MATING OF DISCRETE TREES AND WALKS IN THE QUARTER PLANE

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Transcendance et Combinatoire

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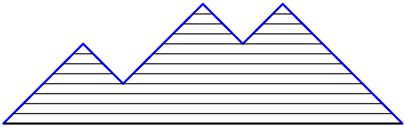
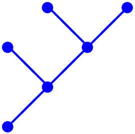
There is a well known way to associate a rooted planar tree to a Dyck path by matching up and down steps:

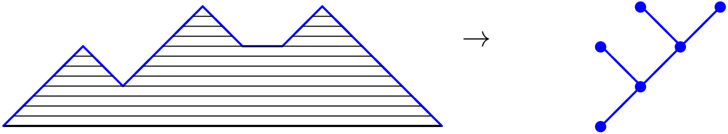


$\rightarrow$

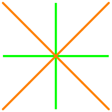


Cut out the striped area under the Dyck path and sew the up and down steps.

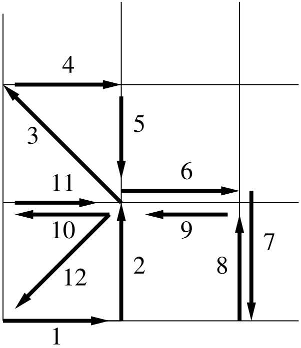
Generalize this to Motzkin paths by shrinking each horizontal step to a point.



We will consider walks with *small steps* in quarter plane $x, y \geq 0$
the



Straight steps are green.
Oblique steps are orange.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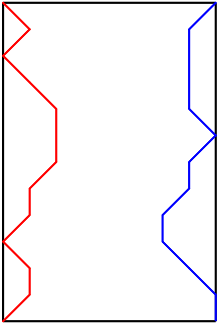
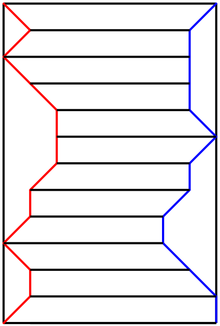
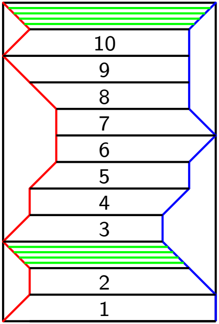
We draw the two paths vertically in an opposite way, with the horizontal coordinate on the left and the vertical coordinate on the right, the paths running from bottom to top.



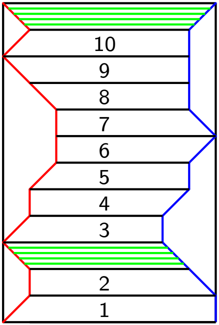
The mating consists in drawing horizontal lines between the vertices of the two paths, as below:



Between the two Motzkin paths we have quadrilaterals corresponding to straight or oblique steps (in green)



We now contract the two Motzkin paths into two trees to get a planar map (we also identify the upper and lower boundaries)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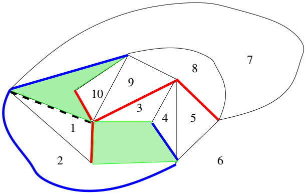
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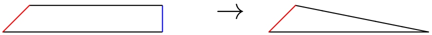
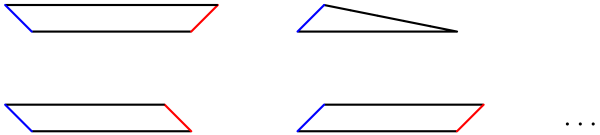
In this contraction the quadrilaterals corresponding to straight steps become triangles as, for example:



The map consists of triangles and quadrilaterals



The main idea of this talk is to contract the quadrilaterals

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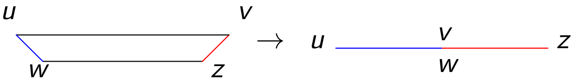
There are two ways to do so, like this



