Formal Solution of a Cauchy Problem with Singular Initial Condition and Mordell's Integral*

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At the beginning, we have been interested in a formal solution $\sum_{n=0}^{\infty} e^{n^2\tau+n\zeta}$ of the following heat equation with a initial condition:

$$\begin{cases} \partial_\tau u - \partial_{\zeta\zeta} u = 0, \\ u(\zeta, 0) = \frac{1}{1-e^\zeta}, \end{cases}$$

where $\zeta \in \mathbb{C} \setminus 2\pi\mathbb{Z}i$ and $\tau > 0$.

If set $q = e^{-\tau} \in (0, 1)$, $z = e^\zeta$, then the above series takes the form

$$\sum_{n=0}^{\infty} q^{-n^2} z^n,$$

which is also a solution of a q-difference equation (see the talk of J. SAULOY, but we will assume that $0 < |q| < 1$):

$$\frac{z}{q} y(\frac{z}{q^2}) - y(z) = -1. \quad (1)$$

So we naturally think that it would be workable to study $\sum_{n=0}^{\infty} q^{-n^2} z^n$ with the analytic theory of q-difference equations.

In the following, I will employ two very different methods for constructing solutions of (1), i.e., consider sums given by means of Heat Kernel and Theta function respectively.

At the end of this talk, we will discover a generalized result on the Mordell's Theorem when comparing the difference between the two sums by a Stokes analysis.

we remember that Jacobi Theta function

$$\theta(z,q)=\sum_{n\in\mathbb{Z}} q^{n^2} z^n$$

satisfies the functional equation

$$qz\theta(q^2z,q)=\theta(z,q),$$

and also the modular relation

$$\theta(z,q)=\sqrt{\frac{\pi}{\log 1/q}}\,e^{\frac{(\log z)^2}{4\log 1/q}}\theta(z^*,q^*),$$

where $0<|q|<1$, and where $q^*=e^{\frac{\pi^2}{\log q}}$ and $z^*=e^{-\pi i\frac{\log z}{\log q}}$ are modular variables.

Part I. Introduction

Part II. Summing $\sum_{n=0}^{\infty} q^{-n^2} z^n$ by means of the Heat Kernel

Part III. Summing $\sum_{n=0}^{\infty} q^{-n^2} z^n$ by means of Theta function

Part IV. Generalization of Mordell's Theorem by comparing sums of $\sum_{n=0}^{\infty} q^{-n^2} z^n$

In Part I., I shall give some motivations of the study of (1) from a point view of the analytic theory of numbers. Namely, I shall give a brief exposition on a paper of Mordell and see how to lead to the q-difference equation (1).

The equation (1) has a unique power series:

$$\sum_{n=0}^{\infty} q^{-n^2} z^n,$$

which is divergent for all $z \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$, as it is assumed that $0 < |q| < 1$.

In Part II. and Part III., I shall present two different summation procedures, each of which allows to define one family of solutions of (1) that are all asymptotic to the above divergent series when $z \longrightarrow 0$ in a suitable domain.

The first procedure (Part II.) will be formed by integrals involving the Heat Kernel, which are no uniform and so are studied in terms of the angular argument variable. The second one is related to Jacobi's Theta function, which is entirely uniform but admits a spiral of simple poles.

In Part IV., I shall compare the functions defined in the previous parts and give a natural generalization of Mordell's Theorem such as

$$\int_{-\infty}^{\infty} \frac{e^{\frac{\pi t^2}{i\omega}+\frac{2\pi t x}{\omega}}}{e^{2\pi t}-1}dt = -\sqrt{\frac{\log 1/q}{\pi}} \frac{e^{\frac{(\log z)^2}{4\log 1/q}}}{\theta(\frac{\lambda}{z})} \sum_{n\in\mathbb{Z}} \frac{q^{n^2}}{1-\lambda q^{2n}}(\frac{\lambda}{z})^n$$

