A}$, i.e. : $\forall H \subsetneq \tilde{A}, y \notin H + \tilde{A}_0(\mathbb{C})$. Chose $\ell n(y) \in exp^{-1}(y)$. Then :

Theorem L (André, 1992)

$$tr.dg.(K(\ell n(y))/K) = 2g.$$

$\tilde{A}$ has a structure of algebraic D-group, with

$$\partial\ell n_{\tilde{A}} : \tilde{A} \to L\tilde{A}$$

Gauss-Manin connection :

$$\partial_{L\tilde{A}} = \partial\ell n_{\tilde{A}} \circ exp : L\tilde{A} \to L\tilde{A}$$

So $\ell n(y) \rightsquigarrow x \in L\tilde{A}(\widehat{K})$ solution of the inhomogeneous LDE : $\partial_{L\tilde{A}}(x) = \partial\ell n_{\tilde{A}} y$.

• $K_{L\tilde{A}} = K(T_B(\tilde{A}))$ = Picard-Vessiot extension for $\partial_{L\tilde{A}}(-) = 0$, with solution space $(L\tilde{A})^\partial = T_B(\tilde{A}) \otimes \mathbb{C} \simeq \mathbb{C}^{2g}$.

We will actually prove that

$$Gal_\partial(K_{L\tilde{A}}(\ell n(y))/K_{L\tilde{A}}) \simeq (L\tilde{A})^\partial.$$

$$\begin{array}{cccc}
\widehat{K} & & & \\
| & & & \\
K_{L\tilde{A}}(\ell n(y)) & & \xi_y & \\
| & \}N & \hookrightarrow & (L\tilde{A})^\partial \\
K_{L\tilde{A}} & & \rho & \\
| & \}J & \hookrightarrow & GL((L\tilde{A})^\partial) \\
K & & &
\end{array}$$

$$\xi_y(\sigma) = \sigma(\ell n(y)) - \ell n(y), \quad \xi_y(\tau\sigma\tau^{-1}) = \tau(\xi_y(\sigma)).$$

Proof (in a "generic" case)

By Deligne, $L\tilde{A}$ is a semi-simple D-module. For simplicity, suppose that it is irreducible.

1. Galois theoretic step .
(Of necessity, base extension to $K_{L\tilde{A}} \rightsquigarrow L\tilde{A}$ becomes "$K_{L\tilde{A}}$-large" for the morphism $[exp]_{\tilde{A}}$.)

$Im(\xi_y) \simeq N$ is a J-submodule of $(L\tilde{A})^\partial$. Assume $N \neq (L\tilde{A})^\partial$. Then $N = 0, x \in L\tilde{A}(K_{L\tilde{A}})$ and

$$\partial\ell n_{\tilde{A}} y = \partial_{L\tilde{A}}(x) \in \partial_{L\tilde{A}}\Big(L\tilde{A}(K_{L\tilde{A}})\Big).$$

2. Galois descent

If a point $z \in L\tilde{A}(K)$ lies in $\partial_{L\tilde{A}}\Big(L\tilde{A}(K_{L\tilde{A}})\Big)$, then, z already lies in $\partial_{L\tilde{A}}(L\tilde{A}(K))$, i.e.

$$Coker(\partial_{L\tilde{A}}, L\tilde{A}(K)) \hookrightarrow Coker(\partial_{L\tilde{A}}, L\tilde{A}(K_{L\tilde{A}}))$$

Indeed, J is reductive, so $H^1(J, (L\tilde{A})^\partial) = 0$.

3. Geometric step

Manin's theorem : if $\partial\ell n_{\tilde{A}} y = \partial_{L\tilde{A}}(x)$ for some $x \in L\tilde{A}(K)$, then $y \in W_A + \tilde{A}_0(\mathbb{C}) + \tilde{A}_{tor}$.

VI. Exponentials on abelian schemes

As in V,

$$K = \mathbb{C}(S),\ \partial,\ A/K,\ A_0/\mathbb{C},\ \tilde{A}.$$
$$0 \to T_B\tilde{\mathcal{A}} \to L\tilde{\mathcal{A}}^{an} \to^{exp} \tilde{\mathcal{A}}^{an} \to 0$$

• $x \in L\tilde{A}(K)$, generating $L\tilde{A}$, i.e. : $\forall H \subsetneq \tilde{A}, x \notin LH + L\tilde{A}_0(\mathbb{C})$. Then :

Theorem E (Be-Pillay, JAMS, 201?)

$$tr.dg.(K(exp(x)/K) = 2g.$$

As in V, we have

$$\partial\ell n_{\tilde{A}} : \tilde{A} \to L\tilde{A}$$
$$\partial_{L\tilde{A}} = \partial\ell n_{\tilde{A}} \circ exp : L\tilde{A} \to L\tilde{A}.$$

So $exp(x) \rightsquigarrow y \in \tilde{A}(\hat{K})$ solution of the inhomogeneous NLDE : $\partial\ell n_{\tilde{A}}(y) = \partial_{L\tilde{A}}x$.