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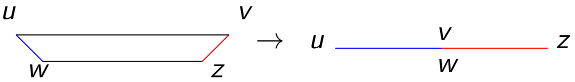
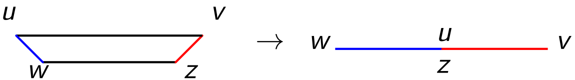
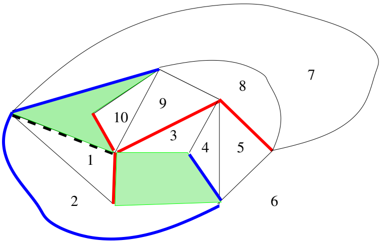
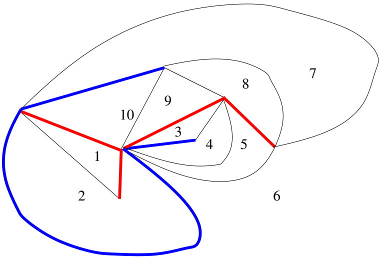


or this





Finally one gets a rooted triangulation of the plane

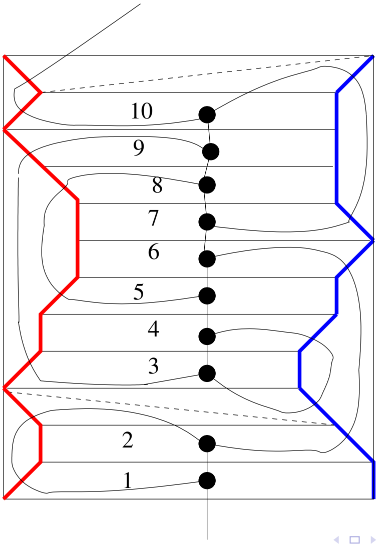


A walk with small steps gives several triangulations of the sphere and each triangulation can be obtained in several ways.

In the sequel I show how to restrict the class of paths and equip the map with some supplementary structure in order to obtain bijections.

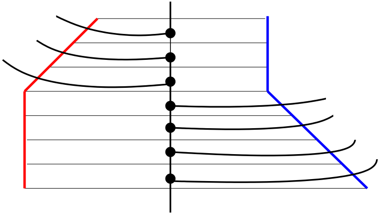
In particular we obtain new bijections and recover several known bijections.

In order to visualize the maps we depict also the dual map putting a dashed line between the opposite vertices which have been identified.

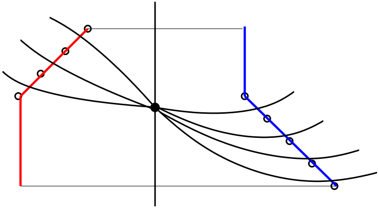


More general steps

It is also possible to use other types of steps. For example here a step $(3,4)$:

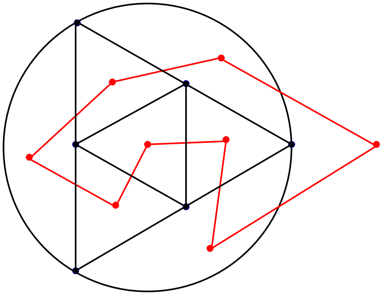


When we erase the internal edges we get a face with nine edges

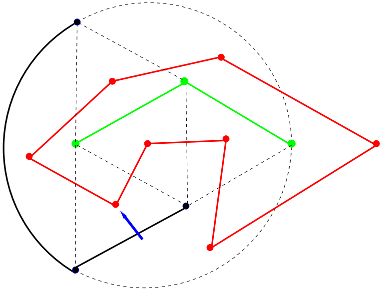


Triangulations with a Hamiltonian cycle: Mullin's construction

A triangulation of the plane with a Hamiltonian cycle going through its faces

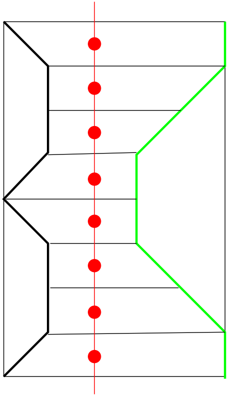


Keep edges not crossed by the cycle:



There remains two trees.

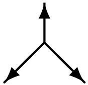
We go through the path and record the successive triangles:



This gives an excursion with straight paths
This is (essentially) Mullin's bijection.

