Quadrant Walks Starting Outside the Quadrant

Manfred Buchacher

joint work with
Manuel Kauers and Amélie Trotignon

Let

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and

$$S(x, y) \in \mathbb{Q}[x, y, \bar{x}, \bar{y}]$$

such that

$$\mathrm{supp}(S) \subseteq \{-1, 0, 1\}^2.$$

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Define

$$K(x, y, t) = xy(1 - tS(x, y)).$$

Solve the functional equation

$$K(x,y,t)Q(x,y,t) = x^{k+1}y^{l+1} + K(x,0,t)Q(x,0,t) + K(0,y,t)Q(0,y,t) - K(0,0,t)Q(0,0,t)$$

over $\mathbb{Q}[x,y][[t]]$.

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over $\mathbb{Q}[x,y][[t]]$.

Is $Q(x,y,t)$ rational, algebraic, D-finite, or differentiably algebraic?

Let

$$\begin{aligned} S(x, y) &= a_{-1}(x)\overline{y} + a_0(x) + a_1(x)y \\ &= b_{-1}(y)\overline{x} + b_0(y) + b_1(y)x \end{aligned}$$

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Define

$$\phi : (x,y) \mapsto (\frac{b_{-1}(y)}{b_1(y)}\overline{x}, y) \quad \text{and} \quad \psi : (x,y) \mapsto (x, \frac{a_{-1}(x)}{a_1(x)}\overline{y}),$$

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and let

$$G = \langle \phi, \psi \rangle.$$

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$$\mathrm{OS}(x^{k+1}y^{l+1}) := \sum_{g \in G} \operatorname{sgn}(g) g(x^{k+1}y^{l+1}).$$

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Theorem

- *If* $|G| < \infty$, *then* $Q(x, y, t)$ *is D-finite.*
- *If* $|G| < \infty$ *and* $\operatorname{OS}(x^{k+1}y^{l+1}) = 0$, *then* $Q(x, y, t)$ *is algebraic.*

Are there any examples for which

$$|G| < \infty \quad \text{and} \quad \mathrm{OS}(x^{k+1}y^{l+1}) = 0,$$

but $Q(x, y, t)$ is not algebraic?

No.

No.

Otherwise the theorem would be wrong.

Maybe.

Maybe.

There might be some in higher dimensions.

Yes.

Yes.

If (k, l) is allowed to lie in $\mathbb{Z}^2 \setminus \mathbb{N}^2$.

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$$\begin{aligned} \mathrm{OS}(x^{k+1}y^{l+1}) &= x^{k+1}y^{l+1} - \bar{x}^{k+1}y^{l+1} + \bar{x}^{k+1}\bar{y}^{l+1} - x^{k+1}\bar{y}^{k+1} \\ &= 0, \quad \text{if } (k, l) = (-1, -1). \end{aligned}$$

$$(1 - t(x + y + \bar{x} + \bar{y}))Q(x, y, t) =$$
$$\overline{xy} - t\bar{x}Q(0, y, t) - t\bar{y}Q(x, 0, t)$$

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What do we mean by $Q(0, y, t)$ and $Q(x, 0, t)$?

$$Q(x,0,t) = [y^0]Q(x,y,t)$$
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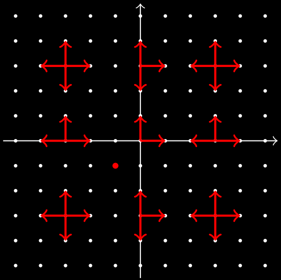

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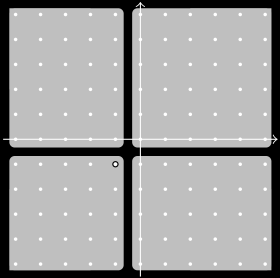

$$Q(x,y,t) =$$

$$[x^< y^<]\frac{xy - \bar{x}y + \overline{xy} - x\bar{y}}{1 - tS} +$$

$$t\bar{y}[x^<]\Big(\big([y^>]\frac{y - \bar{y}}{1 - St}\big)\big([\bar{y}]\frac{xy - \bar{x}y - x\bar{y} + \overline{xy}}{1 - tS}\big)\Big) +$$

$$t\bar{x}[y^<]\Big(\big([x^>]\frac{x - \bar{x}}{1 - St}\big)\big([\bar{x}]\frac{xy - \bar{x}y - x\bar{y} + \overline{xy}}{1 - tS}\big)\Big) +$$

$$\overline{xy}t^2[y^>]\Big(\big([\bar{x}]\frac{(y - \bar{y})[\bar{y}]\frac{xy - \bar{x}y + \overline{xy} - x\bar{y}}{1 - tS}}{1 - tS}\big)\big([x^>]\frac{x - \bar{x}}{1 - tS}\big)\Big) +$$

$$\overline{xy}t^2[x^>]\Big(\big([\bar{y}]\frac{(x - \bar{x})[\bar{x}]\frac{xy - \bar{x}y + \overline{xy} - x\bar{y}}{1 - tS}}{1 - tS}\big)\big([y^>]\frac{y - \bar{y}}{1 - tS}\big)\Big)$$

The coefficient sequence of $Q(1,1,t)$ satisfies the recurrence

$$\begin{aligned}&(2+n)(4+n)(6+n)(-1+2n+n^2)a_{n+2}\\&-4(3+n)(-18+4n+9n^2+2n^3)a_{n+1}\\&-16(1+n)(2+n)(3+n)(2+4n+n^2)a_n=0.\end{aligned}$$

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Its only asymptotic solutions are of the form

$$a_n \sim 4^n n^{-1} \quad \text{and} \quad a_n \sim (-4)^n n^{-3}.$$

$$Q(x,0,t) = [x^{\geq} y^{0}]Q(x,y,t)$$
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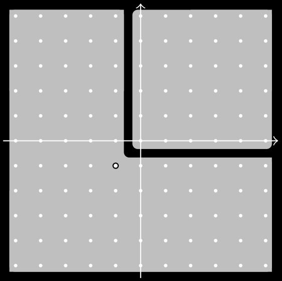

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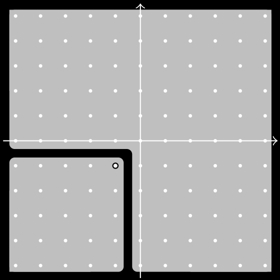

Let $r \in \mathbb{N}$, and $p_0(t), p_1(t), \ldots, p_r(t) \in \mathbb{Q}(t)$.

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How can a differential relation

$$p_0(t)f(t) + p_1(t)f'(t) + \cdots + p_r(t)f^{(r)}(t) = 0$$

imply the transcendence of a function $f(t)$?

Step 1

Step 1

Change your point of view.

Let $f(t)$ be a function, and let $L \in \mathbb{Q}(t)[D]$ be such that

$$L \cdot f = 0.$$

Let $\mathbb{Q}(t)[D]$ be the set of differential operators of the form

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It is naturally endowed with the structure of a non-commutative ring and right-Euclidean domain.

For all $U, V \in \mathbb{Q}(t)[D]$ with $V \neq 0$ there are unique $Q, R \in \mathbb{Q}(t)[D]$ with $\mathrm{ord}(R) < \mathrm{ord}(V)$ such that

$$U = QV + R.$$

We write

$$\mathrm{rquo}(U, V) := Q \quad \text{and} \quad \mathrm{rrem}(U, V) := R.$$

Algorithm (least common left multiple)

Input: $U, V \in \mathbb{Q}(t)[D]$

Output: $\operatorname{lclm}(U, V) \in \mathbb{Q}(t)[D]$

1 *if* $U = 0$ *or* $V = 0$ *return* 0

2 *for* $r = 1, 2, \ldots,$ *do:*

3 *check whether there exists* $(p_0, \ldots, p_r) \in \mathbb{Q}(t)^{r+1} \setminus \{0\}$ *with*

$$\operatorname{rrem}(p_0 + \cdots + p_r D^r, U) = \operatorname{rrem}(p_0 + \cdots + p_r D^r, V) = 0$$

4 *if yes, return* $\frac{p_0}{p_r} + \frac{p_1}{p_r} D + \cdots + D^r$.

Step 2

Step 2

Check if L is completely reducible.

An operator $L \in \mathbb{Q}(t)[D]$ is called irreducible, if

$$L = L_1 L_2$$

for $L_1, L_2 \in \mathbb{Q}(t)[D]$ implies

$$\operatorname{ord}(L_1) = 0 \quad \text{or} \quad \operatorname{ord}(L_2) = 0.$$

An operator $L \in \mathbb{Q}(t)[D]$ is called completely reducible, if there are irreducible operators $L_1, \dots, L_n \in \mathbb{Q}(t)[D]$ such that

$$L = \operatorname{lclm}(L_1, \dots, L_n).$$

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In this case, and if the representation is minimal,

$$V(L) = V(L_1) \oplus \dots \oplus V(L_n).$$

Obviously,

$$V(L_1) + \cdots + V(L_n) \subseteq V(L).$$

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Therefore

$$\dim(V(L_1) + \cdots + V(L_n)) = \sum_i \mathrm{ord}(L_i).$$

Let $A, B \in \mathbb{Q}(t)[D]$. Then

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Therefore

$$V(L_1) \oplus \cdots \oplus V(L_n) = V(L).$$

Step 3

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Check which of the L_i's only have algebraic solutions.

An irreducible operator has either only algebraic solutions or the only algebraic solution is 0.

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If $\mathrm{gcrd}(M, L_i) \neq 1$, then

$$L_i = \mathrm{gcrd}(M, L_i),$$

and L_i has only algebraic solutions.

How to detect an operator which has transcendental solutions?

Inspect its generalized series solutions:

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$$\begin{aligned}
&\exp(s_1 x^{-1/v} + s_2 x^{-2/v} + \cdots + s_u x^{-u/v}) \\
&\qquad \times x^{\alpha}((c_{0,0} + c_{0,1} x^{1/v} + c_{0,2} x^{2/v} + \dots) \\
&\qquad\qquad + (c_{1,0} + c_{1,1} x^{1/v} + \dots c_{1,2} x^{2/v} + \dots) \log(x) \\
&\qquad\qquad + \dots \\
&\qquad\qquad + (c_{m,0} + c_{m,1} x^{1/v} + \dots c_{m,2} x^{2/v} + \dots) \log(x)^m)
\end{aligned}$$

Assume that $k \in \{1, \ldots, n\}$ is such that $L_1, \ldots, L_{k-1}$ only have algebraic solutions, while the non-zero solutions of $L_k, \ldots, L_n$ are transcendental.

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If

$$L \cdot f = 0,$$

there are functions $f_1, \ldots, f_n$ such that

$$f = f_1 + \cdots + f_n \quad \text{and} \quad L_i \cdot f_i = 0.$$

Step 4

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Check if $f_k + \cdots + f_n$ is transcendental.

We can assume that

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Note: Take $P = \mathrm{rquo}(L, M)$.

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This cannot be the case, if one can show that

$$P = \mathrm{rquo}(L, M) \quad \text{with} \quad P \cdot (M \cdot f_k) = 0$$

does not have any non-zero algebraic solutions.

Let $L \in \mathbb{Q}(t)[D]$ be completely reducible.

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How can we find irreducible operators

$$L_1, \ldots, L_n \in \mathbb{Q}(t)[D] \quad \text{s.t.} \quad L = \operatorname{lclm}(L_1, \ldots, L_n)?$$

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$$\begin{gathered}\text{DFactorLCLM}\\ \text{DFactor}\end{gathered}$$

which can be used to find a representation of an operator L as the lclm of irreducible operators $L_1, \ldots, L_n$.

In our situation they turned out to be too expensive.

Instead of applying these functions directly, we first tried to find right factors L_i of L, sufficiently many such that $L = \mathrm{lclm}(L_i)$, and small enough such that $\mathrm{DFactorLCLM}$ and $\mathrm{DFactor}$ can be applied to them, based on the following observation:

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If $L = \mathrm{lclm}(U, V)$ such that $\mathrm{gcrd}(U, V) = 1$, and if ξ is a (non-apparent) singularity of L, then ξ is a non-apparent singularity of U or V. Assuming that ξ is not a singularity of V, there is a generalized series solution of L that is a solution of U but not of V. By computing sufficiently many of its initial terms, U can be found by guessing.

If necessary, the procedure can be applied to find right factors of $\mathrm{adj}(\mathrm{L})$. If A is a right factor of $\mathrm{adj}(\mathrm{L})$, then $\mathrm{adj}(A)$ is a left factor of L, and the corresponding right factor can be computed by division with remainder.

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If the right factors found are too big, one can try to find smaller ones by computing gcrd's and proceeding recursively as just described.

