

q -difference equations

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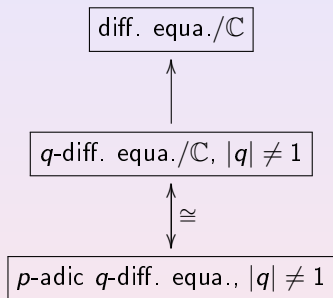
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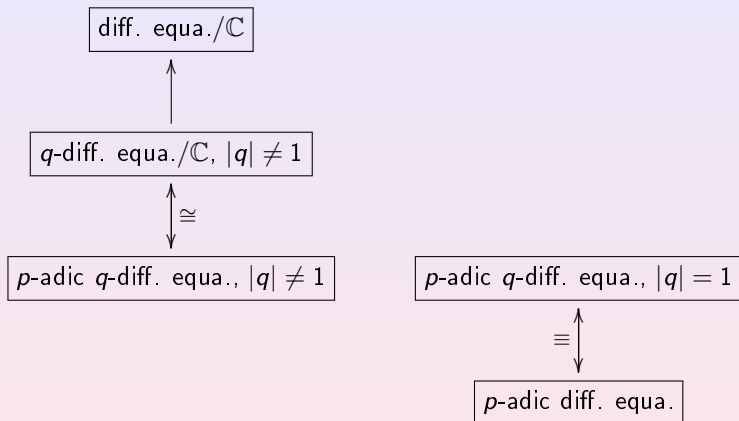
q -diff. equa./ \mathbb{C} , $|q| \neq 1$

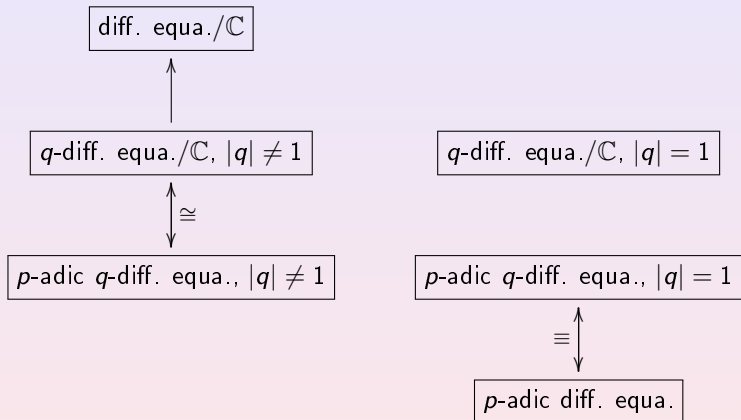
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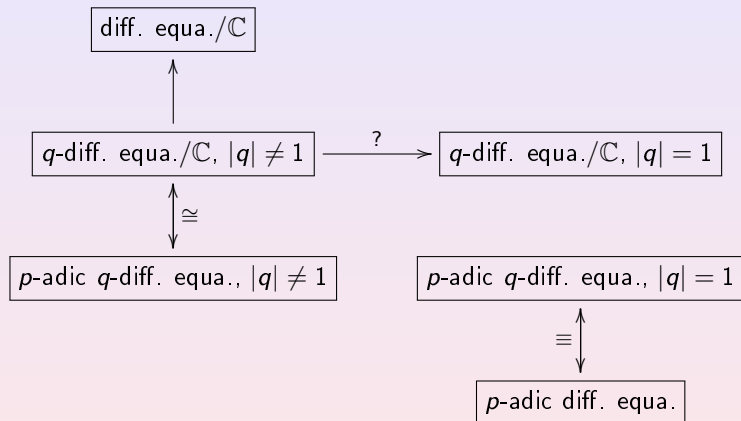