Let $K_{\tilde{A}}$ be the differential extension of $\overline{K}$ generated by all points in

$$\tilde{A}^{\partial} = \{z \in \tilde{A}(\hat{K}), \partial\ell n_{\tilde{A}}(z) = 0.\}$$

Using
. • Pillay's Galois theory
. • + a Galois descent ,
we will actually prove that

$$Gal_\partial(K_{\tilde{A}}(exp(x))/K_{\tilde{A}}) \simeq \tilde{A}^\partial.$$

$$\begin{array}{cccc}
\widehat{K} & & & \\
| & & & \\
K_{\tilde{A}}(exp(x)) & & \xi_x & \\
| & \}N & \hookrightarrow & \tilde{A}^\partial \\
K_{\tilde{A}} & & \rho & \\
| & \}\tilde{J} & \hookrightarrow & Aut(\tilde{A}^\partial) \\
\overline{K} & & &
\end{array}$$

$$\xi_x(\sigma) = \sigma(exp(x)) - exp(x).$$

In generic cases (e.g. when the Kodaira-Spencer rank of A/S is maximal, e.g. when $L\tilde{A}$ is irreducible),

$$K_{\tilde{A}} = \overline{K} :$$

the D-group $\tilde{A}$ is $\overline{K}$-large, and no descent is required ! We then merely need :

1. Galois theoretic step

$Im(\xi_x) \simeq N = H^\partial$ for some algebraic D-subgroup H of $\tilde{A}$. Assume $H \neq \tilde{A}$. Then there is a non trivial D-quotient $\pi : \tilde{A} \to \overline{A}$ sending x to $\overline{x} \in L\overline{A}(K)$, with

$$\partial_{L\overline{A}}(\overline{x}) = \partial\ell n_{\overline{A}}(\overline{y}) \text{ for some } \overline{y} \in \overline{A}(K).$$

3. Geometric step

If $\overline{A} \simeq \tilde{B}$ for some abelian variety quotient B of A, just apply Manin's theorem:
$\overline{x} \in LW_B + L\tilde{B}_0(\mathbb{C})$, so x cannot generate $L\tilde{A}$.

The general case requires Chai's sharpening of Manin's theorem.

That $\overline{A} \simeq \tilde{B}$ happens automatically when W_A contains no non trivial D-subgroup. When $A_0 = 0$, this is equivalent to $\tilde{A}$ being $\overline{K}$-large. In general,

2. Galois descent in Pillay's theory

Write K for $\overline{K}$, and let U be the maximal D-subgroup of $\tilde{A}$ (equivalently D-submodule of $L\tilde{A}$) contained in W_A.

$$0 \to U \to \tilde{A} \to \overline{A} \to 0.$$

- Hrushovski-Sokolovic, Marker-Pillay $\Rightarrow \overline{A}$ is K-large : $\overline{A}^{\partial}(\hat{K}) = \overline{A}^{\partial}(K)$.
- Manin-Chai $\Rightarrow \overline{A}^{\partial}(K) = \overline{A}_{tor} + A_0(\mathbf{C})$.
- $0 \to U^{\partial}(\hat{K}) \to \tilde{A}^{\partial}(\hat{K}) \to \overline{A}^{\partial}(\hat{K}) \to 0$.

Therefore

$K_{\tilde{A}} = K_U$ is a P-V extension of K and $\tilde{J} = Gal_{\partial}(K_{\tilde{A}}/K) := J_U$ is a factor of the reductive group $J = Gal_{\partial}(K_{L\tilde{A}}/K)$. Actually (Deligne), J, hence J_U, is semi-simple.

By Step 1 over $K_{\tilde{A}}$, and rigidity of D-subgroups of $\tilde{A}$, we have :

$\partial_{L\overline{A}}(\overline{x}) = \partial\ell n_{\overline{A}}(\overline{y})$ for some $\overline{y} \in \overline{A}(K_U)$.

and it remains to show that

$L\overline{A}(K)/\partial\ell n_{\overline{A}}(\overline{A}(K)) \hookrightarrow L\overline{A}(K_U)/\partial\ell n_{\overline{A}}(\overline{A}(K_U))$,

i.e. that we may take $\overline{y} \in \overline{A}(K)$.

The cocycle $\widehat{\xi}_{\overline{y}} : J_U \to \overline{A}^{\partial} : \sigma \mapsto \sigma\overline{y} - \overline{y}$ is a group homomorphism. Since $J_U = [J_U, J_U]$, while $\overline{A}^{\partial}$ is abelian, $\xi_{\overline{y}}$ vanishes, so that indeed $\overline{y}$ is defined over K.

Conclusion

• This shows that just as P-V., Pillay's theory can (sometimes) cover relative situations. Does the same hold of Malgrange's theory ?

• The method works in other contexts, e.g., considering the differential equation

$$\partial \ell n(y) = \lambda . \partial \ell n(x)$$

on $\mathbb{G}_m$, with $\lambda \in \mathbb{C}, \lambda \notin \mathbb{Q}$:

if $x_1, ..., x_n \in \mathbb{G}_m(K)$ are multiplicatively independent modulo $\mathbb{G}_m(\mathbb{C})$, then, $x_1^{\lambda}, ..., x_n^{\lambda}$ are algebraically independent over $K = \mathbb{C}(z)$.

For more general (Schanuel-type) results on x^{λ}, see work of M. Bayes, J. Kirby, A. Wilkie, (arXiv: 0810.4457, 2008) and P. Kowalski (Ann. PAL, 156, 2008, 96-109).