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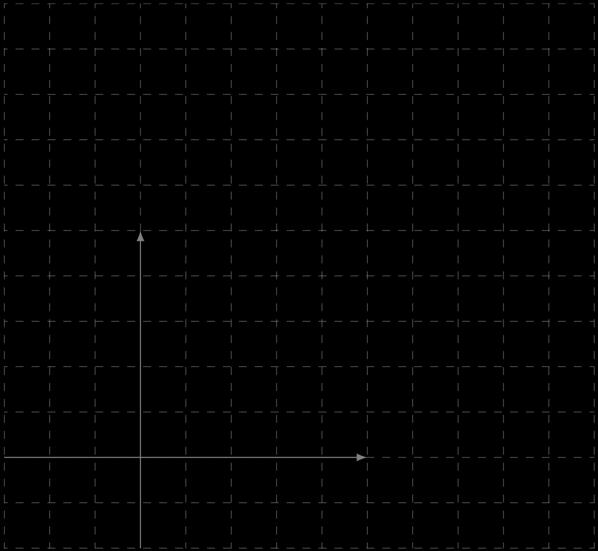
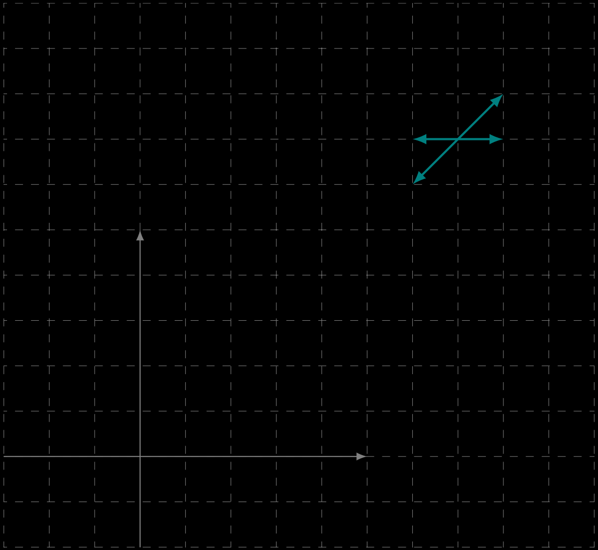
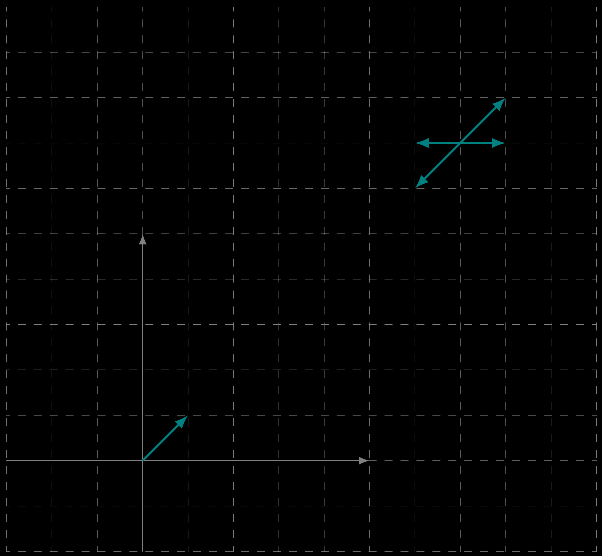
Inhomogeneous Restricted Lattice Walks

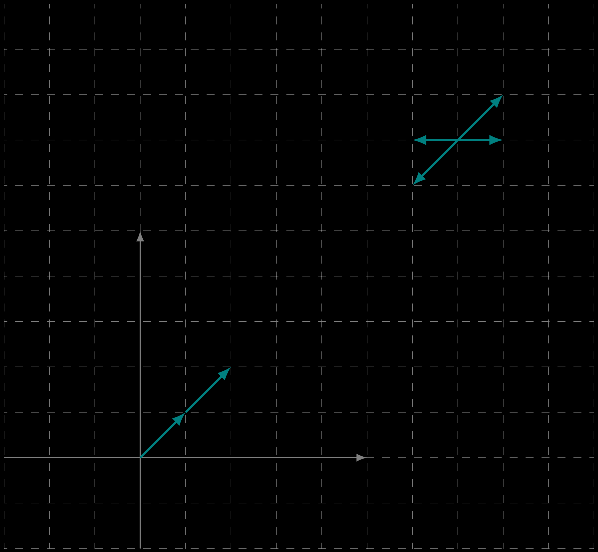
joint work with
Manuel Kauers

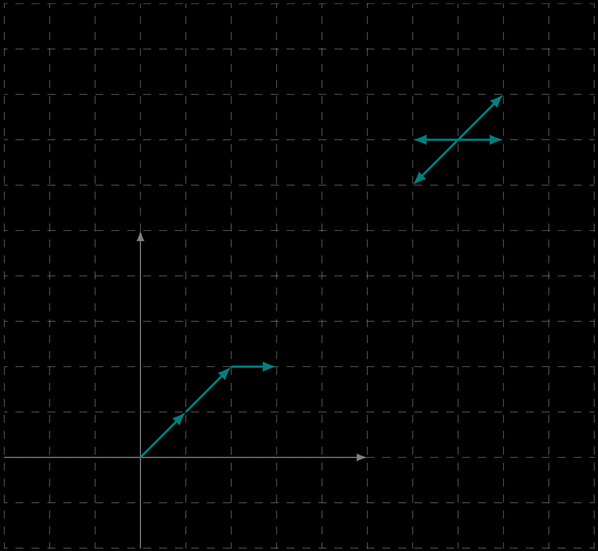
A lattice walk is a sequence of points of $\mathbb{Z}^{\mathrm{d}}$. The consecutive differences of these points are the steps of the walk. If the steps are taken from a fixed set $\mathbf{S}$ we consider the lattice walk to be homogeneous with respect to $\mathbf{S}$.

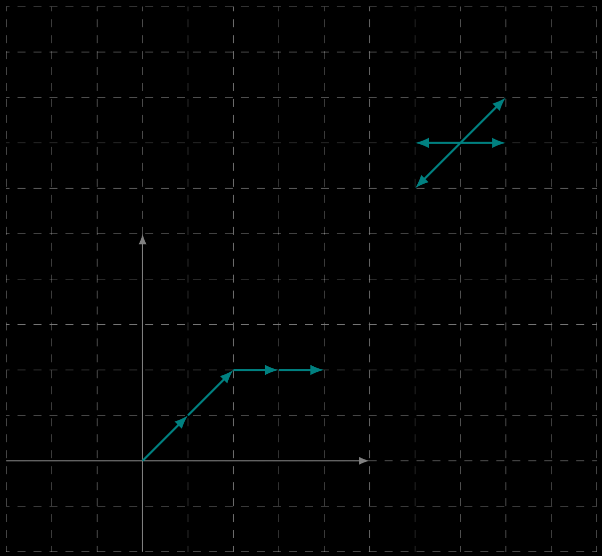


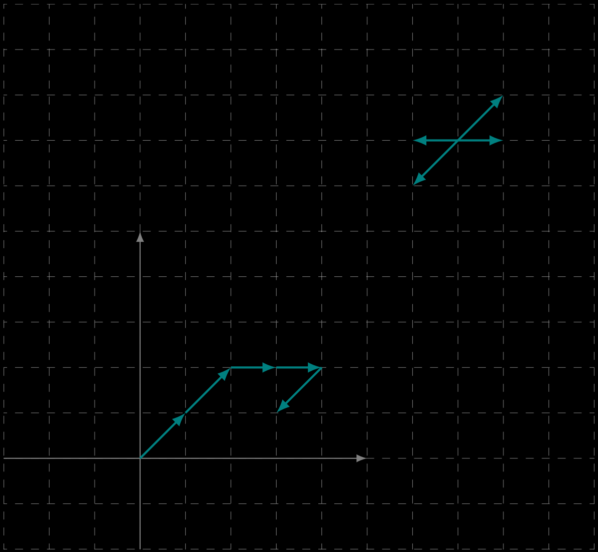


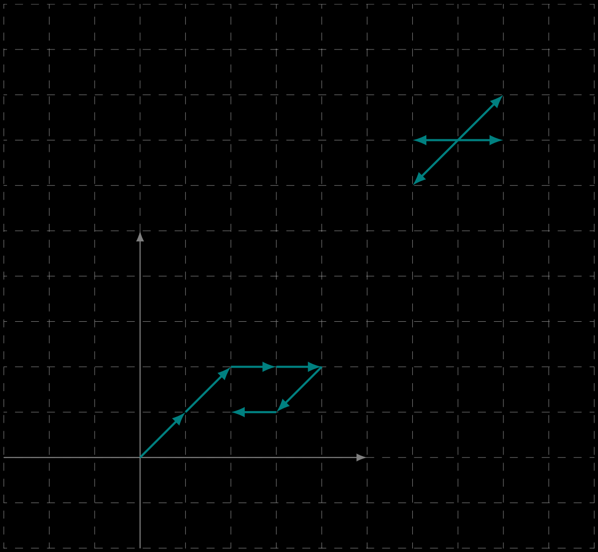


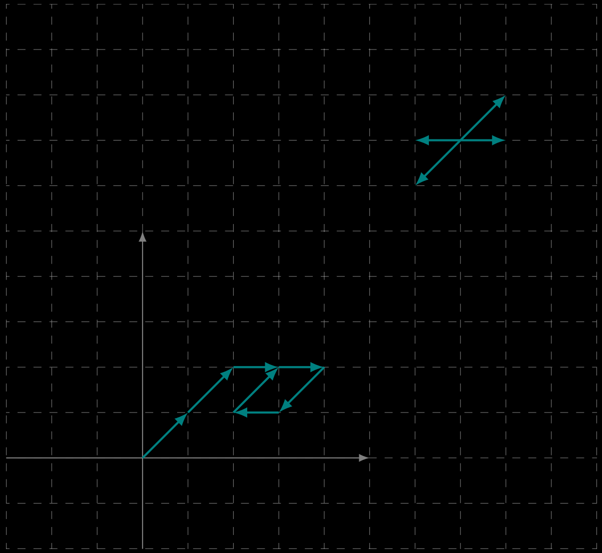


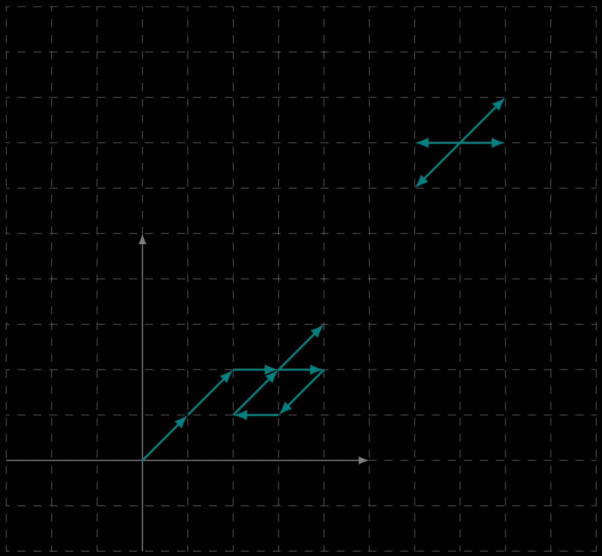


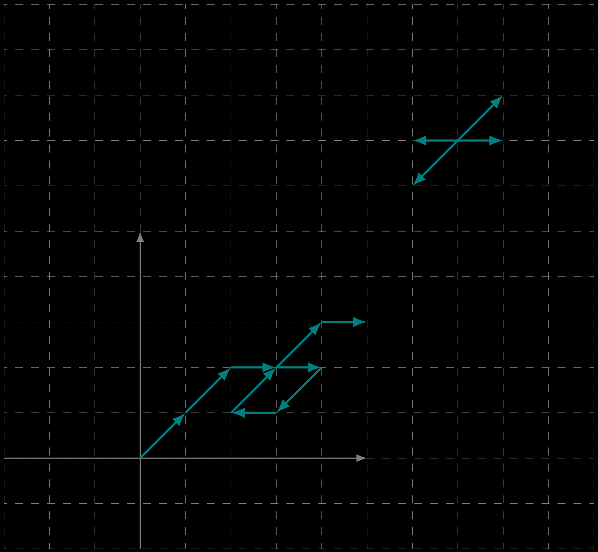




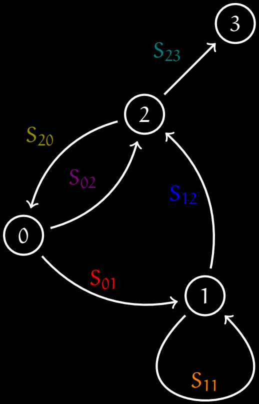








A lattice walk is called inhomogeneous, if the set of admissible steps is governed by a deterministic finite automaton.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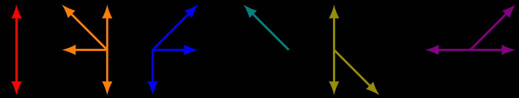

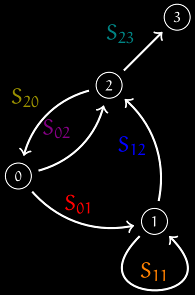
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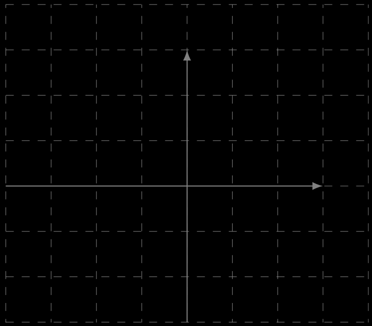