$$+\frac{i}{\theta^*(\frac{\lambda^*}{z^*})} \sum_{n\in\mathbb{Z}} \frac{{q^*}^{n^2}}{1-\lambda^* {q^*}^{2n}}(\frac{\lambda^*}{z^*})^n,$$

where

$$q = e^{\pi i \omega}, \quad q^* = e^{\frac{\pi^2}{\log q}},$$

$$\lambda \in \mathbb{C}^* \setminus q^{2\mathbb{Z}}, \quad \lambda^* = e^{-\pi i \frac{\log \lambda}{\log q}},$$

$$z = e^{-2\pi i x} \in \mathbb{C}^* \setminus (-\lambda q^{2\mathbb{Z}+1}), \quad z^* = e^{-\pi i \frac{\log z}{\log q}}.$$

- L. J. Mordell, The value of the definite integral $\int_{-\infty}^{\infty} \frac{e^{ax^2+bx}}{e^{cx}+d} dx$, Quarterly Journal of Mathematics 68 (1920), 329-342.
- L. J. Mordell, The definite integral $\int_{-\infty}^{\infty} \frac{e^{ax^2+bx}}{e^{cx}+d} dx$ and the analytic theory of Numbers, Acta Math. 61 (1933), 323-360.

Mordell begun his paper(1933) with the following observation:

Professor Siegel in a memoir recently published dealing with the manuscripts left by Riemann has pointed out that Riemann dealt with some integrals of the type

$$I = \int_{-\infty}^{\infty} \frac{e^{ax^2+bx}}{e^{cx}+d} dx$$

in his researches on the zetafunction. Not only can the usual functional equation be thus found, but also an asymptotic formula is obtained for the zetafunction of which the first term gives the well known approximate functional equation due to Hardy and Littlewood...

On the other hand, as said by Mordell himself, the starting point of his investigations was the theory of the positive, definite binary quadratic form

$$ax^2 + 2hxy + by^2,$$

where a, h, b are integers, so that the determinant of the form is

$$h^2 - ab = -D < 0, \ say.$$

Let $F(D)$ be the number of uneven classes of forms of given determinant $-D$, that is, classes of forms in which a and b are not both even.

The formulae for the class number known since Dirichlet(1839), shows that when $-D$ is negative and has no squared factors > 1,

$$F(D) = \frac{2}{\pi}\sqrt{D}\left(\left(\frac{-D}{1}\right) + \frac{1}{3}\left(\frac{-D}{3}\right) + \frac{1}{5}\left(\frac{-D}{5}\right) + \cdots\right).$$

Let $q = e^{\pi i \omega}$ with $\Im\omega > 0$ and let

$$\Omega(\omega) = \sum_{n=1}^{\infty} F(n) q^n$$

be **the generating function for** $F(n)$.

Mordell(1916) discovered the following expression for the generating function for $\Omega(\omega)$:

$$\Omega(\omega) = \frac{i}{4\pi} \frac{f_{01}'(0)}{\theta_{01}},$$

where $f_{01}(x)$ denotes the unique integral function defined by the functional equations

$$f_{01}(x+1) = f_{01}(x),$$

$$f_{01}(x+\omega) + f_{01}(x) = \theta_{01}(x)$$

and where $\theta_{01} = \theta_{01}(0, \omega)$, $\theta_{01}(x, \omega)$ being one of the four Jacobi functions:

$$\theta_{01}(x, \omega) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e^{2n\pi i x}.$$