Reversed Y-walks

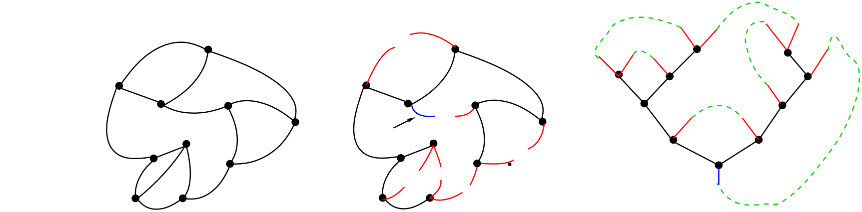
Walks with steps in the set $\{(0,1),(-1,-1),(1,-1)\}$.



rY-walks of length $4n$ in the quarterplane, starting and ending at 0 are counted by $C_{2n}C_n$ ($C_l = \frac{1}{l+1}\binom{2l}{l}$ Catalan number)

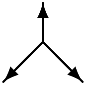
(C_{2n} for the vertical Dyck path, C_n for the horizontal path)

Finding a family of maps counted by $C_{2n}C_n$



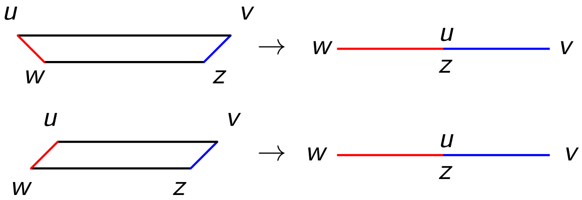
(a) (b) (c)

A cubic map with a complete rooted spanning tree



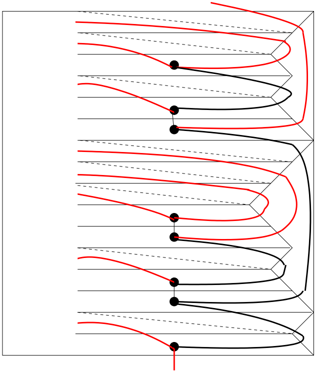
The rY-walks have two types of oblique steps, so we must give the rules for contracting the associated quadrilaterals. These rules are simple, we show them below.

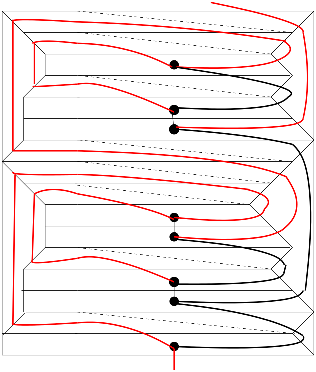
Contraction Rules



in both cases identify the NW and SE corners.

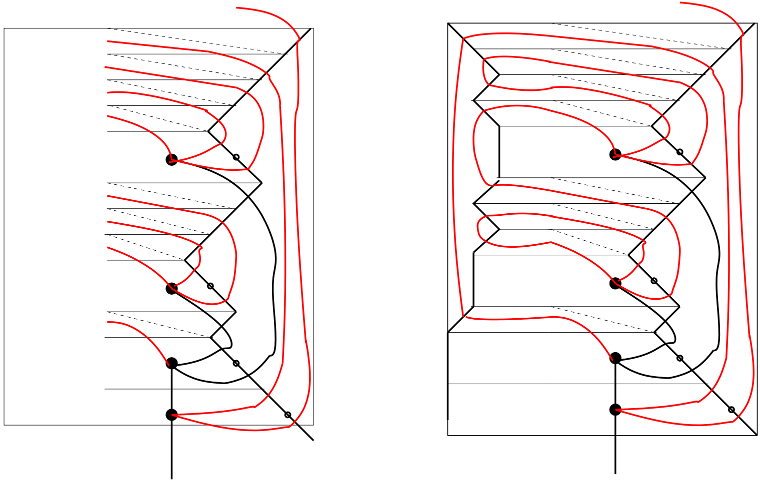
rY-walks and their associated maps.


(a)


(b)

(a) Construction of the spanning tree using the vertical coordinates.
(b) Matching the leaves using the horizontal coordinates.

Instead of the step $(0,1)$ we could consider $(0,2)$ and get a quartic map

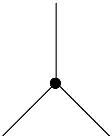


One can even generalize to step set $(1,-1); (-1,-1); (0,k), k \geq 1$ and get a bijection between maps with a complete spanning tree and generalized rY walks.