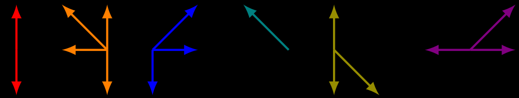

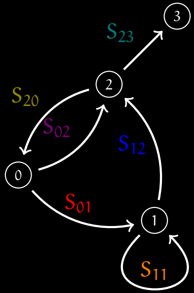

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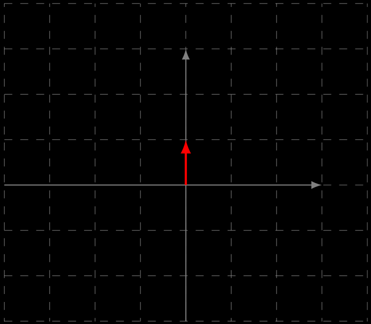

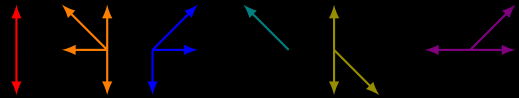

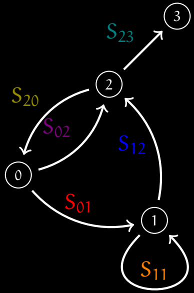
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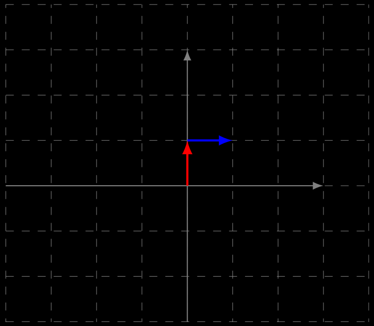

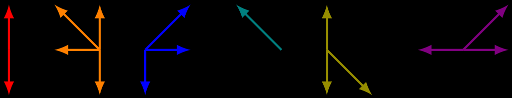

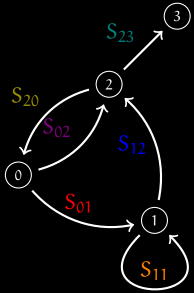

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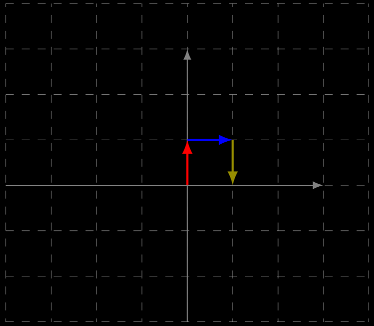

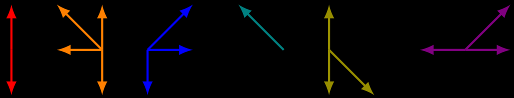

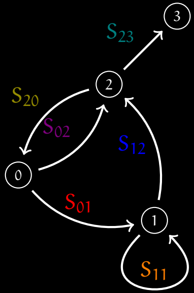
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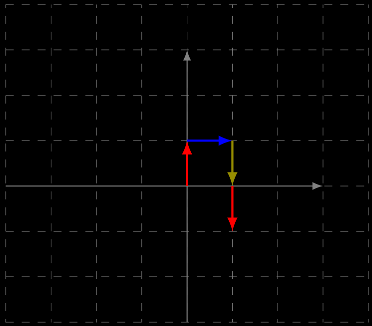

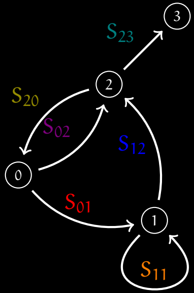
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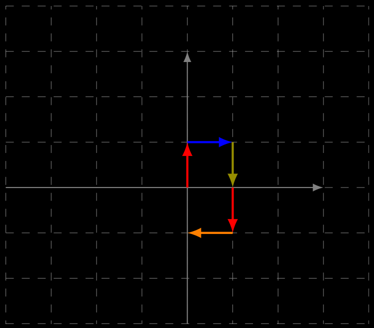

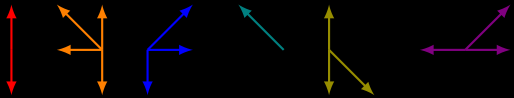

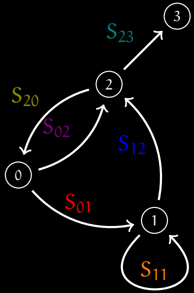

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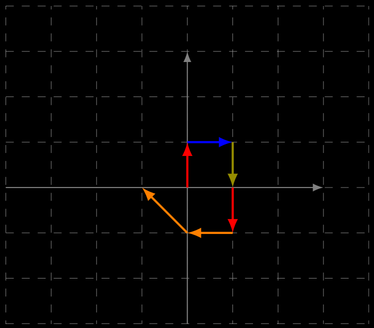

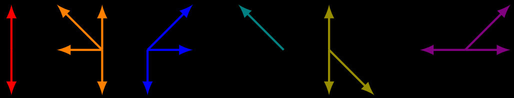

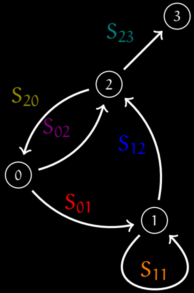

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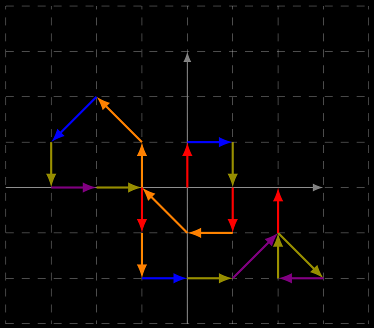
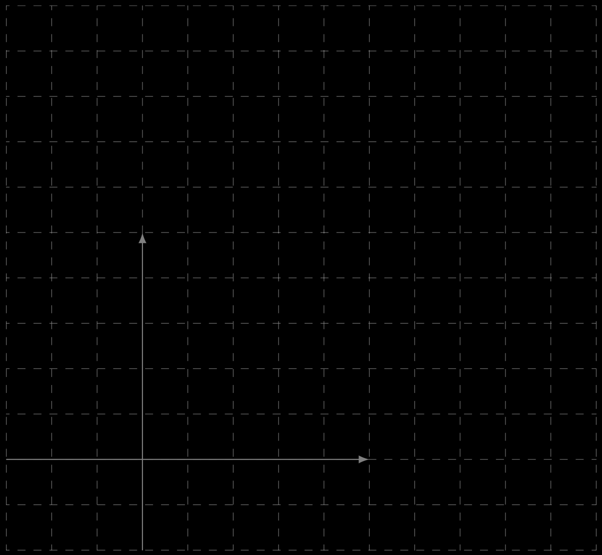
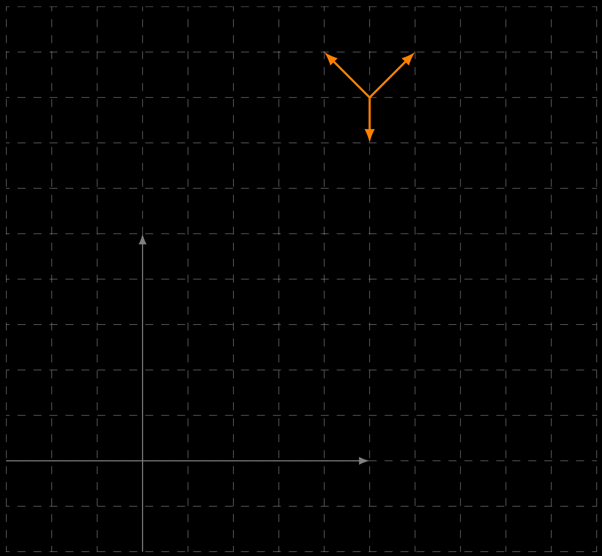
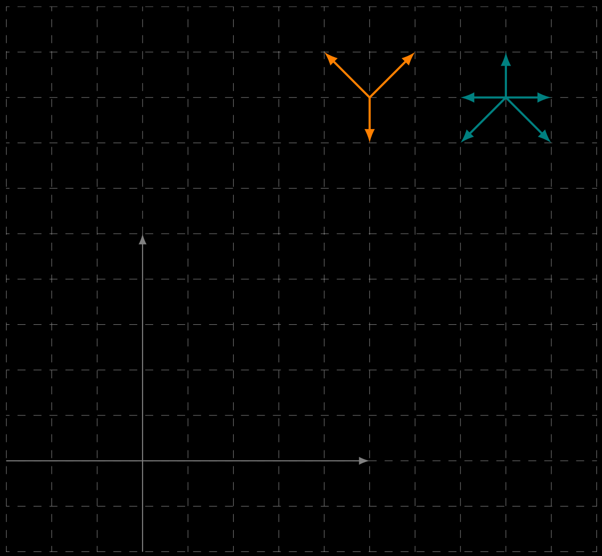
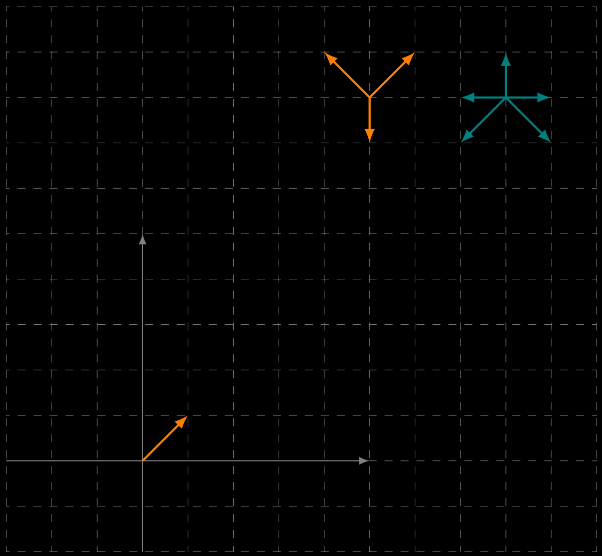


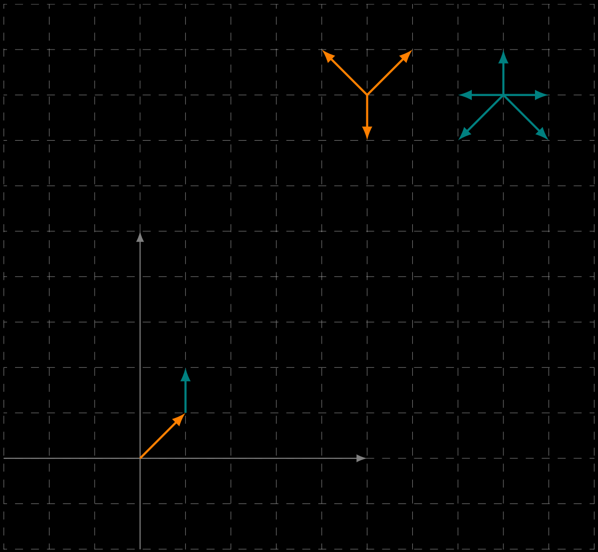
A few examples of what it can mean to be inhomogeneous...

