Differential Galois theory of linear difference equations

C. Hardouin (Lab. E. Picard, Toulouse)
with M.F. Singer (NCSU)

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$$u_i(x+1) = b_i(x)u_i(x), \text{ for all } i = 1, ..., n.$$

The functions $u_1, ..., u_n$ are algebraically differentially dependent over the field of meromorphic 1-periodic functions if and only if there exists a non zero linear homogeneous differential polynomial $L(Y_1, ..., Y_n)$ with constant coefficients and $g \in \mathbb{C}(x)$ s.t.

$$L(\frac{\partial(b_1(x))}{b_1(x)}, ..., \frac{\partial(b_n(x))}{b_n(x)}) = g(x+1) - g(x)$$

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Ex : For $\Gamma(x)$, $L(\frac{1}{x}) = g(x+1) - g(x)$???

Theorem[Ishizaki, 1998] Let $(a,b) \in \mathbb{C}(x)^* \times \mathbb{C}(x)$, $q \in \mathbb{C}^*$, $|q| \neq 1$ and let $z(x) \notin \mathbb{C}(x)$ meromorphic over $\underline{\mathbb{C}}$, solution of the equation

$$z(qx) = a(x)z(x) + b(x). \tag{1}$$

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Theorem[H.- Singer, 2007] $z(x)$ differentially algebraic iff $a(x) = cx^n$ and

$b = \sigma_q(f) - af$, for $f \in \mathbb{C}(x)$ if $a \neq q^r$, or

$b = \sigma_q(f) - af + dx^r$, for $f \in \mathbb{C}(x)$, $d \in \mathbb{C}$ if $a = q^r$ where $r \in \mathbb{Z}$.

Theorem[Roques, 2007] Let $y_1(x), y_2(x)$ two linearly independent soutions of

$$y(q^2x) - \frac{2ax-2}{a^2x-1}y(qx) - \frac{x-1}{a^2x-q^2x}y(x) = 0$$

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Galois theory of linear difference equations

k-field, σ-automorphism.
Ex : $k = \mathbb{C}(x), \sigma(x) = x + 1, \sigma(x) = qx$.

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Decomposition ring : $k[Y, \frac{1}{det(Y)}]$, $Y = (y_{i,j})$ indet., $\sigma(Y) = AY$,

$$M \text{ ideal } \sigma - \text{maximal.}$$

σ-Picard-Vessiot Ring $R = k[Y, \frac{1}{det(Y)}]/M = k[Z, \frac{1}{det(Z)}].$

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Ex:

$$k = \mathbb{C}, \sigma(y) = -y$$

$$R = \mathbb{C}[y, \frac{1}{y}]/(y^2 - 1)$$

σ-Galois group:

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<u>Ex.</u>

$$\sigma(y) - y = f, f \in k => \sigma \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$$

$$\phi \in \mathrm{Gal}_\sigma => \phi(\mathrm{y}) = \mathrm{y} + \mathrm{c}_\phi, \mathrm{c}_\phi \in \mathrm{C}$$

$$\mathrm{Gal}_\sigma = (\mathrm{C}, +) \text{ ou } = \{0\}$$

- $\phi \in \mathrm{Gal}_\sigma, \sigma(\mathrm{Z}) = \mathrm{AZ} => \phi(\mathrm{Z}) = \mathrm{Z}[\phi], [\phi] \in \mathrm{GL_n(C)}$

  $\mathrm{Gal}_\sigma \hookrightarrow \mathrm{GL_n(C)}$ and the image is Zariski closed.