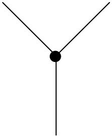
Prographs and Kenyon-Miller-Sheffield-Wilson bijection

Prographs are formed with vertices oriented from bottom to top

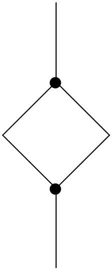
Products:



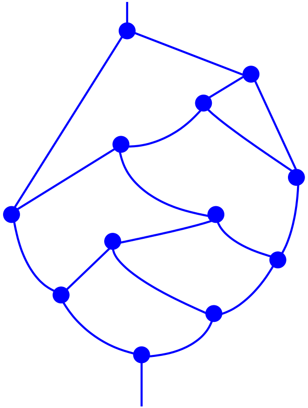
Coproducts:



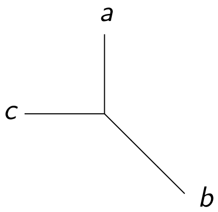
Consider prographs with one input and one output



A more complicated exampe:



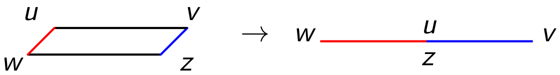
(N. Borie) Prographs are in bijection with *tandem walks* with steps



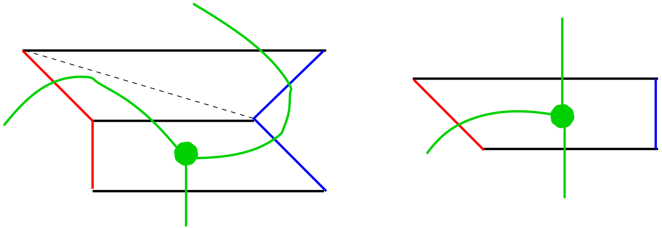
There are three types of cells:

type a type b type c

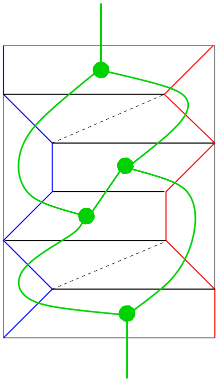
Contraction Rules



Interpret a step $c = (-1, 0)$ as formed by a step $(0, 1)$ followed by a step $(-1, -1)$. Tandem walks form a subset of rY walks. The contraction rules are compatible.

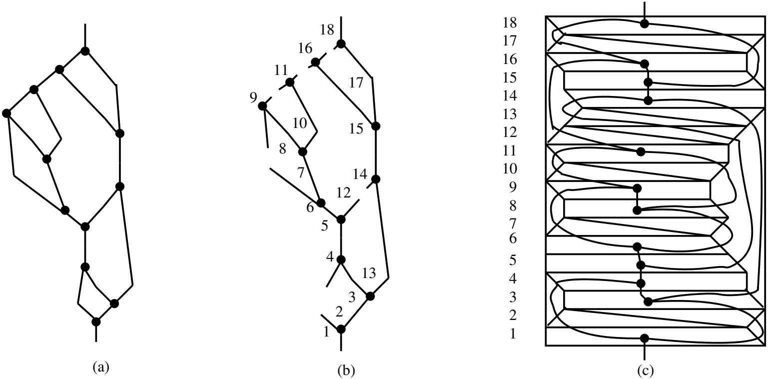


This construction associates to a tandem walk in the quarter plane, starting and ending at 0, a prograph, obtained as the dual of the triangulation. Each a step corresponds to a coproduct while a c-step corresponds to a product.
Here is an example with the word $abacbc$:



This correspondance between tandem walks and prographs is a bijection.

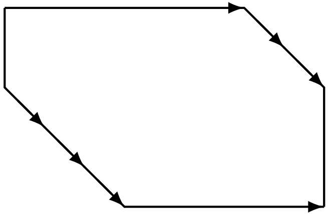
Figure: From prographs to tandem walks.
(a) A prograph.
(b) Cutting left inputs and exploring the tree.
(c) The resulting path, recovering the prograph as the dual of the triangulation.



Bipolar maps (Kenyon Miller Sheffield Wilson)

Bipolar maps can also be accomodated and are in bijection with walks with steps $(-1, 1)$ and $(i, -j)$.