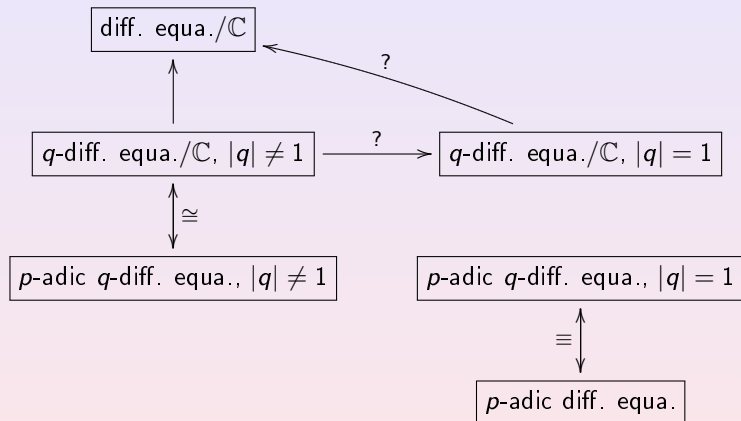
p -adic q -diff. equa., $|q| \neq 1$

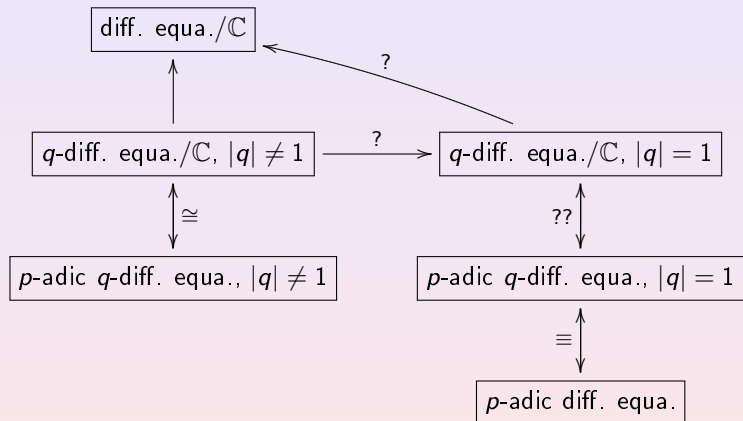












Let $a \in \mathbb{Z}_p \setminus \mathbb{Z}$.

$$x \frac{d}{dx} \circ \left(x \frac{d}{dx} - (a + x) \right) y(x) = 0$$

Solution:

$$y(x) = \sum_{n \geq 0} \frac{x^n}{(1-a)_n} \in \mathbb{Q}_p[[x]],$$

where $(1-a)_n = (1-a)(2-a) \cdots (n-a)$.

\mathbb{Z} is dense in $\mathbb{Z}_p \Rightarrow \exists a \in \mathbb{Z}_p$ s.t $y(x)$ is divergent

Rmk. a is an **exponent**

$$\text{Let } q, \alpha \in \mathbb{C}, |q| = |\alpha| = 1, d_q f(x) = \frac{f(qx) - f(x)}{(q-1)x}$$

$$x d_q \circ \left(\alpha q^{-1} x d_q + \frac{\alpha q^{-1} - 1}{q-1} - x \right) y(x) = 0$$

$$y(x) = \sum_{n \geq 0} \frac{(1-q)^n x^n}{(\alpha; q)_n},$$

where $(\alpha; q)_n = (1-\alpha)(1-q\alpha) \cdots (1-q^{n-1}\alpha)$.

Remark:

- When $|\alpha| = |q| = 1$ this series may diverge.
- $\alpha^{-1}q$ is an exponent.

Example

$$q \in \mathbb{C}, |q| = 1, \mathbf{K} = \mathbb{C}(\{x\}).$$

The series

$$\Phi(x) = \sum_{n \geq 0} \frac{(1-q)^n x^n}{(1-q\lambda) \cdots (1-q^n \lambda)},$$

$\lambda \in \mathbb{C}^*$, $\lambda \notin q^{\mathbb{Z} < 0}$, is solution of

$$\mathcal{L} = (\sigma_q - 1) \circ [\lambda \sigma_q - ((q-1)x + 1)].$$

The modules associated to \mathcal{L} is analytically isomorphic to:

$$\begin{aligned} \mathbf{K}^2 &\longrightarrow \mathbf{K}^2 \\ \mathbf{y}(x) &\longmapsto \begin{pmatrix} 1 & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \mathbf{y}(qx) \end{aligned}$$

if and only if $\Phi(x)$ is convergent (\rightarrow diophantine condition).