In order to express "modular" relations connecting $\Omega(\omega)$ and $\Omega(-\frac{1}{\omega})$, Mordell(1919) used the integrals $\int_{-\infty}^{\infty} \frac{te^{\pi i\omega t^2}}{e^{2\pi t}\pm 1}$dt:

$$\int_{-\infty}^{\infty} \frac{te^{\pi i\omega t^2}}{e^{2\pi t}-1}dt = -2\Omega(\omega) + \frac{2}{\omega^2}\sqrt{(-i\omega)}\Omega(-\frac{1}{\omega}) + \frac{1}{4}\theta_{00}^3(0,\omega),$$

$$\int_{-\infty}^{\infty} \frac{te^{\pi i\omega t^2}}{e^{2\pi t}+1}dt = \sum_{n=1}^{\infty}(-1)^n F(4n-1)q^{\frac{1}{4}(4n-1)}$$

$$+\frac{2}{\omega^2}\sqrt{(-i\omega)}\sum_{n=1}^{\infty}(-1)^{n-1}F(n)q_1^n,$$

where $q_1 = e^{-\pi i/\omega}$ and $\theta_{00}(x,\omega) = \sum_{n=-\infty}^{\infty} q^{n^2}e^{2n\pi ix}$.

Mordell published his paper about the definite integral $\int_{-\infty}^{\infty} \frac{e^{ax^2+bx}}{e^{cx}+d}dx$ in 1933.

Theorem(Mordell,1933) Let $\Im\omega > 0$. Let f be the integral function of x defined as follows:

$$if(x,\omega) = \sum_{m\ odd}^{\pm\infty} \frac{(-1)^{\frac{1}{2}(m-1)} q^{\frac{1}{4}m^2} e^{m\pi ix}}{1+q^m}.$$

Let θ_{11} be the following Jacobi theta function:

$$i\theta_{11}(x,\omega) = \sum_{m\ odd}^{\pm\infty} (-1)^{\frac{1}{2}(m-1)} q^{\frac{1}{4}m^2} e^{m\pi ix}.$$

Then

$$\int_{-\infty}^{\infty} \frac{e^{\pi i\omega t^2 - 2\pi tx}}{e^{2\pi t}-1} dt = \frac{f(\frac{x}{\omega}, -\frac{1}{\omega}) + i\omega f(x,\omega)}{\omega\theta_{11}(x,\omega)},$$

where the path of integration may be taken as either the real axis of t indented by the lower half of a small circle described about the origin as center, say the path $(-\infty, \underline{0}, \infty)$, or as a straight line parallel to the real axis of t and below it at a distance less than unity. Such a path may be denoted by $P_{0,-1}$.

The above-used integral function f can be uniquely defined by two equations such as

$$f(x+1)+f(x)=0,$$

$$f(x+\omega)+f(x)=\theta_{11}(x).$$

By the way, the integral function $f_{01}(x)$ is another function of the type $f(x)$.

On the other hand,

$$\theta_{11}(x+1)=-\theta_{11}(x),$$

$$\theta_{11}(x+\omega)=-e^{-\pi i(2x+\omega)}\theta_{11}(x).$$

If set $g(x) = \frac{f(x)}{\theta_{11}(x)}$, then

$$g(x+1) = g(x),$$

$$e^{-\pi i(2x+\omega)} g(x+\omega) - g(x) = -1.$$

As before, let $q = e^{\pi i \omega}$. If $z = e^{-2\pi i \omega}$ and $G(z) = g(x)$, then:

$$G(e^{2\pi i} z) = G(z),$$

$$\frac{z}{q} G(\frac{z}{q^2}) - G(z) = -1.$$

So we shall consider the q-difference equation (1):

$$\frac{z}{q}y(\frac{z}{q^2}) - y(z) = -1,$$

which admits a unique power series solution, i.e.,

$$\hat{y}(z) := \sum_{n=0}^{\infty} q^{-n^2} z^n.$$

As $0 <\mid q \mid< 1$, the formal solution $\hat{y}(z)$ diverges for all $z \neq 0$, so we can say the equation (1) admits an irregular singular point at $z = 0$.

It is well-known that a similar phenomenon occurs for the non-Fucshian differential equation, the so-called Euler's differential equation (see J. SAULOY's talk):

$$z^2 y'(z) - y(z) = 1,$$

which has the power series solution $\sum_{n\geq 0} n! z^n$, obviously divergent for any $z \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$.