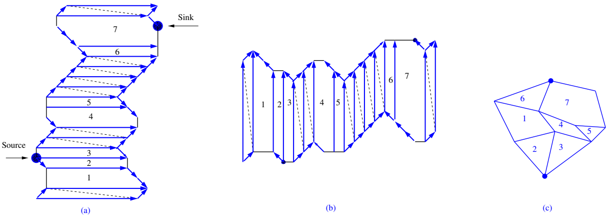
Orient the sides of a face from west to east, e.g. :



For a quadrilateral corresponding to a step $(-1, 1)$ make the usual contraction:

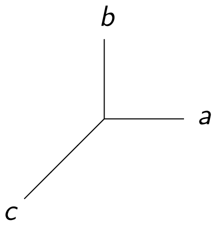
We recover the Kenyon-Miller-Sheffield-Wilson bijection.

Figure: (a) A walk with steps
(1,−1);(0,2);(−1,0);(0,1);(1,−1);(1,−1);(−1,1);(0,1);(1,−1);(1,−1);(1,−1);(1,−1);(−1,0);(−2,1);(1,−1
(b) After a reflection through the main diagonal.
(c) The final bipolar map

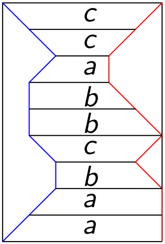


Kreweras walks and Bernardi's bijection (2009)

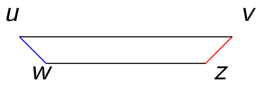
Kreweras steps:



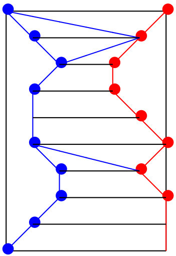
Here is an example *aabcbbacc*:



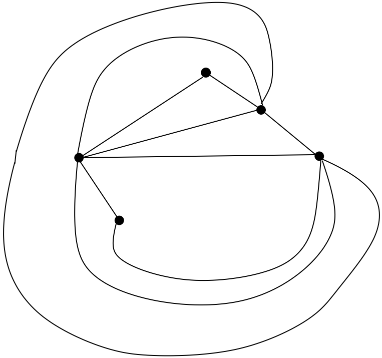
Rules for the contractions:



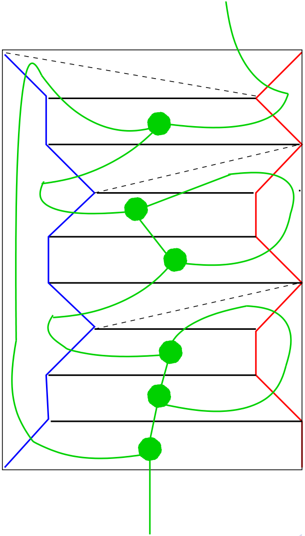
Consider red and blue segments (uw and vz above). They are paired with two segments red and blue whose heights are i and j. If $i > j$ (red is higher) identify u and z, if not identify v and w .



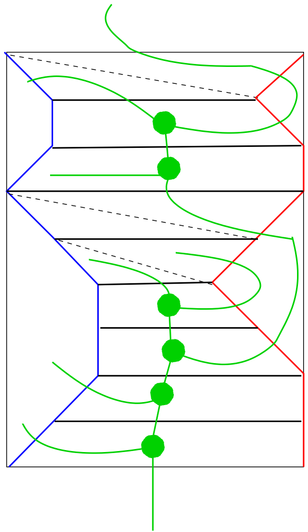
The final triangulation:



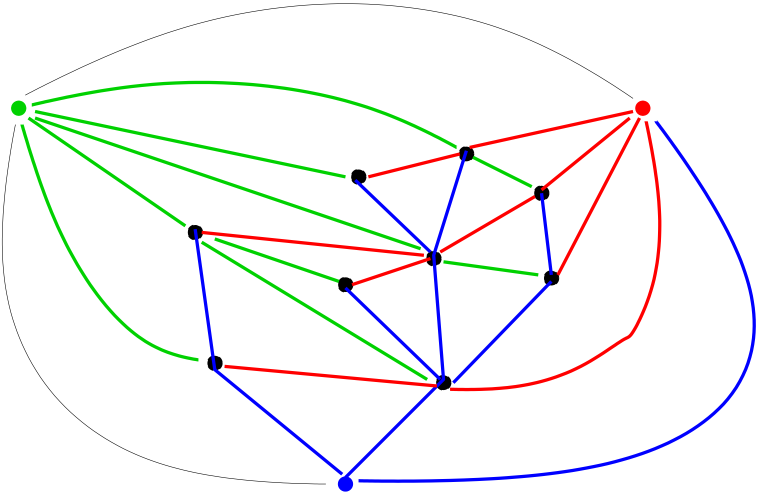
The preceding construction is not a bijection, we need more information to recover the walk.
Consider the dual map



Orient the edges from bottom to top and cut the edges which are not correctly oriented. We get a covering tree which allows to recover the original walk.