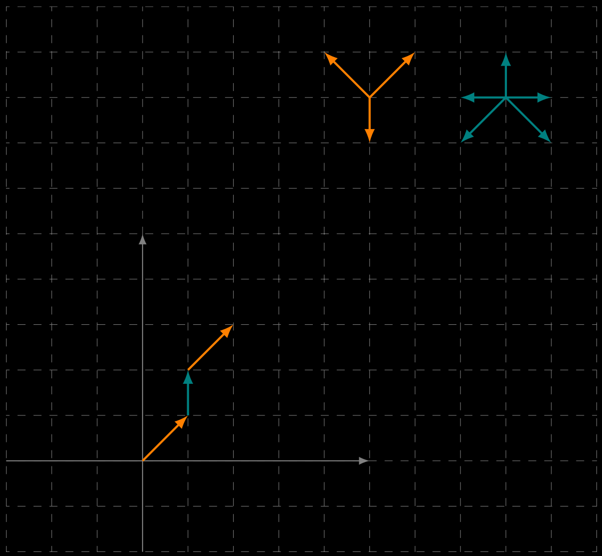


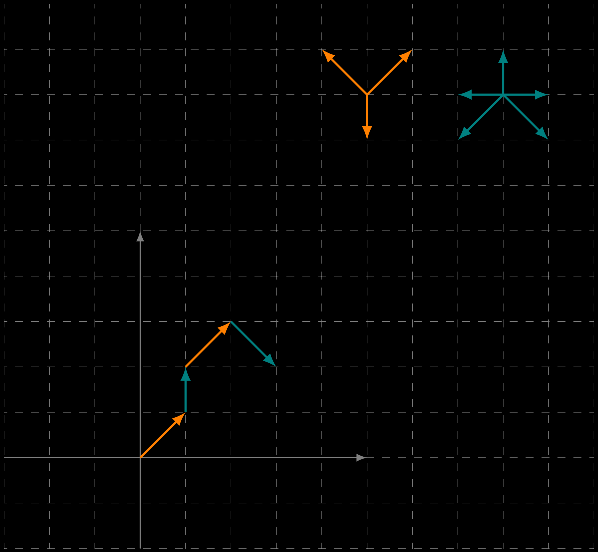


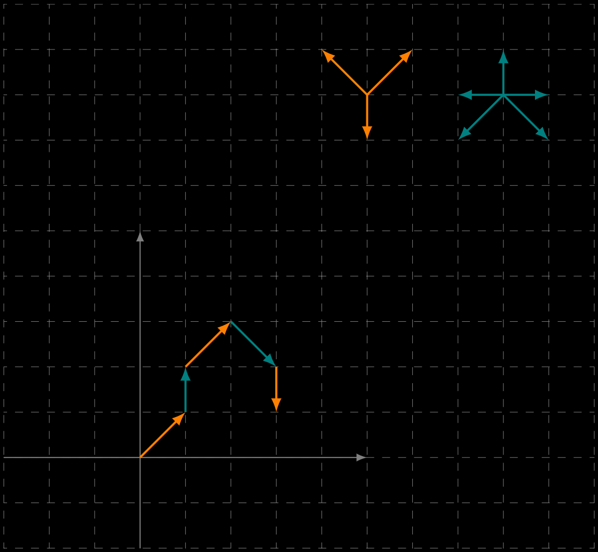


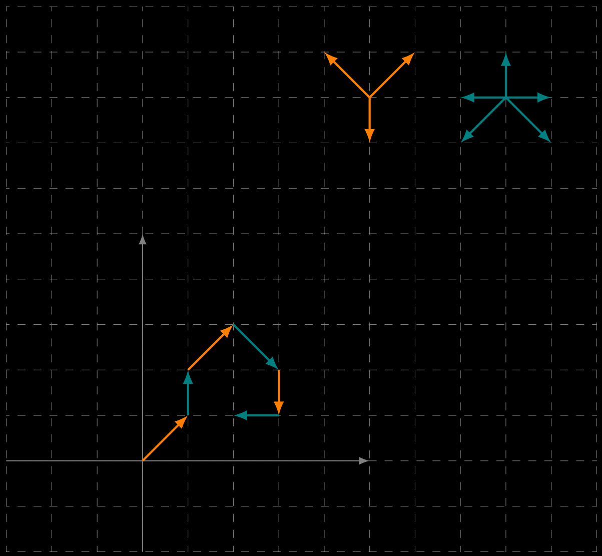


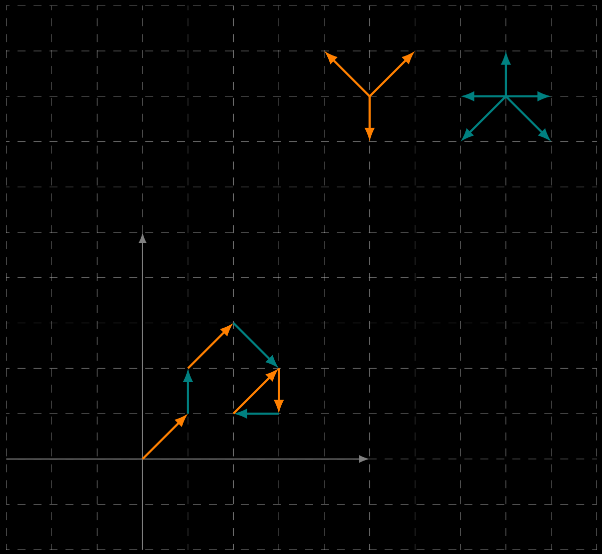


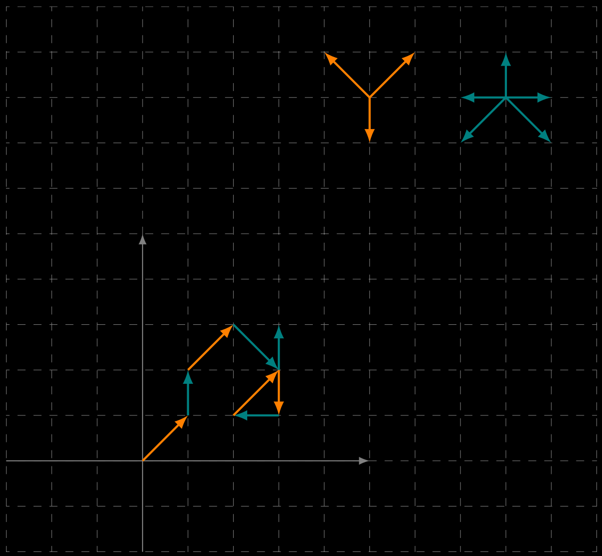


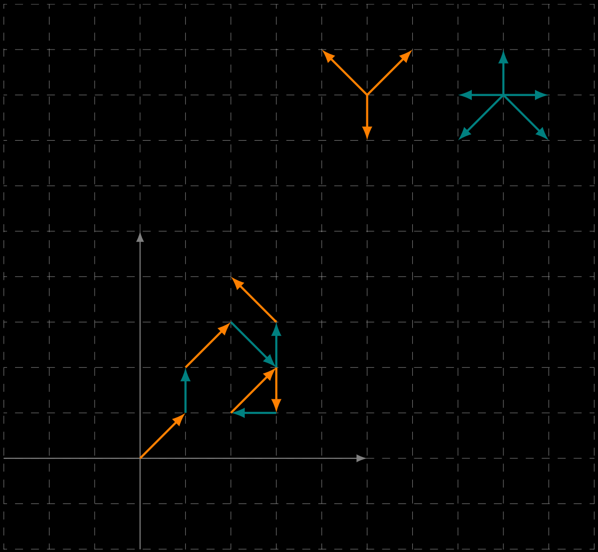


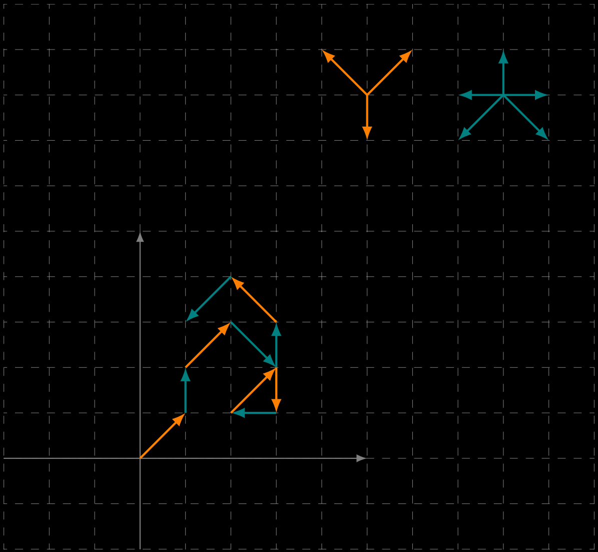


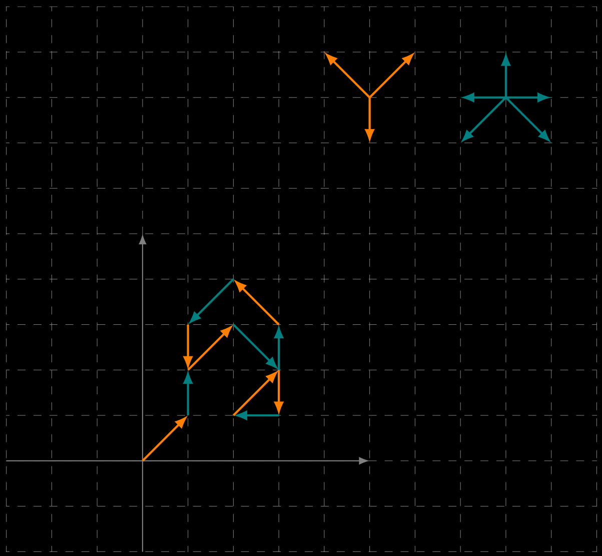


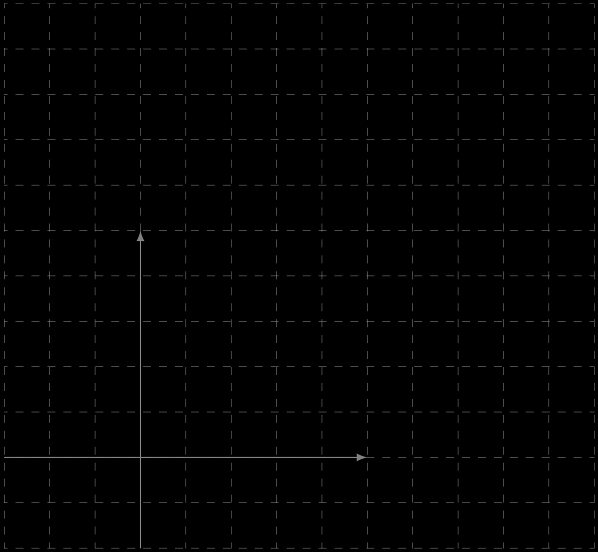


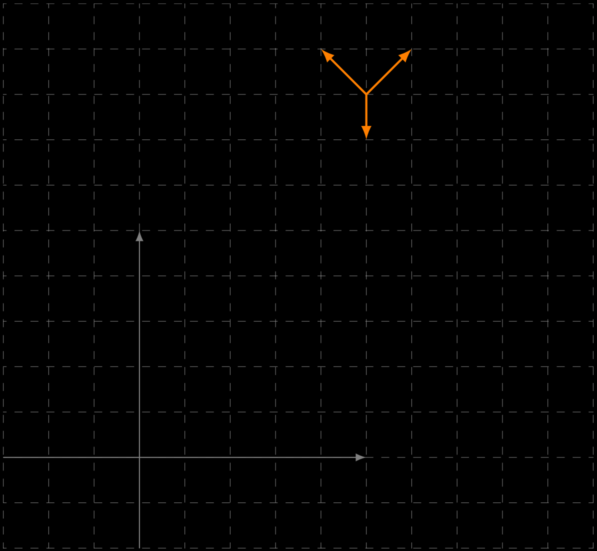


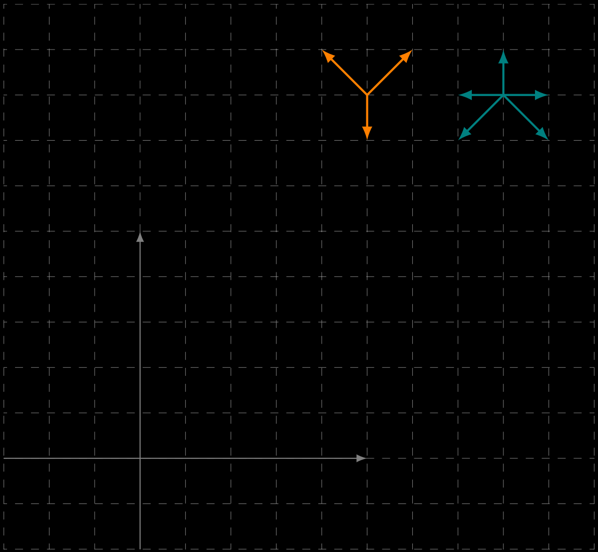


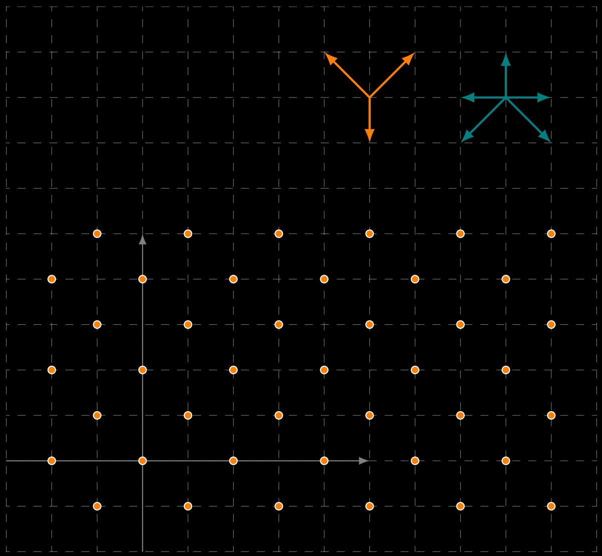


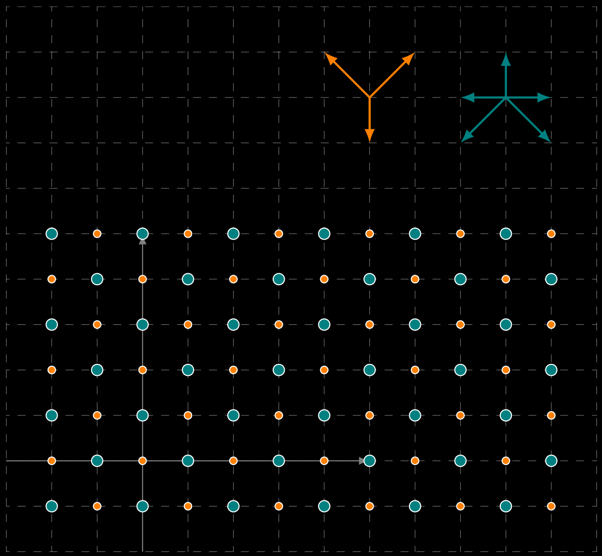


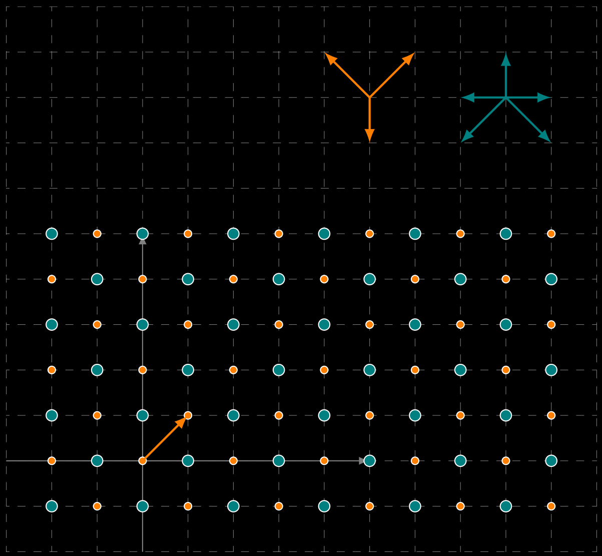


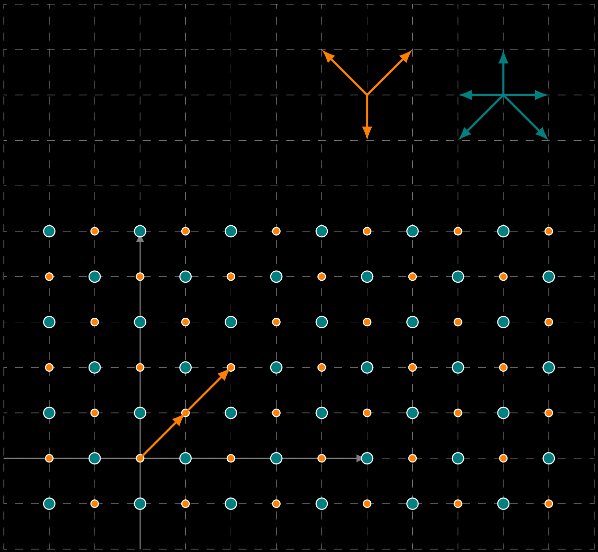


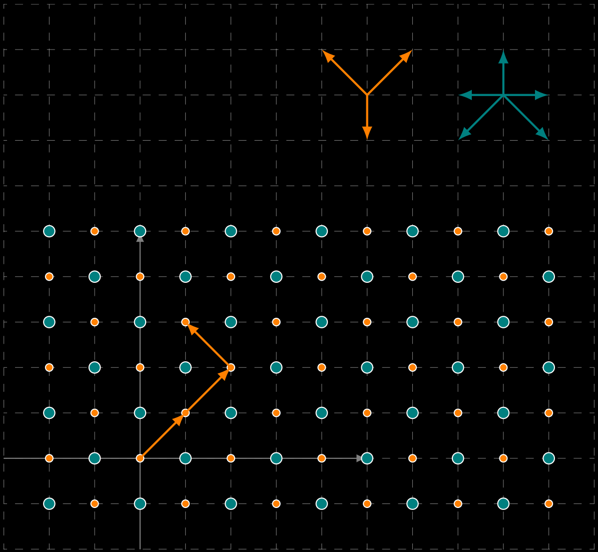


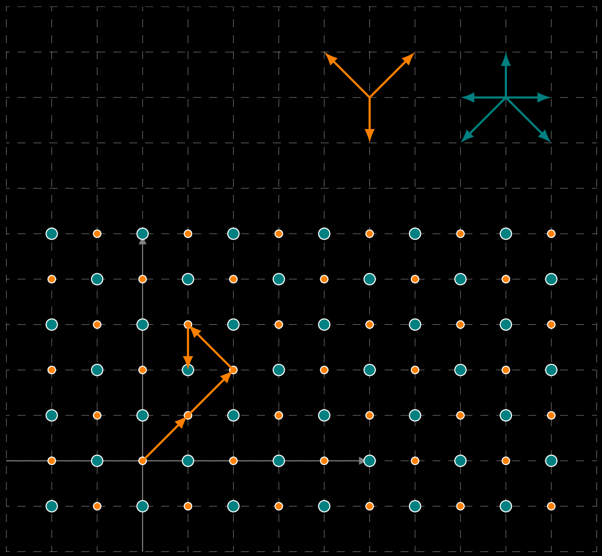


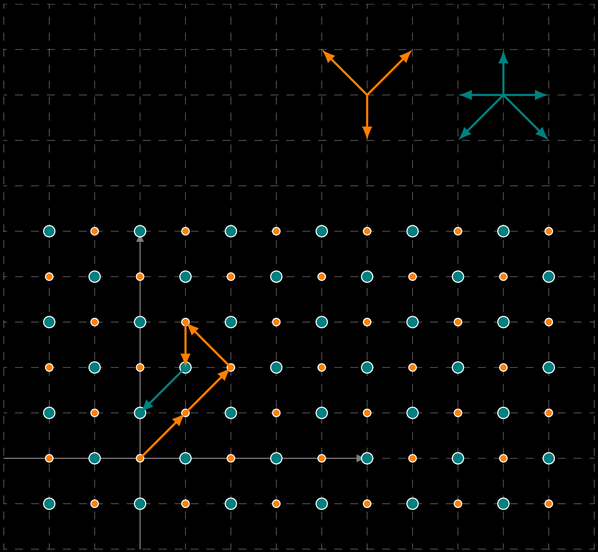


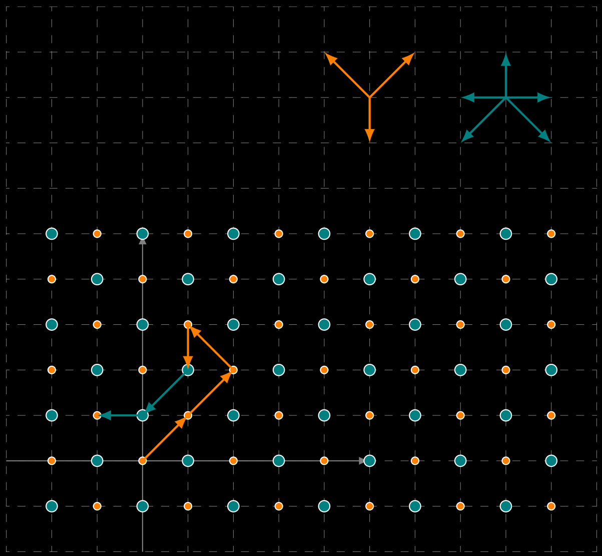


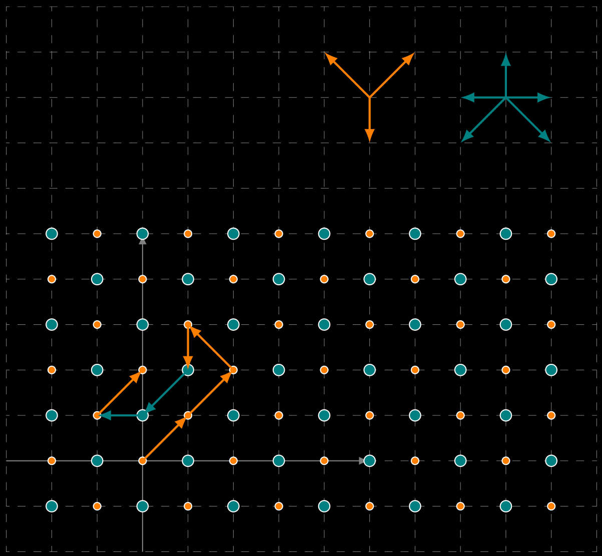


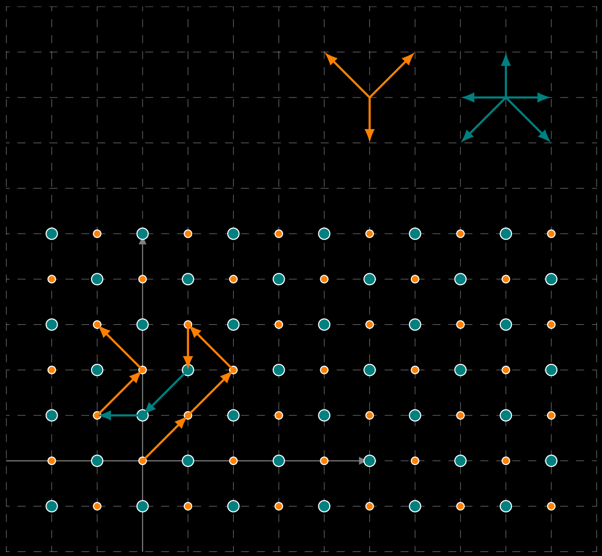


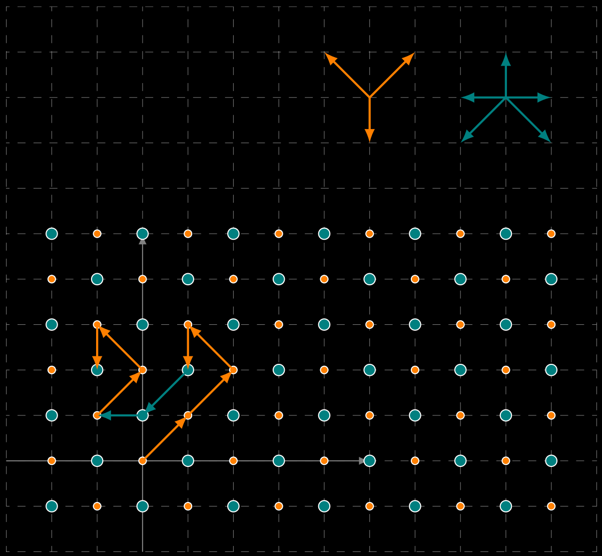


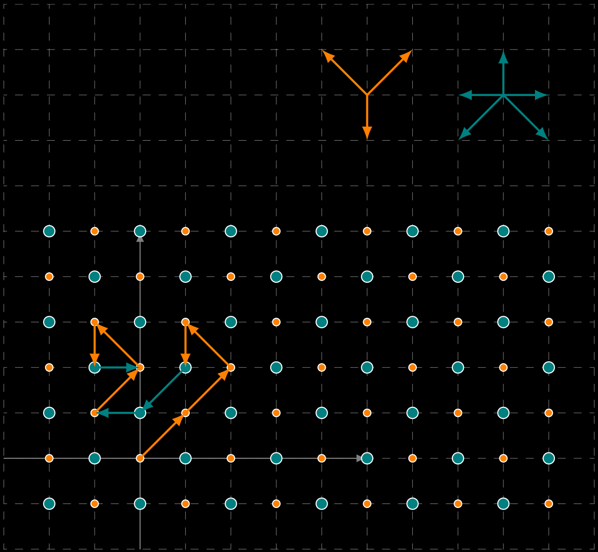


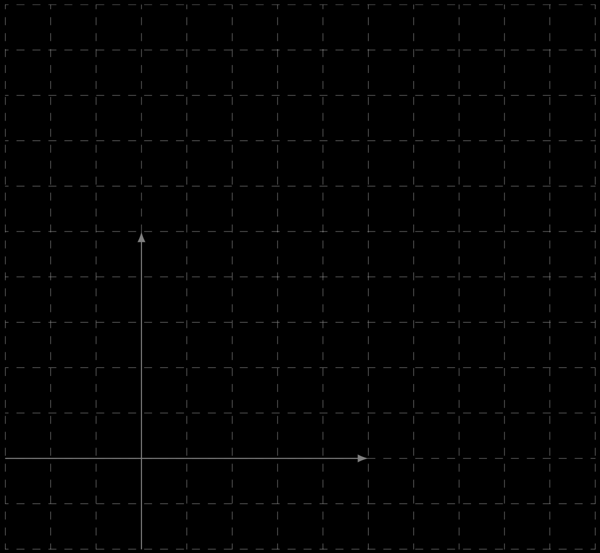


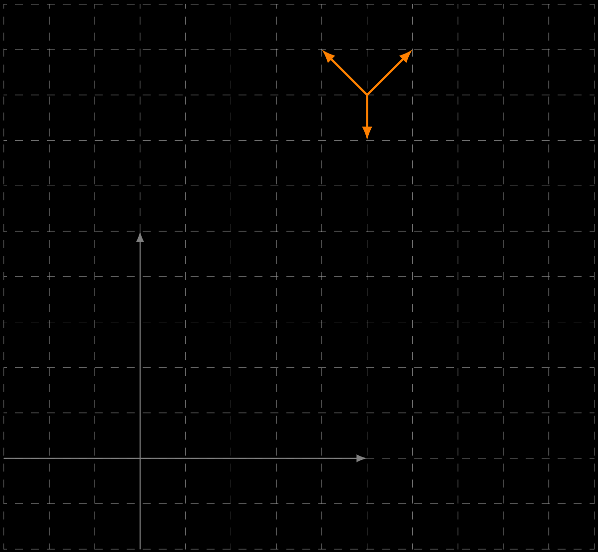


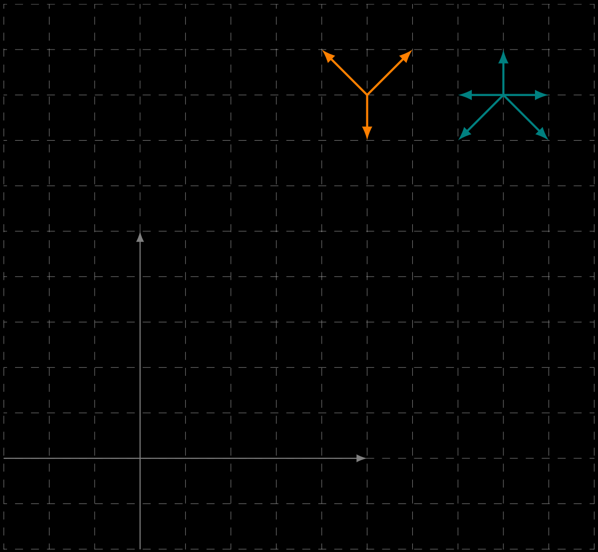


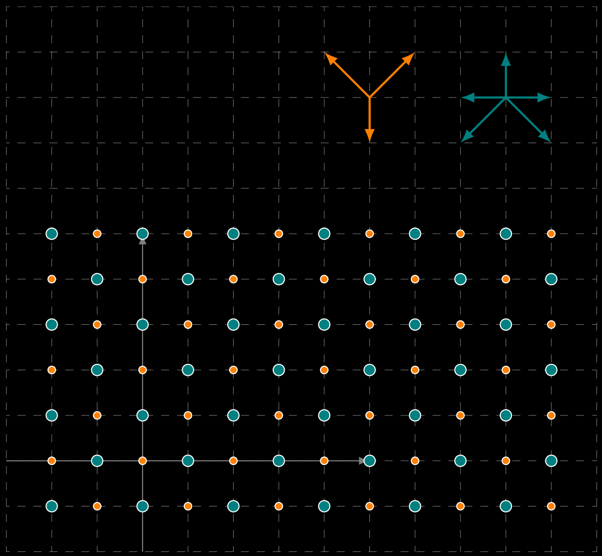


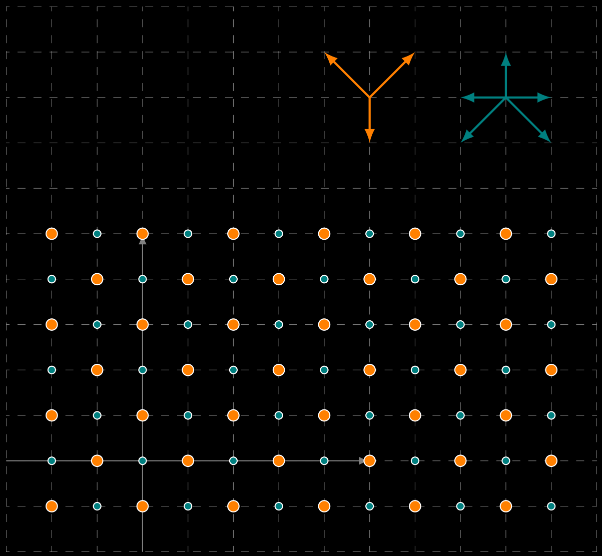


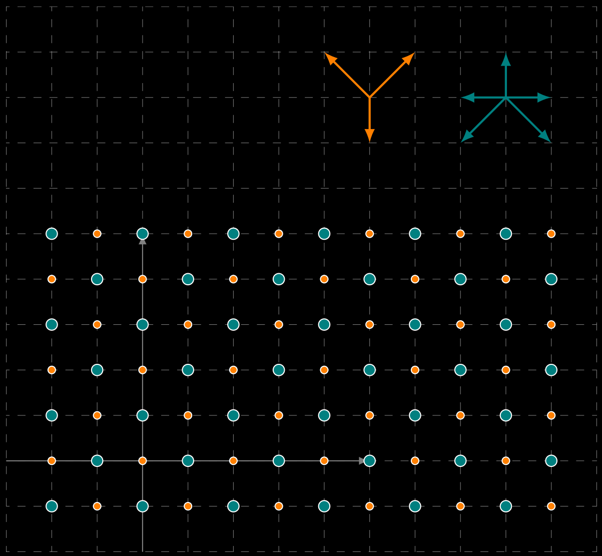


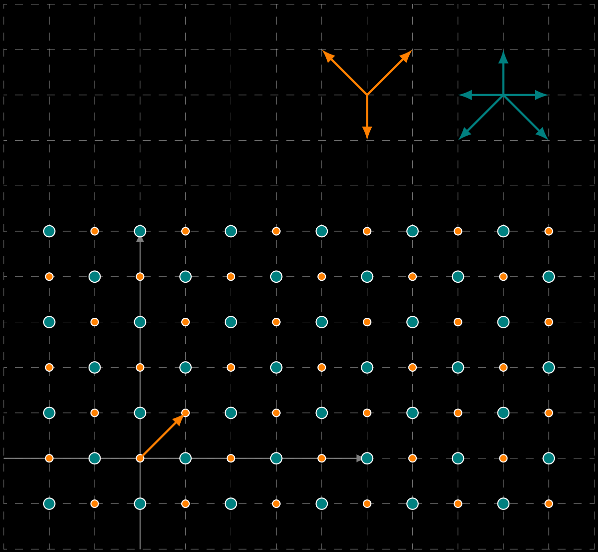


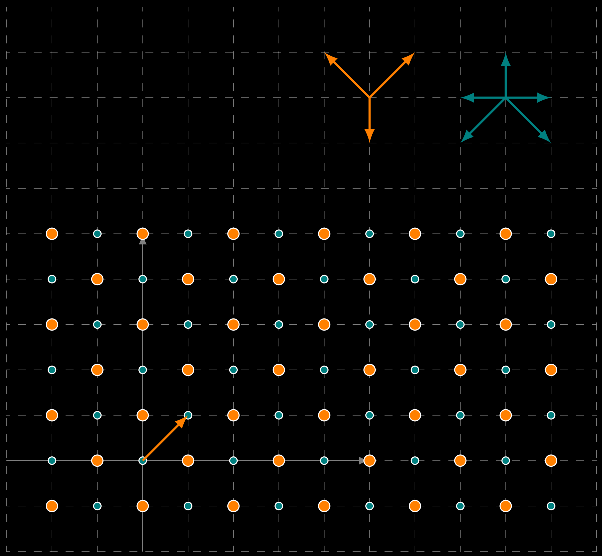


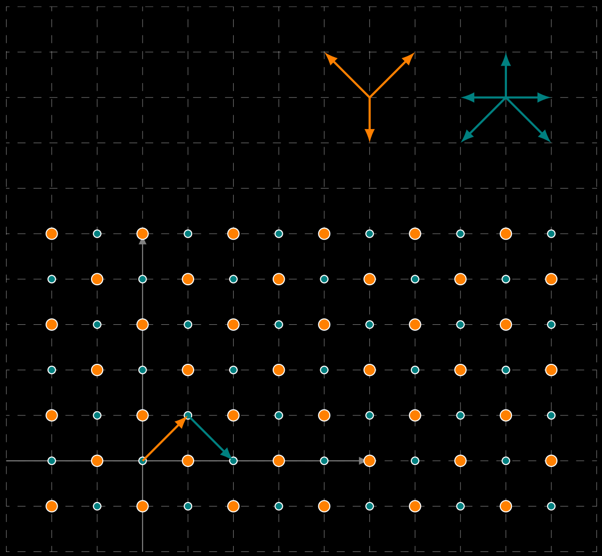


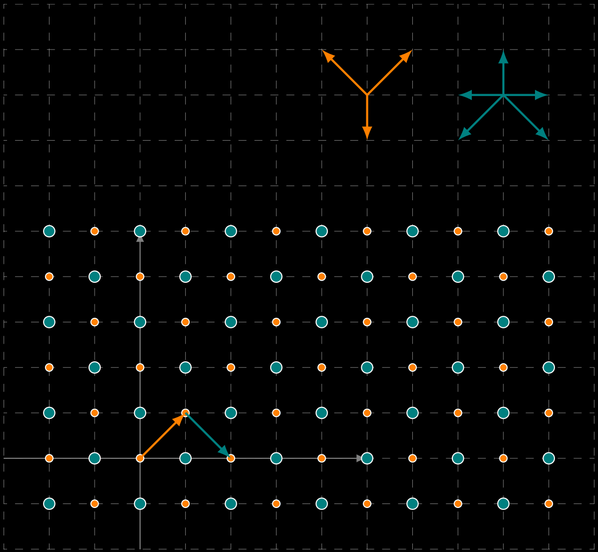


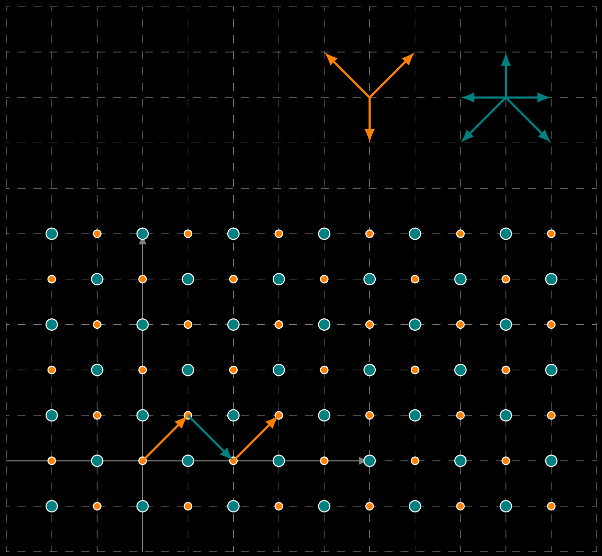


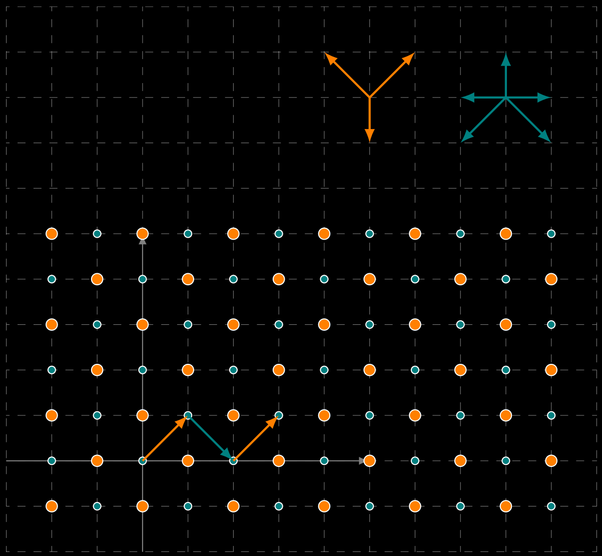