$\dashrightarrow$**Based on q-Gevrey asymptotic analysis**

From now on, assume $\omega \in \mathbb{R}^+ i$ and write $\tau = -\pi i \omega > 0$, $\zeta = -2\pi i x$, i.e.,

$$q = e^{-\tau} \in (0,1),\ \ z = e^{\zeta}.$$

Then the q-series $\sum_{n=0}^{\infty} q^{-n^2} z^n$ takes the form

$$\sum_{n=0}^{\infty} e^{n^2\tau + n\zeta},$$

which, as we know, is the formal solution of a Cauchy problem related to the Heat equation:

$$\begin{cases} \partial_\tau u - \partial_{\zeta\zeta} u = 0, \\ u(\zeta, 0) = \frac{1}{1-e^{\zeta}}. \end{cases}$$

Suppose ζ be over a line parallel to the real axis but not passing by any complex number of the form $2\pi mi$, $m = 0, \pm 1, \pm 2, \cdots$, i.e., $\zeta \in (-\infty + \alpha i, \infty + \alpha i)$ for $\alpha \in \mathbb{R} \backslash 2\pi\mathbb{Z}$.

By means of the Heat Kernel, the above-stated initial value problem on the line $(-\infty + \alpha i, \infty + \alpha i)$ has the following solution:

$$u_\alpha(\zeta, \tau) = \int_{-\infty+\alpha i}^{\infty+\alpha i} \frac{e^{-\frac{(\zeta-\xi)^2}{4\tau}}}{\sqrt{4\pi\tau}} \frac{1}{1 - e^\xi} d\xi.$$

And it is easily seen that the right integral remains convergent when extending the interval of definition of the variable ζ to the whole complex plane $\mathbb{C}$, which is still denoted by $u_\alpha(\zeta, \tau)$.

Theorem 1(Zhang, 1999). For any $\alpha \in \mathbb{R} \setminus 2\pi\mathbb{Z}$ and any z over the Riemann surface $\tilde{\mathbb{C}}^*$ of the logarithm, we define

$$f_\alpha(z, q) = u_\alpha(\log z, \ln 1/q).$$

Then (i) $f_\alpha(z, q)$ is holomorphic over $\tilde{\mathbb{C}}^*$, and if α and β belong to a common interval of the set $\{\alpha \in \mathbb{R} \setminus 2\pi\mathbb{Z}\}$, then $f_\alpha(z, q) = f_\beta(z, q)$.

(ii) $f_\alpha(ze^{-2\pi i}, q) - f_\alpha(z, q) = i\sqrt{\frac{\pi}{\ln 1/q}} e^{\frac{(\log z - 2\pi k_\alpha i)^2}{4 \ln q}}$, where $2\pi k_\alpha$ is the integer between α and $\alpha + 2\pi$.

(iii) $f_\alpha(z, q)$ is the unique solution of (1) which admits $\sum_{n=0}^{\infty} q^{-n^2} z^n$ as q-Gevrey asymptotic expansion at $z = 0$ along the direction $(0, \ \infty e^{\alpha i})$.

Let $f_-(z,q)$ be the function associated with $f_\alpha(z,q)$ for $\alpha \in (-2\pi,\ 0)$, then

$$f_-(z,q) = \frac{1}{\sqrt{4\pi \ln 1/q}} \int_{-\infty}^{\infty} \frac{e^{\frac{(\log z-\xi)^2}{4\ln q}}}{1-e^{\xi}} d\xi,$$

where the path of integration may be taken as either the real axis of ξ intended by the lower half of a small circle described about the origin as center, denoted by the path $(-\infty,\ \underline{0},\ \infty)$, or as a straight line parallel to the real axis of t and below it at a distance less than 2π.

So the function f_- is a variant of the function given by the integral in Mordell's Theorem.