  $\mathrm{Gal}_\sigma = \mathrm{G(C)}, \mathrm{G}$ a lin. alg. group $/\mathrm{C}$

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- $R =$ coordinate ring of a G-torsor

  $$R^{\mathrm{Gal}_\sigma} = k$$

  $dim(G) = Krulldim_k(R)(\simeq$ trans. degree of fraction field)

The structure of Gal_σ measure the algebraic relations between the solutions

Ex. $f_1, \ldots, f_n \in k$, k a difference field with an alg. closed field of constants

$$\begin{aligned} \sigma(y_1) - y_1 &= f_1 \\ &\vdots \\ \sigma(y_n) - y_n &= f_n \end{aligned}$$

Picard-Vessiot Ring$= K[y_1, \ldots, y_n]$

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Picard-Vessiot Ring= $K[y_1, ..., y_n]$

Prop $y_1, ..., y_n$ alg. dep. over k if and only if

$\exists g \in k$ and a linear form with constant coeff. L $t.q.$ $L(y_1, ..., y_n) = g$

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Ex. $y(x+1) - y(x) = \frac{1}{x}$
$\frac{1}{x} \neq g(x+1) - g(x) => y(x)$ is not algebraic over $\mathbb{C}(x)$

Linear differential algebraic groups

(k, δ) = a differential field differentially closed

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Definition: A sub-group $G \subset GL_n(k) \subset k^{n^2}$ is a linear differential algebraic group if it is Kolchin-closed in $GL_n(k)$, i. e., G is the sub-set in $GL_n(k)$ of the zeros of a collection of differential polynomials in n^2 variables.

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Ex. Let $C = Ker(\delta)$ and let $G(k)$ a linear algebraic group defined over k. Then $G(C)$ is a linear *differential* algebraic group (add the differential equations $\{\delta y_{i,j} = 0\}_{i,j=1}^{n}$!)

Ex. Differential sub-groups of $\mathbb{G}_a^n(k) = (k^n, +)$
The linear differential algebraic sub-groups are of the form

$$\mathbf{G}_a^{\mathcal{L}} = \{(z_1, ..., z_n) \in k^n | L(z_1, ..., z_n) = 0, \forall L \in \mathcal{L}\}$$

where $\mathcal{L}$ is a set of linear homogeneous differential polynomials.

Ex. H is a Zariski-dense proper differential sub-group of $G \subset GL_n(k)$, a simple algebraic group , defined over $C = Ker(\delta)$

$$=> \exists g \in GL_n(k) \ t.q. \ gHg^{-1} = G(C), C = Ker(\delta).$$

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Decomposition Ring: $k\{Y, \frac{1}{det(Y)}\} = k[Y, \delta Y, \delta^2 Y, ..., \frac{1}{det(Y)}]$

$$Y = (y_{i,j}) \text{ differential indeterminate}$$

$$\sigma(Y) = AY, \sigma(\delta Y) = A(\delta Y) + (\delta A)Y, ...$$

$$M = \ \sigma\delta - \text{maximal ideal}$$

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$\sigma\delta$-Picard-Vessiot Ring$= R = k\{Y, \frac{1}{det(Y)}\}/M = k\{Z, \frac{1}{det(Z)}\}$

$$k - \sigma\delta - \text{field}$$

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$$R=k\{Z,\frac{1}{det(Z)}\}-\sigma\delta-\text{ Picard-Vessiot ring}$$

- R is reduced
- If $C=k^\sigma=\{c\in k|\sigma(c)=c\}$ is differentially closed $=>$ R is unique and $R^\sigma=C$.

$\sigma\delta$-Galois group:

$$\mathrm{Gal}_{\sigma,\delta}(\mathrm{R}/\mathrm{k}) = \{\phi : \mathrm{R} \to \mathrm{R} : \phi \text{ is a } \sigma\delta\mathrm{k} - \text{automorphism}\}$$

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$$\mathrm{Gal}_{\sigma\delta} \hookrightarrow \mathrm{GL_n}(\mathrm{C}) \text{ and the image is Kolchin closed.}$$

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- $\mathrm{Gal}_{\sigma\delta}$ is Zariski dense in Gal_σ

- $R =$ coordinate ring of a G-torsor
  - $R^{\mathrm{Gal}_{\sigma\delta}} = k$
  - If G is connected. Then $diff.dim_C(G) = diff.tr.deg_k(F)$(where F=fraction field of R)

Ex.

$$k = \tilde{\mathbb{C}}, \sigma(y) = -y => R = k[y, \frac{1}{y}]/(y^2 - 1)$$

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Ex.

$$\sigma(y) - y = f, f \in k, \mathrm{Gal}_{\sigma\delta} \subset \mathbb{G}_{\mathrm{a}}$$

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$k - \sigma\delta -$ corps, $C = k^{\sigma}$ differentially closed.