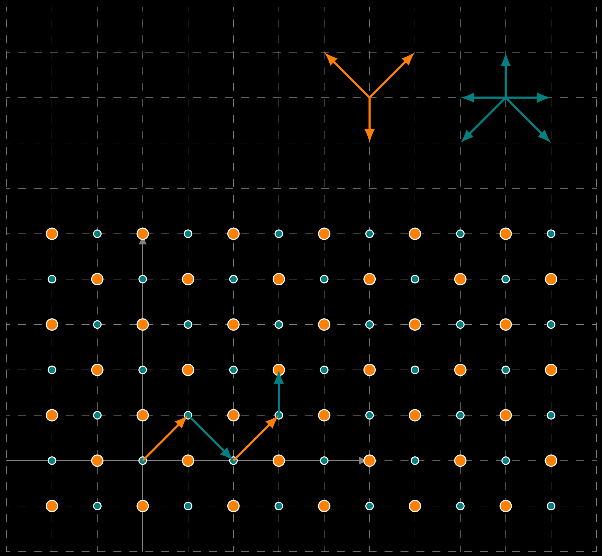


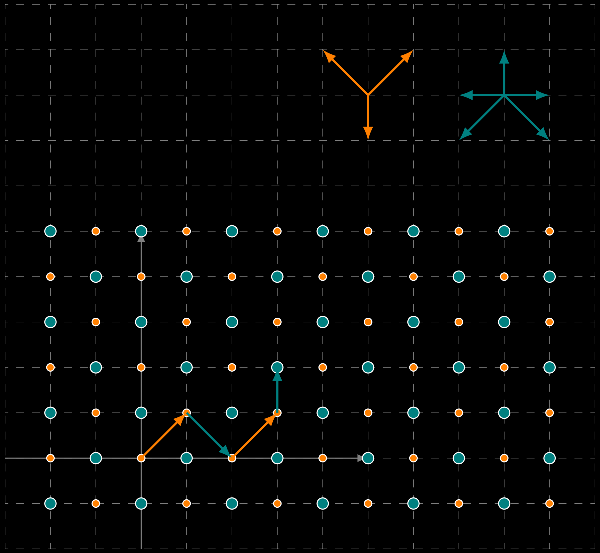


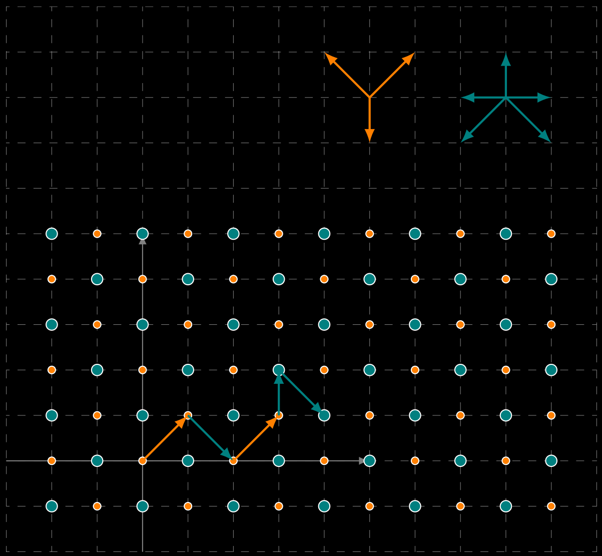


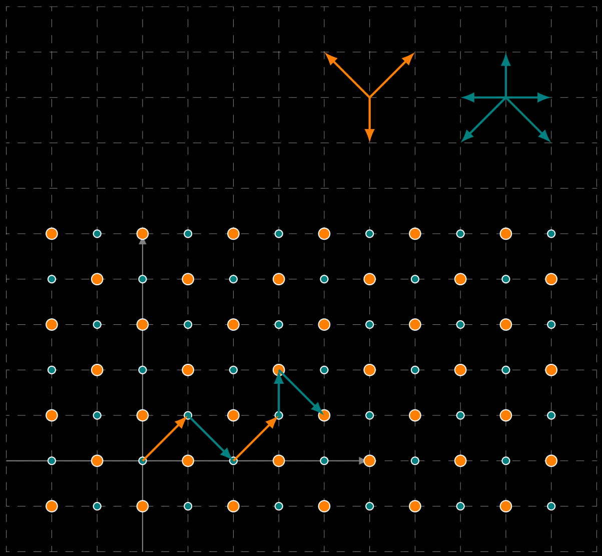


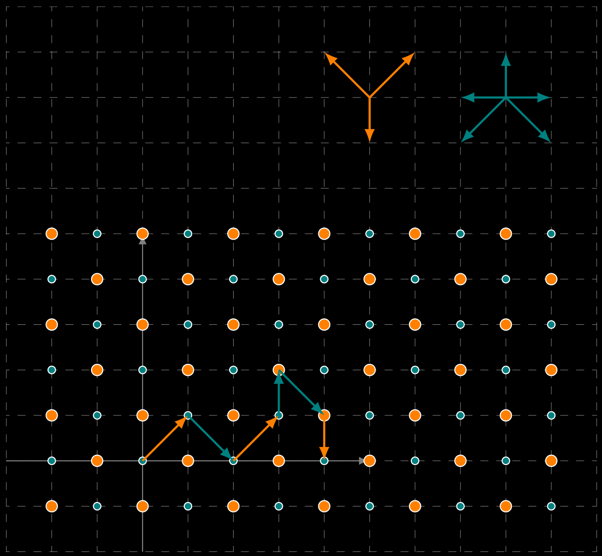












Another example: walks obeying two-step rules.

A first step in finding exact or asymptotic formulas for the number of lattice walks is to decide the nature of the generating function and to find an expression for it.

Generating functions of unrestricted lattice walks are rational.

Let F be the generating function of walks that start at the origin, counted by length and endpoint, and let F_q be the one of those associated with paths in the automaton that end at final state q.

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$$F = \sum_{q \text{ final}} F_q$$

and the F_q's uniquely solve a linear system of functional equations of the form

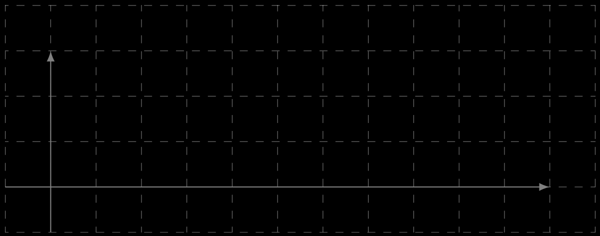
$$F_q = [q = q_0] + t \sum_p S_{pq} F_p,$$

where S_{pq} is the step polynomial of the step set $\mathbf{S}_{pq}$.

Generating functions of lattice walks restricted to the half-space $\mathbb{Z}^{d-1} \times \mathbb{Z}_{\geq 0}$ are algebraic.

Example

Walks in the half-plane $\mathbb{Z} \times \mathbb{Z}_{\geq 0}$ which start at the origin and take its steps from $\mathbf{S} = \{(1,1),(1,-1)\}$.



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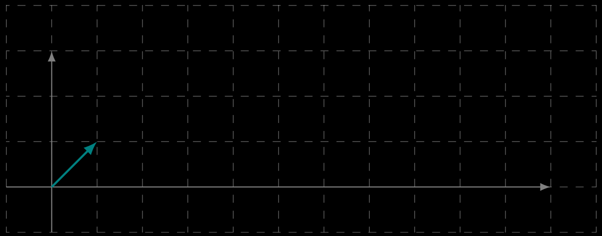

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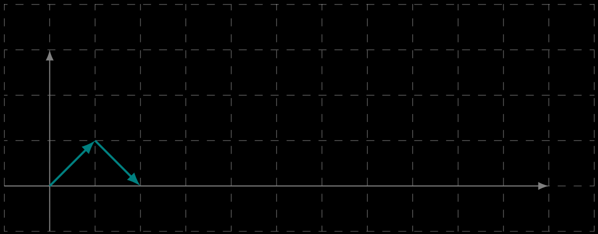

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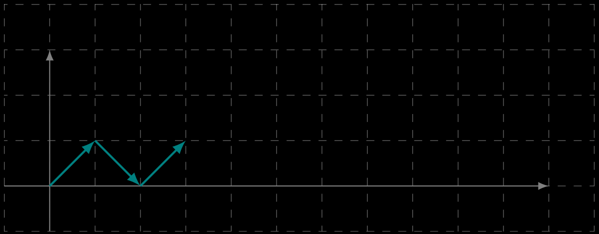