Namely, if we set $q = e^{\pi i \omega}$ and $z = e^{-2\pi i x}$, then

$$-\frac{1}{\sqrt{(-i\omega)}} e^{-\frac{\pi x^2}{i\omega}} \int_{-\infty}^{\infty} \frac{e^{\frac{\pi t^2}{i\omega} + \frac{2\pi t x}{\omega}}}{e^{2\pi t} - 1} dt = f_-(z, q),$$

so that one can write

$$-\sqrt{(-i\omega)} e^{\frac{\pi x^2}{i\omega}} \int_{-\infty}^{\infty} \frac{e^{\pi i \omega t^2 - 2\pi t x}}{e^{2\pi t} - 1} dt = f_-(z^*, q^*),$$

where $q^* = e^{-\frac{\pi i}{\omega}}$ and $z^* = e^{2\pi i \frac{x}{\omega}}$.

Since

$$f_{\alpha+2\pi}(z,q) = f_\alpha(ze^{-2\pi i}, q),$$

we get

$$f_-(zq^{-2\pi i}) - f_-(z,q) = i\sqrt{\frac{\pi}{\ln 1/q}}\, e^{\frac{(\log z)^2}{4\ln q}},$$

where the function given in the right hand is infinitely small or said flat as $z \to 0$ and is the solution of the homogeneous q-difference equation associated with (1):

$$\frac{z}{q} y(\frac{z}{q^2}) - y(z) = 0.$$

On the other hand, the above relation can be seen as a Stokes phenomenon in the singular direction argument $0 \in 2\pi\mathbb{Z}$ in the Riemann surface of logarithm.

Remarks. ★ The solution f_α can be formulated in terms of Fourier analysis. Namely, from the Gaussian integral, it follows that

$$q^{-n^2} = \frac{1}{\sqrt{4\pi \ln 1/q}} \int_{-\infty}^{\infty} e^{\frac{t^2}{4 \ln q} + nt} dt.$$

So the power series $\sum_{n=0}^{\infty} q^{-n^2} z^n$ may be associated with the integral of the type:

$$\frac{1}{\sqrt{4\pi \ln 1/q}} \int_{-\infty}^{\infty} \frac{e^{\frac{t^2}{4 \ln q}}}{1 - ze^t} dt,$$

which gives rise to the integral function f_α.

★ Here is the so-called Gq-summation:

$$\hat{f}(z) := \sum_{n \geq o} a_n z^n$$

$$\Rightarrow \varphi(\xi) := \sum_{n \geq o} a_n q^{n^2} \xi^n$$

$$\Rightarrow f(\xi) := \frac{1}{\sqrt{4\pi \ln 1/q}} \int_{-\infty+\alpha i}^{\infty+\alpha i} e^{\frac{(\log z - \xi)^2}{4 \ln q}} \varphi(\xi) d\xi$$

where f is asymptotic to the starting power series $\hat{f}$. This procedure can be applied to any power series solution when the q-difference equation admits a unique slope for its Newton polygon.

★ If one repeats Heat Kernel integrals, one can obtain a q-Gevrey asymptotic solution of each power series solutions of any irregular singular linear q-difference equation: multisummability in q-difference equation cases.

$\dashrightarrow$**Based on an elliptic summation approach**

Suppose now $0 < q < 1$. we will note Jacobi theta function

$$\theta(z) = \theta(z, q) = \sum_{n\in\mathbb{Z}} q^{n^2} z^n, \ z \in \mathbb{C}^*.$$

From the functional equation

$$\theta(z) = qz\theta(q^2 z),$$

we deduce that

$$\theta(q^{2n} z) = q^{-n^2} z^{-n} \theta(z),$$

so, provided that $\theta(\lambda q^{2n}) \neq 0$ for all integers n,

$$\sum_{n\in\mathbb{Z}} \frac{1}{\theta(\lambda q^{2n})} = \sum_{n\in\mathbb{Z}} \frac{q^{n^2} \lambda^n}{\theta(\lambda)} = 1,$$

for any $\lambda \in \mathbb{C}^*$.