Groups measure differential relations

$$k - \sigma\delta - \text{corps}, \ C = k^{\sigma} \text{ differentially closed.}$$

Differential sub-groups of $\mathbb{G}_a^n(k) = (k^n, +)$ are given by

$$\mathbf{G}_a^{\mathcal{L}} = \{(z_1, ..., z_n) \in k^n | L(z_1, ..., z_n) = 0, \ \forall L \in \mathcal{L}\}$$

where $\mathcal{L}$ is a collection of homogeneous linear differential polynomials.

Differential relations between the solutions of a linear difference equation
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$$\Downarrow$$

Proposition Let R a $\sigma\delta$-Picard-Vessiot ring, extension de k containing $z_1, \dots, z_n$ s.t.

$$\sigma(z_i) - z_i = f_i, i = 1, \dots, n.$$

where $f_i \in k$. Then $z_1, \dots, z_n$ are differentially dependent over k iff there exists an homogeneous linear differential polynomial L over C s.t.

$$L(z_1, \dots, z_n) = g, g \in k$$

or equivalently, $L(f_1, \dots, f_n) = \sigma(g) - g$.

Corollary Let $f_1, ..., f_n \in \mathbb{C}(x)$, $\sigma(x) = x + 1$, $\delta = \frac{d}{dx}$ and let $z_1, ..., z_n$ s.t.

$$\sigma(z_i) - z_i = f_i, i = 1, ..., n.$$

where $f_i \in k$. Then $z_1, ..., z_n$ are differentially dependent over $\mathcal{F}(x)$ ($\mathcal{F}$ is the field of 1-periodic field) iff there exists a linear differential polynomial L over $\mathbb{C}$ s.t.

$$L(z_1, ..., z_n) = g, g \in \mathbb{C}(x)$$

Equivalently, $L(f_1, ..., f_n) = \sigma(g) - g$.

-Similar results for q-differences $\sigma y_i = f_i y_i$

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- If $z(x)$ satisfies a differential polynomial equation then

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- $L(\frac{1}{x})$ has a pole $=>$ $g(x)$ has a pole

The Γ function is hypertranscendent

- $z(x) = \Gamma'(x)/\Gamma(x)$ satisfies $\sigma(z) - z = \frac{1}{x}$
- If $z(x)$ satisfies a differential polynomial equation then

$$\exists L \in \mathbb{C}[\frac{d}{dx}], g \in \mathbb{C}(x) \ s.t. \ L(\frac{1}{x}) = g(x+1) - g(x)$$

- $L(\frac{1}{x})$ has a pole $=>$ $g(x)$ has a pole
- If $g(x)$ has a pole, then $g(x+1) - g(x)$ has at least two poles but $L(\frac{1}{x})$ has exactly one pole.

Let H be a proper, Zariski-dense differential sub-group of $G \subset GL_n(k)$, a simple algebraic group defined over C

$$=> \exists g \in SL_n(k)\ s.t.\ gHg^{-1} = G(C), C = Ker(\delta).$$

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$$\Downarrow$$

<u>Proposition</u> Let $A \in Gl_n(k)$. If the σ-Galois group of $\sigma(Y) = AY$ is a simple linear algebraic group, non-commutative of dimension t. Let $R = k\{Z, \frac{1}{det(Z)}\}$ the $\sigma\delta$-P.V. ring.
The differential trans. degree of R over k is strictly smaller than t

$$\Updownarrow$$

$$\exists B \in gl_n(k) \ t.q. \ \sigma(B) = ABA^{-1} + \delta(A)A^{-1}$$

$$(\text{in that case } (\delta Z - BZ)Z^{-1} \in gl_n(k^\sigma))$$

Comments

- General Theory: Integrable Systems of Equations w.r.t.

$$\Sigma = \{\sigma_1, ..., \sigma_r\}, \Delta = \{\partial_1, ..., \partial_s\}$$

  and differential dependency w.r.t a set of auxilliary derivations $\delta_1, ..., \delta_t$.

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- Inverse Problem