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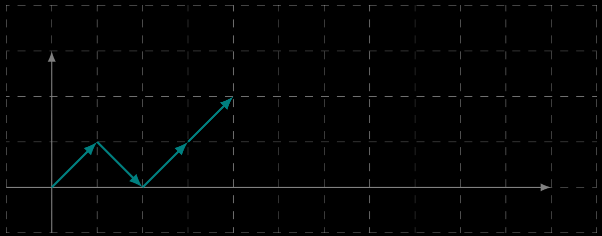

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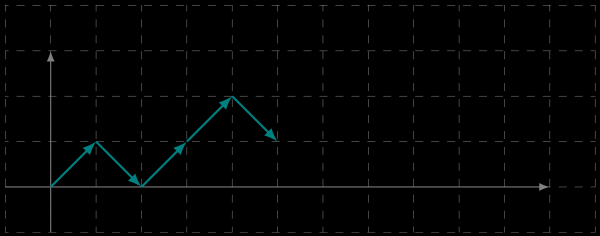

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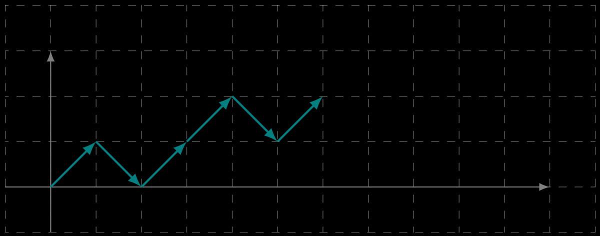

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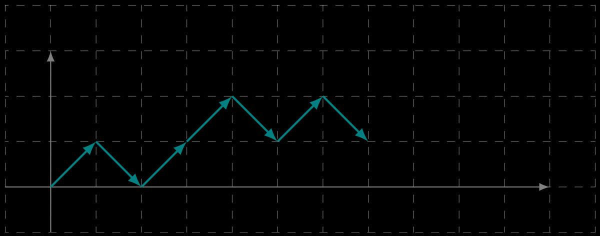

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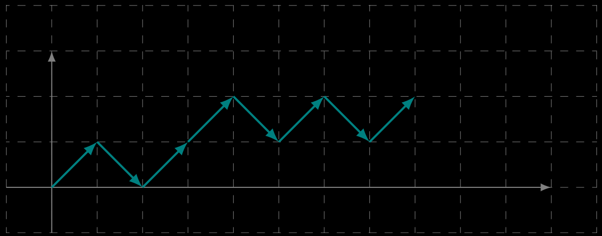

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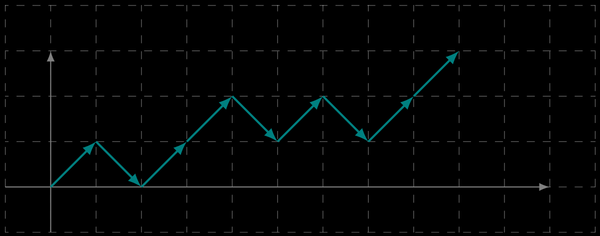

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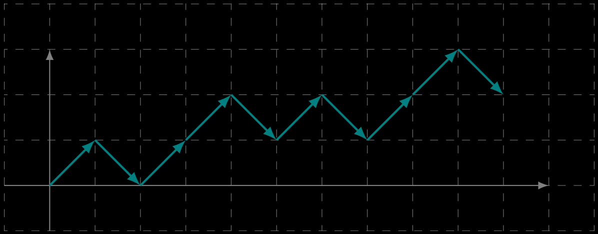

The functional equation for the corresponding generating function

$$F(x,t) = 1 + t(\bar{x} + x)F(x,t) - t\bar{x}F(0,t)$$

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when evaluated at the power series root $x(t)$ of $1 - t(\bar{x} + x)$. Therefore,

$$F(0,t) = x(t)/t \quad \text{and} \quad F(x,t) = \frac{1 - \bar{x}x(t)}{1 - t(\bar{x} + x)}.$$

Theorem

Let $\mathbb{K}$ be a field of characteristic zero, and let Δ be given by

$$\Delta : \mathbb{K}[x][[t]]^n \to \mathbb{K}[x][[t]]^n$$
$$\Delta f(x,t) = (f(x,t) - f(0,t))/x.$$

Then, for $a \in \mathbb{K}[x,t]^n$ and $B_i \in \mathbb{K}[x,t]^{n\times n}$,

$$f = a + t\sum_{i=0}^{k} B_i \Delta^i f$$

has a unique solution f in $\mathbb{K}[x][[t]]^n$, and its components are algebraic over $\mathbb{K}[x,t]$.

Sketch of Proof

Existence and Uniqueness

The equation