On the other hand, from Jacobi's triple product formula

$$\theta(z) = \prod_{n\geq 0}(1 - q^{2n+2})(1 + zq^{2n+1})(1 + \frac{q^{2n+1}}{z}),$$

we deduce that $\theta(\lambda q^{2n}) \neq 0$ holds for all $\lambda \in \mathbb{C}^*\backslash(-q^{2\mathbb{Z}+1})$.

Because of the above deduction, we can get, for any integer $m \in \mathbb{Z}$ and any $\lambda \in \mathbb{C}^* \backslash (-q^{2\mathbb{Z}+1})$,

$$\sum_{\xi \in \lambda q^{2\mathbb{Z}}} \frac{\xi^m}{\theta(\xi)} = q^{-m^2} \sum_{n \in \mathbb{Z}} \frac{\lambda^{m+n} q^{(m+n)^2}}{\theta(\lambda)} = q^{-m^2}.$$

Thus the divergent power series $\sum_{n \geq 0} q^{-n^2} z^n$ may be written as a double series and by this way we are led to $\sum_{\xi \in \lambda q^{2\mathbb{Z}}} \frac{1}{1-\xi z} \frac{1}{\theta(\xi)}$.

So if $\lambda \in \mathbb{C}^*\backslash q^{2\mathbb{Z}}$, we can define

$$g_\lambda(z,q) = \sum_{\xi \in \lambda q^{2\mathbb{Z}}} \frac{1}{1-\xi} \frac{1}{\theta(\frac{\xi}{z})},$$

or equivalently,

$$g_\lambda(z,q) = \frac{1}{\theta(\frac{\lambda}{z})} \sum_{n\in\mathbb{Z}} \frac{q^{n^2}}{1-\lambda q^{2n}} (\frac{\lambda}{z})^n,$$

for all $z \in \mathbb{C}^*\backslash(-\lambda q^{2\mathbb{Z}+1})$.

Theorem 2(Zhang, 2002). If $\lambda \in \mathbb{C}^* \setminus q^{2\mathbb{Z}}$, we define

$$g_\lambda(z,q) = \frac{1}{\theta(\frac{\lambda}{z})} \sum_{n \in \mathbb{Z}} \frac{q^{n^2}}{1 - \lambda q^{2n}} (\frac{\lambda}{z})^n,$$

for all $z \in \mathbb{C}^* \setminus (-\lambda q^{2\mathbb{Z}+1})$. Then

(i) g_λ is holomorphic over $\mathbb{C}^* \setminus (-\lambda q^{2\mathbb{Z}+1})$ and admits $(-\lambda q^{2\mathbb{Z}+1})$ as a set of simples poles.

(ii) $g_\lambda(ze^{2\pi i}, q) = g_\lambda(z,q)$.

(iii) $g_\lambda(z,q)$ is the unique solution of (1) which admits the power series $\sum_{n=0}^{\infty} q^{-n^2} z^n$ as asymptotic expansion for $z \to 0$ in $\mathbb{C}^* \setminus (-\lambda q^{2\mathbb{Z}+1})$ in the following sense:

there exist $C > 0$, $A > 0$ such that for all $N \in \mathbb{N}^*$ and $\epsilon > 0$ small enough

$$| g_\lambda(z,q) - \sum_{n=0}^{N-1} q^{-n^2} z^n | \leq \frac{C}{\epsilon} A^N q^{-N^2} | z |^N,$$

where $z \in \mathbb{C}^* \setminus \bigcup_{n\in\mathbb{Z}} \{q^{2n-1}z : |z+\lambda| \leq \epsilon\}$.

This asymptotic relation is stronger than Poincaré's asymptotic manner, but little weaker than q-Gevrey asymptotics.