$e_q(x) = \sum_{n \geq 0} \frac{x^n}{(q; q)_n}$ is solution of

$$y(qx) = (1+x)y(x).$$

It may have radius of convergence between 0 and 1

For any $r \in [0, 1]$ there exists $q \in \mathbb{C}$, $|q| = 1$, not a root of unity, such that the radius of convergence of $e_q(x)$ is equal to r .

$$\log e_q(x) = \sum_{n \geq 1} \frac{(-1)^{n+1}}{n(1-q^n)} x^n \quad (\text{Hardy-Wright})$$

For a complete study of $e_q(x)$; cf. Lubinsky, 1998.

$$E_q(x) = \sum_{n \geq 1} \frac{(q; q)_n}{(1-q)^n} x^{n+1} \text{ is solution of}$$

$$x^2 d_q y(x) - y(x) = -x.$$

It converges for $|x| < |1 - q|$, independently of the choice of q .

Hint. A weak version of the equidistribution criteria allows to prove that

$$\limsup_{n \rightarrow \infty} |(q; q)_n|^{1/n} = 1;$$

cf. Driver-Lubinsky-Petruska-Sarnak, 1991.

The series

$$\Phi(x) = \sum_{n \geq 0} \frac{(1-q)^n x^n}{(1-q\lambda) \cdots (1-q^n \lambda)},$$

$\lambda \in \mathbb{C}^*$, $\lambda \notin q^{\mathbb{Z}_{<0}}$, solution of

$$\mathcal{L} = (\sigma_q - 1) \circ [\lambda \sigma_q - ((q-1)x + 1)],$$

can have any radius of convergence in $[0, 1]$.

Lemma

$$\Phi(x) = (1-\lambda) \left(\sum_{n \geq 0} \frac{x^n}{(q; q)_n} \right) \left(\sum_{n \geq 0} q^{n(n+1)/2} \frac{1}{(1-q^n \lambda)} \frac{(-x)^n}{(q; q)_n} \right)$$

The series $\sum_{n \geq 0} (\lambda; q)_n x^n$, solution of

$$(\sigma_q - 1) \circ (x \lambda \sigma_q - (x - 1)) y(x) = 0,$$

is always convergent of radius of convergence ≥ 1 .

cf. Driver-Lubinsky-Petruska-Sarnak, 1991, and Petruska, 1992.

Analytic classification with $|q| = 1$

\mathcal{B}_q = category of q -difference modules over $\mathbf{K} = \mathbb{C}(\{x\})$.

$\widehat{\mathcal{B}}_q$ = category of q -difference modules over $\widehat{\mathbf{K}} = \mathbb{C}((x))$.

Proposition (Soibelman-Vologosky)

If $e_q(x)$ is convergent, then

$$\text{Pic}(\mathbf{K}, \sigma_q) = \text{Pic}(\mathcal{O}(\mathbb{C}^*), \sigma_q) = \mathbb{C}^*/q^{\mathbb{Z}} \times \mathbb{Z}.$$

Let $\mathcal{M} = (M, \Sigma_q)$ be an object of \mathcal{B}_q , having Newton polygon $\{(\mu_i, r_i) : i = 1, \dots, \kappa\}$.

Theorem (DV)

If $e_q(x)$ is convergent, then the q -difference module \mathcal{M} can be decomposed in a direct sum:

$$\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \dots \oplus \mathcal{M}_\kappa,$$

such that the Newton polygon of \mathcal{M}_i has only the slope μ_i and rank r_i .

Remark. Notice that if $\mu_i \in \mathbb{Z}$ then $\mathcal{M}_i \otimes_{\mathbf{K}} \hat{\mathbf{K}}$ admits a basis \underline{e} such that $\Sigma_q \underline{e} = \underline{e} A x^{-\mu_i}$, with $A \in GL_\nu(\mathbb{C})$.

\mathcal{B}_q^f = full subcategory of \mathcal{B}_q containing the objects whose Newton polygon has only the zero slope.