$$f = a + t \sum_{i=0}^{k} B_i \Delta^i f$$

has a unique solution $f \in \mathbb{K}[x][[t]]^n$ because the coefficients of f can be computed recursively via

$$[t^0]f = [t^0]a$$

$$[t^{n+1}]f = [t^{n+1}]a + \sum_{i=0}^{k} \sum_{j=0}^{n} [t^j] B_i \Delta^i [t^{n-j}] f.$$

Algebraicity

1 Rewrite the equation in the form

$$\left(x^k I_n - t\sum_{i=0}^{k} x^{k-i} B_i\right) f(x,t) = x^k a - t\sum_{j=0}^{k-1}\left(\sum_{i=j+1}^{k} \frac{x^{k+j-i}}{j!} B_i\right) f^{(j)}(0,t).$$

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2 Eliminate $f(x,t)$ by

a) replacing x by a root $x(t)$ of $\det\left(x^k I_n - t\sum_{i=0}^{k} x^{k-i} B_i\right)$, and
b) multiplying the equation by elements of the co-kernel of the matrix.

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b) multiplying the equation by elements of the co-kernel of the matrix.

3 Solve the resulting linear systems for the $f^{(j)}(0,t)$'s and for $f(x,t)$.

To avoid difficulties arising from the linear system possibly being singular one solves a perturbation of the original equation:

$$\tilde{f} = a(x, t^2) + \epsilon t E \Delta^k \tilde{f} + t^2 \sum_{i=0}^{k} B_i(x, t^2) \Delta^i \tilde{f}.$$

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Algebraicity of $\tilde{f}$ then implies algebraicity of f, since

$$f(x, t^2) = [\epsilon^0] \tilde{f}(x, t).$$

Details

Step 1

Step 1

Rewrite the Equation.

The rewriting of the equation is based on

$$\Delta^i f(x,t) = \frac{1}{x^i}\left(f(x,t) - \sum_{j=0}^{i-1} \frac{x^j}{j!} f^{(j)}(0,t)\right).$$

Step 2

Step 2

Eliminate $f(x, t)$.