Remark. ★ Since $\lambda \mapsto g_\lambda(z,q)$ is left invariant by $\lambda \mapsto q^2\lambda$, one can calculate the cocycle $g_\lambda - g_\mu$ as follows:

$$g_\lambda(z,q) - g_\mu(z,q) = \frac{K(\lambda,\mu,z)}{\theta(z)},$$

where $\lambda,\ \mu \in \mathbb{C}^*\backslash q^{2\mathbb{Z}}$, $z \in \mathbb{C}^*\backslash(-\lambda q^{2\mathbb{Z}+1}) \bigcap \mathbb{C}^*\backslash(-\mu q^{2\mathbb{Z}+1})$ and

$$K(\lambda,\mu,z) = \frac{(q^2;q^2)_\infty^3 \theta(-\frac{\lambda}{\mu}q)\theta(\frac{\lambda\mu}{z})\theta(\frac{1}{z})}{\theta(-q\lambda)\theta(-\frac{\mu}{q})\theta(\frac{\lambda}{z})\theta(\frac{\mu}{z})}.$$

Such elliptic cocycles play the role of Stokes multipliers and allow to classify the corresponding q-difference equation.

$\dashrightarrow$ **Based on Stokes phenomenon analysis**

The main idea is to use the simple fact that if y_1 and y_2 are two solutions of (1), then $y_1 - y_2$ will be a solution of the associated homogeneous equation and be flat or asymptotically zero. Let us consider

$$h_\lambda(z,q) := \frac{1}{i}\sqrt{\frac{\ln 1/q}{\pi}}\, e^{-\frac{(\log z)^2}{4\ln q}} \left(f_-(z,q) - g_\lambda(z,q)\right),$$

where $\lambda \in \mathbb{C}^* \backslash q^{2\mathbb{Z}}$ and $\mathbb{C}^* \backslash (-\lambda q^{2\mathbb{Z}+1})$.

Remember that $f_-(z,q)$ and $g_\lambda(z,q)$ are two solutions of (1) and

$$f_-(ze^{-2\pi i}, q) - f_-(z,q) = i\sqrt{\frac{\pi}{\ln 1/q}}\, e^{\frac{(\log z)^2}{4\ln q}},$$

$$g_\lambda(ze^{-2\pi i}, q) = g_\lambda(z,q).$$

By considering the initial equation (1), we find the following relations:

$$h_\lambda(\frac{z}{q^2}, q) = h_\lambda(z, q),$$

$$e^{-\frac{\pi i}{\ln q} \log z} e^{-\frac{\pi^2}{\ln q}} h_\lambda(ze^{-2\pi i}, q) - h_\lambda(z, q) = 1.$$

If we set $q^* = e^{\frac{\pi^2}{\ln q}}$ and $z^* = e^{-\pi i \frac{\log z}{\ln q}}$, then

$$(\frac{z}{q^2})^* = z^* e^{2\pi i}, \ (ze^{-2\pi i})^* = \frac{z^*}{q^{*2}}.$$

Consider $h_\lambda^*(z^*, q) = -h_\lambda(z, q)$. It follows that

$$h_\lambda^*(z^* e^{2\pi i}, q) = h_\lambda^*(z^*, q),$$

$$\frac{z^*}{q^*} h_\lambda^*(\frac{z^*}{q^{*2}}, q) - h_\lambda^*(z^*, q) = -1.$$

By observing that $h_\lambda^*(z^*, q)$ is holomorphic over $\mathbb{C}^* \backslash (-\lambda^* q^{*2\mathbb{Z}+1})$ and admits simple poles in the q-spiral $(-\lambda^* q^{*2\mathbb{Z}+1})$, we conclude that

$$h_\lambda^*(z^*, q) = g_{\lambda^*}(z^*, q^*),$$

and then get our main theorem:

Theorem. The following relation holds for every $\lambda \in \mathbb{C}^* \setminus q^{2\mathbb{Z}}$

$$f_-(z,q) = g_\lambda(z,q) - i\sqrt{\frac{\pi}{\ln 1/q}}\, e^{\frac{(\log z)^2}{4\ln q}} g_{\lambda^*}(z^*, q^*), \qquad (2)$$

where $q^* = e^{\pi^2/\ln q}$, $z^* = e^{-\pi i \frac{\log z}{\ln q}}$, and $\lambda^* = e^{-\pi i \frac{\log \lambda}{\ln q}}$, or equivalently

$$\frac{1}{\sqrt{4\pi \ln 1/q}} \int_{-\infty}^{\infty} \frac{e^{\frac{(\log z - \xi)^2}{4\ln q}}}{1 - e^{\xi}} d\xi = \frac{1}{\theta(\frac{\lambda}{z})} \sum_{n\in\mathbb{Z}} \frac{q^{n^2}}{1 - \lambda q^{2n}} (\frac{\lambda}{z})^n$$

$$-i\sqrt{\frac{\pi}{\ln 1/q}}\, e^{\frac{(\log z)^2}{4\ln q}} \frac{1}{\theta^*(\frac{\lambda^*}{z^*})} \sum_{n\in\mathbb{Z}} \frac{q^{*n^2}}{1 - \lambda^* q^{*2n}} (\frac{\lambda^*}{z^*})^n.$$

On the other hand, we remember that

$$-\frac{1}{\sqrt{(-i\omega)}} e^{-\frac{\pi x^2}{i\omega}} \int_{-\infty}^{\infty} \frac{e^{\frac{\pi t^2}{i\omega} + \frac{2\pi t x}{\omega}}}{e^{2\pi t} - 1} dt = f_-(z, q),$$

so that we find the following generalization of Mordell's Theorem: for any $\lambda \in \mathbb{C}^* \setminus q^{2\mathbb{Z}}$ and any $z \in \mathbb{C}^* \setminus (-\lambda q^{2\mathbb{Z}+1})$,

$$\int_{-\infty}^{\infty} \frac{e^{\frac{\pi t^2}{i\omega} + \frac{2\pi t x}{\omega}}}{e^{2\pi t} - 1} dt = -\frac{\sqrt{(-i\omega)} e^{\frac{\pi x^2}{i\omega}}}{\theta(\frac{\lambda}{z})} \sum_{n \in \mathbb{Z}} \frac{q^{n^2}}{1 - \lambda q^{2n}} (\frac{\lambda}{z})^n$$

$$+ \frac{1}{\theta^*(\frac{\lambda^*}{z^*})} \sum_{n \in \mathbb{Z}} \frac{q^{*n^2}}{1 - \lambda^* q^{*2n}} (\frac{\lambda^*}{z^*})^n,$$

where, as before, the path of integration in the left hand is taken as the real axis of t indented by the lower half of a small circle described about the origin as center.

Remarks. ★ The Mordell's Integral can be included as a particular case:

$$\lambda = \frac{1}{q} e^{\pi i}.$$

★ In the above formulae, q is firstly assumed to be in (0,1). By the standard argument of analytical continuation, the formulae remain valid for all $q = e^{\pi i \omega}$ with $\Im(\omega) > 0$.

Lately, we consider a more general integral of the type

$$I(\nu, x) = I(\nu, x; \omega) = -\frac{1}{\sqrt{(-i\omega)}} e^{-\frac{\pi x^2}{i\omega}} \int_{-\infty}^{\infty} \frac{e^{\frac{\pi t^2}{i\omega} + \frac{2\pi x t}{\omega}}}{e^{2\pi t} - e^{2\pi\nu}} dt,$$

where k is a positive integral and $\nu \in \mathbb{C}$ with $\Im\nu \in (-1, 0]$ and where the path of integration may be any straight line parallel to the real axis of t and just below the point ν at a distance less then unity, i.e., the line $(-\infty + \nu - i\epsilon, \infty + \nu - i\epsilon)$ with $\epsilon \in (0, 1)$.

Some analogous results may be obtained on these integrals.

END.

Thank you for your attention.