q-difference equations

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$\boxed{q\text{-diff. equa.}/\mathbb{C},\ |q| \neq 1}$

$$\begin{tikzcd}
\boxed{q\text{-diff. equa.}/\mathbb{C},\ |q| \neq 1} \arrow[d, leftrightarrow, "\cong"] \\
\boxed{p\text{-adic } q\text{-diff. equa.},\ |q| \neq 1}
\end{tikzcd}$$

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\boxed{p\text{-adic } q\text{-diff. equa., } |q| \neq 1} & \boxed{p\text{-adic } q\text{-diff. equa., } |q| = 1} \arrow[d, leftrightarrow, "\equiv"'] \\
 & \boxed{p\text{-adic diff. equa.}}
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Let $a \in \mathbb{Z}_p \smallsetminus \mathbb{Z}$.

$$x\frac{d}{dx} \circ \left(x\frac{d}{dx} - (a+x) \right) y(x) = 0$$

Solution:

$$y(x) = \sum_{n\geq 0} \frac{x^n}{(1-a)_n} \in \mathbb{Q}_p[[x]]\,,$$

where $(1-a)_n = (1-a)(2-a)\cdots(n-a)$.

$\mathbb{Z}$ is dense in $\mathbb{Z}_p \Rightarrow \exists a \in \mathbb{Z}_p$ s.t $y(x)$ is divergent

Rmk. a is an exponent

Let $q, \alpha \in \mathbb{C}$, $|q| = |\alpha| = 1$, $d_q f(x) = \dfrac{f(qx) - f(x)}{(q-1)x}$

$$x d_q \circ \left(\alpha q^{-1} x d_q + \frac{\alpha q^{-1} - 1}{q - 1} - x \right) y(x) = 0$$

$$y(x) = \sum_{n \geq 0} \frac{(1-q)^n x^n}{(\alpha; q)_n},$$

where $(\alpha; q)_n = (1-\alpha)(1-q\alpha)\cdots(1-q^{n-1}\alpha)$.

<u>**Remark**</u>:

- When $|\alpha| = |q| = 1$ this series may diverge.
- $\alpha^{-1} q$ is an exponent.

Example

$q \in \mathbb{C}$, $|q| = 1$, $\mathbf{K} = \mathbb{C}(\{x\})$.

The series

$$\Phi(x) = \sum_{n \geq 0} \frac{(1-q)^n x^n}{(1-q\lambda)\cdots(1-q^n\lambda)},$$

$\lambda \in \mathbb{C}^*$, $\lambda \notin q^{\mathbb{Z}_{<0}}$, is solution of

$$\mathcal{L} = (\sigma_q - 1) \circ [\lambda\sigma_q - ((q-1)x + 1)].$$

The modules associated to $\mathcal{L}$ is analytically isomorphic to:

$$\begin{array}{ccc} \mathbf{K}^2 & \longrightarrow & \mathbf{K}^2 \\ \mathbf{y}(x) & \longmapsto & \begin{pmatrix} 1 & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \mathbf{y}(qx) \end{array}$$

if and only if $\Phi(x)$ is convergent ($\rightarrow$ diophantine condition).

$e_q(x) = \sum_{n\geq 0} \frac{x^n}{(q;q)_n}$ is solution of

$$y(qx) = (1+x)y(x)\,.$$

It may have radius of convergence between 0 and 1

For any $r \in [0,1]$ there exists $q \in \mathbb{C}$, $|q| = 1$, not a root of unity, such that the radius of convergence of $e_q(x)$ is equal to r.

$$\log e_q(x) = \sum_{n\geq 1} \frac{(-1)^{n+1}}{n(1-q^n)} x^n \qquad \text{(Hardy-Wright)}$$

For a complete study of $e_q(x)$; *cf.* Lubinsky, 1998.

$E_q(x) = \sum_{n\geq 1} \frac{(q;q)_n}{(1-q)^n} x^{n+1}$ is solution of

$$x^2 d_q y(x) - y(x) = -x\,.$$

It converges for $|x| < |1-q|$, independently of the choice of q.

Hint. A weak version of the equidistribution criteria allows to prove that

$$\limsup_{n\to\infty} |(q;q)_n|^{1/n} = 1\,;$$

cf. Driver-Lubinsky-Petruska-Sarnak, 1991.

The series

$$\Phi(x) = \sum_{n \geq 0} \frac{(1-q)^n x^n}{(1-q\lambda)\cdots(1-q^n\lambda)},$$

$\lambda \in \mathbb{C}^*$, $\lambda \notin q^{\mathbb{Z}_{<0}}$, solution of

$$\mathcal{L} = (\sigma_q - 1) \circ [\lambda\sigma_q - ((q-1)x+1)],$$

can have any radius of convergence in $[0,1]$.

Lemma

$$\Phi(x) = (1-\lambda)\left(\sum_{n \geq 0} \frac{x^n}{(q;q)_n}\right)\left(\sum_{n \geq 0} q^{n(n+1)/2} \frac{1}{(1-q^n\lambda)} \frac{(-x)^n}{(q;q)_n}\right)$$

The series $\sum_{n\geq 0}(\lambda;q)_n x^n$, solution of

$$(\sigma_q - 1)\circ(x\lambda\sigma_q - (x-1))\,y(x) = 0\,,$$

is always convergent of radius of convergence ≥ 1.

cf. Driver-Lubinsky-Petruska-Sarnak, 1991, and Petruska, 1992.