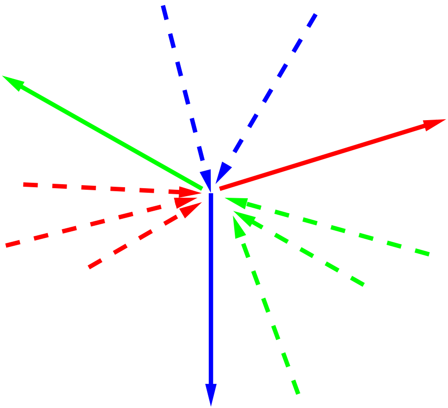


Schnyder woods (Li, Sun, Watson 2017)

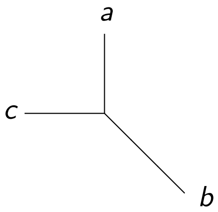


A triangulation with three trees rooted on three vertices of the external face.

In each internal vertex we have *Schnyder condition* :



Consider tandem walks with steps



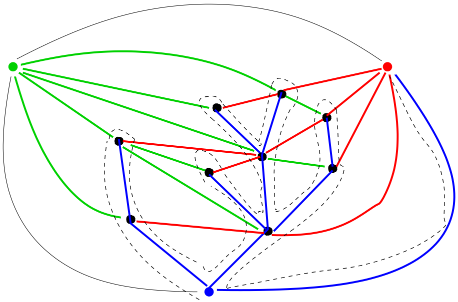
type a

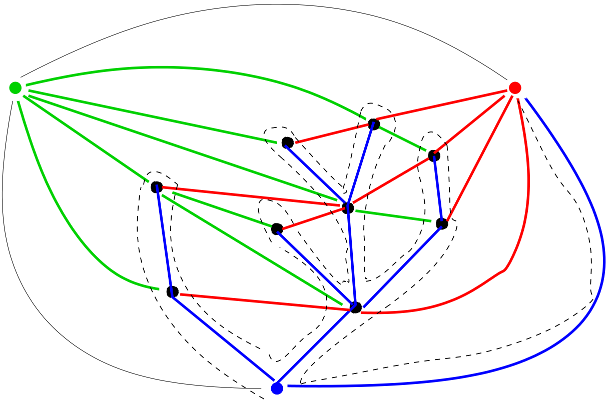
type b

type c

Make a contour walk of the blue tree

1. when you go up a blue edge make an a step
2. when you go down a b step
3. when you cross a red edge for the second time make a c step
4. do not go down the last blue step and do not make the last b step.

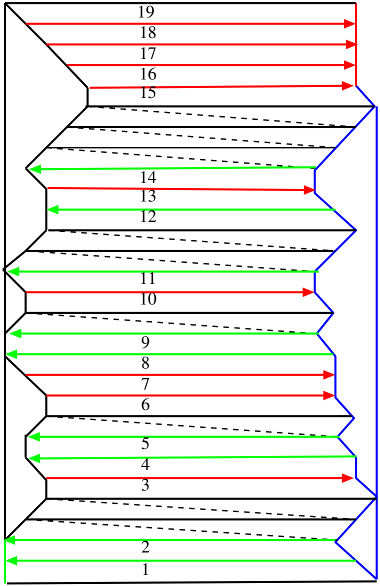




Here is the word you get in the example

$$aabbacabaccabacbbaacbbbacccc(b)$$

There is never a step c following immediately a step b. The upper and lower edges of the rectangle, together with the a step not paired with the last b step, form the external triangle.
This construction recovers the Li Sun Watson bijection between such walks and Schnyder woods.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CONCLUSION

The construction has a very simple principle, many variants are possible and could hopefully be used to produce many more bijections.

THANK YOU!