Corollary

If $e_q(x)$ is convergent, then \mathcal{B}_q is equivalent to the category of \mathbb{Q} -graded objects of \mathcal{B}_q^f .

Problem

Some isoanalytic classes correspond 1 : 1 to their isoformal classes.
Which one? Is this phenomenon generic?

$\mathcal{M} = (M, \Sigma_q) \in \mathcal{B}_q$ with N.P. $\{(\mu, r)\}$ and $\mu \in \mathbb{Z}$
 $\Rightarrow \exists \underline{e}$ basis of $\widehat{\mathcal{M}} = \mathcal{M} \otimes_{\mathbf{K}} \mathbb{C}((x))$ s.t.

$$\Sigma_q \underline{e} = \underline{e} \frac{A}{x^\mu}, \quad \text{with } A \in GL_r(\mathbb{C}).$$

$\Lambda = \{\lambda_1, \dots, \lambda_r\}$ = eigenvalue of A , called exponents.

If $\mu \in \mathbb{Q}$ then $r\mu \in \mathbb{Z}$ and we can reduce to the previous situation by extending the scalars to $\mathbf{K}(x^{1/r'})$, for some r' dividing r . In this way we can still define the set Λ .

Consider:

$$\Phi_{\mathcal{M}}(x) = \sum_{n \geq 1} \left(\prod_{\substack{\lambda_i, \lambda_j \in \Lambda \\ \lambda_i/\lambda_j \notin q^{\mathbb{Z}} \leq 1}} \frac{1}{(1 - q\lambda_i/\lambda_j)(1 - q^2\lambda_i/\lambda_j)(1 - q^n\lambda_i/\lambda_j)} \right) x^n.$$

Let \mathcal{B}_q^{Dio} the full subcategory of \mathcal{B}_q of the objects are direct sum of isoclinic objects \mathcal{M} such that $\Phi_{\mathcal{M}}(x)$ is convergent.

Theorem

The category \mathcal{B}_q^{Dio} is the largest full subcategory of \mathcal{B}_q such that the scalar extension $- \otimes_{\mathbf{K}} \widehat{\mathbf{K}}$ induces an equivalence of category with its essential image in $\widehat{\mathcal{B}}_q$.

Simple objects of \mathcal{B}_q^{Dio} :

- rank one modules associated with equation of the form:

$$y(qx) = \frac{\lambda}{x^\mu} y(x), \quad \lambda \in \mathbb{C}^*, \quad \mu \in \mathbb{Z}.$$

Isomorphism class: $\rightarrow (\lambda q^{\mathbb{Z}}, \mu)$.

(simple objects of slope μ)

- \mathbf{K} -modules obtain by restriction of scalars $\mathbf{K} \hookrightarrow \mathbf{K}(x^{1/r})$ from rank 1 $q^{1/r}$ -modules associated to the equation:

$$y(q^{1/r} x^{1/r}) = \frac{\lambda}{x^{\mu/r}} y(x^{1/r}), \quad \lambda \in \mathbb{C}^*, \quad \mu \in \mathbb{Z}, \quad (\mu, r) = 1.$$

(simple objects of slope $\mu/r \in \mathbb{Q}$)

Indecomposable objects of \mathcal{B}_q^{Dio} :

Iterated nontrivial extensions of simple a object by itself.

Remark. This is the analogue for $|q| = 1$ of:

1. van der Put-Reversat, 2006, for $|q| \neq 1$
2. Soibelman-Vologodsky, 2003, formal case, q not a root of unity.

→ Extension of scalars to $\mathcal{O}(\mathbb{C}^*)$ → fiber bundles on elliptic curves:

- $|q| \neq 1$:
 - 1 Baranovsky-Ginzburg (1996)
 - 2 Sauloy (2004)
 - 3 van der Put-Reversat (2006)
- $|q| = 1$:
 - 1 Polishchuk-Schwarz (2002, ...)
 - 2 Soibelman-Vologosky (2003)
 - 3 Mahanta-van Suijlekom (2008)