Analytic classification with $|q| = 1$

$\mathcal{B}_q$= category of q-difference modules over $\mathsf{K} = \mathbb{C}(\{x\})$.
$\widehat{\mathcal{B}}_q$= category of q-difference modules over $\widehat{\mathsf{K}} = \mathbb{C}((x))$.

Proposition (Soibelman-Vologosky)

If $e_q(x)$ is convergent, then

$$Pic\,(\mathsf{K}, \sigma_q) = Pic(\mathcal{O}(\mathbb{C}^*, \sigma_q) = \mathbb{C}^*/q^{\mathbb{Z}} \times \mathbb{Z}\,.$$

Let $\mathcal{M} = (M, \Sigma_q)$ be an object of $\mathcal{B}_q$, having Newton polygon $\{(\mu_i, r_i) : i = 1, \dots, \kappa\}$.

Theorem (DV)

If $e_q(x)$ is convergent, then the q-difference module $\mathcal{M}$ can be decomposed in a direct sum:

$$\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \cdots \oplus \mathcal{M}_\kappa \,,$$

such that the Newton polygon of $\mathcal{M}_i$ has only the slope μ_i and rank r_i.

Remark. Notice that if $\mu_i \in \mathbb{Z}$ then $\mathcal{M}_i \otimes_{\mathbf{K}} \hat{\mathrm{K}}$ admits a basis $\underline{e}$ such that $\Sigma_q \underline{e} = \underline{e} A x^{-\mu_i}$, with $A \in Gl_\nu(\mathbb{C})$.

$\mathcal{B}_q^f$ = full subcategory of $\mathcal{B}_q$ containing the objects whose Newton polygon has only the zero slope.

Corollary

If $e_q(x)$ is convergent, then $\mathcal{B}_q$ is equivalent to the category of $\mathbb{Q}$-graded objects of $\mathcal{B}_q^f$.

Problem

Some isoanalytic classes correspond $1:1$ to their isoformal classes. Which one?Is this phenomenon generic?

$\mathcal{M} = (M, \Sigma_q) \in \mathcal{B}_q$ with N.P. $\{(\mu, r)\}$ and $\mu \in \mathbb{Z}$
$\Rightarrow \exists\ \underline{e}$ basis of $\widehat{\mathcal{M}} = \mathcal{M} \otimes_{\mathbf{K}} \mathbb{C}((x))$ s.t.

$$\Sigma_q \underline{e} = \underline{e}\frac{A}{x^\mu}\,, \quad \text{with } A \in GL_r(\mathbb{C})\,.$$

$\Lambda = \{\lambda_1, \dots, \lambda_r\}$ = eigenvalue of A, called exponents.

If $\mu \in \mathbb{Q}$ then $r\mu \in \mathbb{Z}$ and we can reduce to the previous situation by extending the scalars to $\mathbf{K}(x^{1/r'})$, for some r' dividing r. In this way we can still define the set Λ.

Consider:

$$\Phi_{\mathcal{M}}(x) = \sum_{n\geq 1}\left(\prod_{\substack{\lambda_i,\lambda_j\in\Lambda \\ \lambda_i/\lambda_j \notin q^{\mathbb{Z}\leq 1}}} \frac{1}{(1-q\lambda_i/\lambda_j)(1-q^2\lambda_i/\lambda_j)(1-q^n\lambda_i/\lambda_j)}\right)x^n\,.$$

Let $\mathcal{B}_q^{Dio}$ the full subcategory of $\mathcal{B}_q$ of the objects are direct sum of isoclinic objects $\mathcal{M}$ such that $\Phi_{\mathcal{M}}(x)$ is convergent.

Theorem

The category $\mathcal{B}_q^{Dio}$ is the largest full subcategory of $\mathcal{B}_q$ such that the scalar extension $-\otimes_{\mathbf{K}} \widehat{\mathbf{K}}$ induces an equivalence of category with its essential image in $\widehat{\mathcal{B}}_q$.

Simple objects of $\mathcal{B}_q^{Dio}$:

- rank one modules associated with equation of the form:
  $y(qx) = \dfrac{\lambda}{x^{\mu}} y(x)$, $\lambda \in \mathbb{C}^*$, $\mu \in \mathbb{Z}$.
  Isomorphism class: $\rightarrow (\lambda q^{\mathbb{Z}}, \mu)$.
  (simple objects of slope μ)
- $\mathbf{K}$-modules obtain by restriction of scalars $\mathbf{K} \hookrightarrow \mathbf{K}(x^{1/r})$ from rank 1 $q^{1/r}$-modules associated to the equation:
  $y(q^{1/r}x^{1/r}) = \dfrac{\lambda}{x^{\mu/r}} y(x^{1/r})$, $\lambda \in \mathbb{C}^*$, $\mu \in \mathbb{Z}$, $(\mu, r) = 1$.
  (simple objects of slope $\mu/r \in \mathbb{Q}$)

Indecomposable objects of $\mathcal{B}_q^{Dio}$:

Iterated nontrivial extensions of simple a object by itself.

Remark. This is the analogue for $|q| = 1$ of:

1. van der Put-Reversat,2006, for $|q| \neq 1$
2. Soibelmann-Vologodsky, 2003, formal case, q not a root of unity.

$\rightarrow$ Extension of scalars to $\mathcal{O}(\mathbb{C}^*)$ $\rightarrow$ fiber bundles on elliptic curves:

- $|q| \neq 1$:
  1. Baranovsky-Ginzburg (1996)
  2. Sauloy (2004)
  3. van der Put-Reversat (2006)
- $|q| = 1$:
  1. Polishchuk-Schwarz (2002, ...)
  2. Soibelmann-Vologosky (2003)
  3. Mahanta-van Suijlekom (2008)