Lemma

Let $P \in \mathbb{K}[x][t]^{n\times n}$ such that $P_0 = [t^0]P \in \mathbb{K}^{n\times n}$ and $\det P_0 \neq 0$, and let $K = x^k I_n - tP \in \mathbb{K}[x][t]^{nxn}$ for some $k \in \mathbb{N}$. Then, for any eigenvalue λ of P_0 and any k-th root of unity ω, there are algebraic series

$$y(t) = \omega\lambda^{1/k}t^{1/k} + O(t^{2/k}) \in \mathbb{K}[[t^{1/k}]],$$

$$v(t) \in \mathbb{K}[[t^{1/k}]]^n$$

such that

$$\det K(y(t), t) = 0,$$

$$v(t)K(y(t), t)) = 0 \quad \text{and} \quad v(0)P_0 = \lambda v(0).$$

$$\tilde{f} = a(x, t^2) + \epsilon t E \Delta^k \tilde{f} + t^2 \sum_{i=0}^{k} B_i(x, t^2) \Delta^i \tilde{f}$$

$$\left(x^k I_n - tP\right)\tilde{f}(x,t) = x^k a - t\sum_{j=0}^{k-1} Q_j \tilde{f}^{(j)}(0,t)$$

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$$P = E + O(t) \quad \text{and} \quad Q_j = x^j E + O(t)$$

Let $\lambda_0, \ldots, \lambda_{n-1} \in \mathbb{K} \setminus \{0\}$ be pairwise different and define

$$E = \mathrm{diag}(\lambda_0, \ldots, \lambda_{n-1}).$$

Let ω be a primitive k-th root of unity, $i \in \{0, \ldots, k-1\}$ and $j \in \{0, \ldots, n-1\}$.

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By the lemma, there are a series

$$y_{ij}(t) = \omega^i \lambda_j^{1/k} t^{1/k} + O(t^{2/k}) \in \mathbb{K}[[t^{1/k}]]$$

$$v_{ij}(t) = \lambda_j^{-1} e_j + O(t^{1/k}) \in \mathbb{K}[[t^{1/k}]]^n$$

such that

$$\det K(y_{ij}(t), t) = 0 \quad \text{and} \quad v_{ij}(t) K(y_{ij}(t), t) = 0.$$

Step 3

Step 3

Show that the linear system is non-singular.

Lemma

Let $\lambda_0, \dots, \lambda_{n-1} \in \mathbb{K} \setminus \{0\}$ and let ω be a primitive k-th root of unity. For $u, v = 0, \dots, nk-1$ define

$$c_{u,v} = (\omega^{u \bmod k} \lambda_{\lfloor u/k \rfloor})^{\lfloor v/n \rfloor} \delta_{\lfloor u/k \rfloor, v \bmod n}.$$

Then

$$\det((c_{u,v})_{u,v=0}^{nk-1}) = \pm \left(\prod_{0 \le i < j < k} (\omega^j - \omega^i) \right)^n \prod_{l=0}^{n-1} \lambda_l^{\binom{k}{2}}.$$

References:

Manfred Buchacher and Manuel Kauers, *Inhomogeneous Restricted Lattice Walks*, Proceedings of FPSAC'19, 2019

Mireille Bousquet-Mélou and Arnaud Jehanne, *Polynomial equations with one catalytic variable, algebraic series and map enumeration*, Journal of Combinatorial Theory, 2006

Qing-Hu Hou and Toufik Mansour, *Kernel method and linear recurrence system*, Journal of Computational and Applied Mathematics, 2008

Dorin Popescu, *General Néron desingularization and approximation*, Nagoya Mathematical Journal, 1986

Time-Inhomogeneous Lattice Walks Restricted to $\mathbb{Z}^2_{\geq 0}$

joint work (in progress) with
Manuel Kauers

Let

$$S_0, S_1 \in \mathbb{Q}[x, y, \overline{x}, \overline{y}]$$

be such that

$$\mathrm{supp}(S_0), \mathrm{supp}(S_1) \subseteq \{-1, 0, 1\}^2.$$

Solve the system of functional equations

$$\begin{aligned} F_0 &= 1 + tS_1F_1 - t\bar{y}[\bar{y}]S_1F_1(x,0) - t\bar{x}[\bar{x}]S_1F_1(0,y) \\ &\qquad + t\bar{x}\bar{y}[\bar{x}\bar{y}]S_1F_1(0,0) \\ F_1 &= \qquad tS_0F_0 - t\bar{y}[\bar{y}]S_0F_0(x,0) - t\bar{x}[\bar{x}]S_0F_0(0,y) \\ &\qquad + t\bar{x}\bar{y}[\bar{x}\bar{y}]S_0F_0(0,0) \end{aligned}$$

for F_0 and F_1 over $\mathbb{Q}[x,y][[t]]$.

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for F_0 and F_1 over $\mathbb{Q}[x,y][[t]]$.

Are F_0 and F_1 rational, algebraic, D-finite, or differentiably algebraic?

The algebraic kernel method.

A classical example.

$$S(x, y) = x + y + \overline{x} + \overline{y}$$

$$S(x, y) = x + y + \overline{x} + \overline{y}$$

$$G = \{(x, y), (\overline{x}, y), (\overline{x}, \overline{y}, (x, \overline{y}))\}$$

$$xyK(x,y,t)F(x,y,t) = xy - txF(x,0,t) - tyF(0,y,t)$$

$$
\begin{aligned}
xyK(x,y,t)F(x,y,t) &= xy - txF(x,0,t) - tyF(0,y,t)\\
\bar{x}yK(x,y,t)F(\bar{x},y,t) &= \bar{x}y - t\bar{x}F(\bar{x},0,t) - tyF(0,y,t)\\
\overline{xy}K(x,y,t)F(\bar{x},\bar{y},t) &= \overline{xy} - t\bar{x}F(\bar{x},0,t) - t\bar{y}F(0,\bar{y},t)\\
x\bar{y}K(x,y,t)F(x,\bar{y},t) &= x\bar{y} - txF(x,0,t) - t\bar{y}F(0,\bar{y},t)
\end{aligned}
$$

$$xyF(x,y,t) - \bar{x}yF(\bar{x},y,t) + \overline{xy}F(\bar{x},\bar{y},t) - x\bar{y}F(x,\bar{y},t)$$

$$= \frac{xy - \bar{x}y + \overline{xy} - x\bar{y}}{K(x,y,t)}$$

$$xyF(x, y, t) = [x^{>}y^{>}]\frac{xy - \bar{x}y + \overline{xy} - x\bar{y}}{1 - t(x + y + \bar{x} + \bar{y})}$$

Another example.

$$S_0(x, y) = \overline{x} + x\overline{y} + xy$$
$$S_1(x, y) = \overline{x} + \overline{y} + y + x\overline{y} + xy$$

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$$G = \{(x, y), (\frac{\bar{x}}{\bar{y} + y}, y), (\frac{\bar{x}}{\bar{y} + y}, \bar{y}), (x, \bar{y})\}$$

$$F_0 = 1 + tS_1F_1 - t(1+x)\overline{y}F_1(x,0) - t\overline{x}F_1(0,y)$$
$$F_1 = tS_0F_0 - x\overline{y}F_0(x,0) - t\overline{x}F_0(0,y)$$

$$\mathrm{OS}(xyF_0) = \mathrm{OS}(xy) + tS_1\mathrm{OS}(xyF_1)$$
$$\mathrm{OS}(xyF_1) = tS_0\mathrm{OS}(xyF_0)$$

$$\mathrm{OS}(xyF_0) = \frac{\mathrm{OS}(\mathrm{xy})}{1 - t^2 S_0 S_1}$$

$$xyF_0 = [x^{>}y^{>}]\frac{\mathrm{OS}(\mathrm{xy})}{1 - t^2 S_0 S_1}$$

Again another example.

$$S_0(x, y) = \overline{xy} + y$$
$$S_1(x, y) = y + x\overline{y}$$

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$$S_1(x, y) = y + x\overline{y}$$

There is no suitable group!

$$F_0 = 1 + t(x\bar{y} + y)F_1 - tx\bar{y}F_1(x, 0)$$
$$F_1 = t(y + \bar{x}\bar{y})F_0 - t\bar{x}\bar{y}(F_0(x, 0) + F_0(0, y) - F_0(0, 0))$$

$$xy(1 - t^2 S_0 S_1) F_1 = -txyS_0 - tF_0(x, 0) - tF_0(0, y) + tF_0(0, 0)$$

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$$S_0 S_1 = \bar{x} + \bar{y}^2 + y^2 + x$$

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$$S_0 S_1 = \overline{x} + \overline{y}^2 + y^2 + x$$
$$G = \{(x, y), (\bar{x}, y), (\bar{x}, \bar{y}), (x, \bar{y})\}$$

$$xyF_1 = [x^> y^>] \frac{x^2 y - \bar{x}^2 y + \bar{x}^2 \bar{y} - x^2 \bar{y}}{1 - t^2 S_0 S_1}.$$

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- There is no (suitable) finite group.
- The sections do not cancel.
- The orbit sum is zero.
- The extraction of the positive part fails.

Definition [Bostan, Bousquet-Mélou, Melczer]

Let $S(x,y) \in \mathbb{Q}(x,y)$. Define a relation $\sim$ on $\overline{\mathbb{Q}(x,y)}^2$ by

$$(u_1, u_2) \sim (v_1, v_2) \quad :\Longleftrightarrow$$
$$u_1 = v_1 \text{ or } u_2 = v_2 \quad \text{and} \quad S(u_1, u_2) = S(v_1, v_2),$$

and let $\approx$ its transitive closure. The orbit of $S(x,y)$ is the equivalence class of $(x,y) \in \overline{\mathbb{Q}(x,y)}^2$ with respect to $\approx$.

Algorithm [Bostan, Bousquet-Mélou, Melczer]

Input: A bivariate polynomial $S(x, y) \in \mathbb{Q}(x, y)$.
Output: The set of (pairs of) minimal polynomials of elements of the orbit of $S(x, y)$.

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1. Set $S = S(x, y) - S(X, Y)$, $\mathrm{done} = \{\}$, and $\mathrm{todo} = \{(-X + x, -Y + y)\}$.
2. Remove an element $\{P, Q\}$ from todo, append it to done.
3. Compute the list P_{new} of factors of $\mathrm{res}_Y(Q, S)$, not free of and not equal to Y.
4. Compute the list Q_{new} of factors of $\mathrm{res}_X(P, S)$, not free of and not equal to X.
5. Enlarge todo by the elements of $\{P\} \times Q_{new}$ and $P_{new} \times \{Q\}$ which are not in done.
6. Repeat steps 2 to 5 until todo is empty.
7. Return done.

Theorem (Primitive Element Theorem)

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Corollary

Let E/F be a separable field extension of finite degree. Then there is a polynomial $m \in F[X]$ such that

$$E \cong F[X]/\langle m \rangle.$$

Theorem (Shape Lemma)

Let K be a field of characteristic 0, $I \subseteq K[x_1, \ldots, x_n]$ a 0-dimensional radical ideal in normal x_n-position, $g_n \in K[x_n]$ the monic generator of $I \cap K[x_n]$ and d its degree. Then there are $g_1, \ldots, g_{n-1} \in K[x_n]$ such that $\{x_1 - g_1, \ldots, x_{n-1} - g_{n-1}, g_n\}$ is a reduced Gröbner basis of I with respect to lex order.

Algorithm

Input: A set $\{p_1, \ldots, p_n\}$ of irreducible and pairwise distinct polynomials over $\mathbb{Q}(x, y)$.

Output: The minimal polynomial of a generator of the splitting field of these polynomials over $\mathbb{Q}(x, y)$, and a representation of their roots as polynomials in this generator over $\mathbb{Q}(x, y)$.

1 Let $d_i = \deg p_i$, and let z, t and, for $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, d_i\}$, x_{ij} be variables.

2 Define $q_1 = 1 - t \prod_{i=1}^{n} \prod_{1 \le j_1 < j_2 \le d_i} (x_{ij_1} - x_{ij_2})$, and

3 define $q_2 = z - \sum_{i=1}^{n} \sum_{j=1}^{d_i} a_{ij} x_{ij}$ with random integer coefficients a_{ij}.

4 Compute a Gröbner basis of the ideal

$$I = \langle p_i(x_{ij}), q_1, q_2 \mid i \in \{1, \ldots, n\}, j \in \{1, \ldots, d_j\} \rangle$$

in $\mathbb{Q}(x, y)[t, \{x_{ij}\}, z]$ with respect to a lexicographic order where t is the largest and z the smallest variable.

5 Repeat steps 3 and 4 until the Gröbner basis contains a non-zero polynomial g in $\mathbb{Q}(x, y)[z]$ and polynomials of the form $x_{ij} - g_{ij}$ with g_{ij} in $\mathbb{Q}(x, y)[z]$ for each i and j.

6 Return g and the g_{ij}'s.

We can find the minimal polynomial $m \in \mathbb{Q}(x, y)[z]$ of a primitive element α of the splitting field of the orbit over $\mathbb{Q}(x, y)$ and a representation of its elements by elements of $\mathbb{Q}(x, y)[\alpha]^2$.

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Plugging the orbit elements into one of the kernel equations, forming a linear combination with undetermined coefficients, and setting coefficients of sections equal to zero, results in a linear system over $\mathbb{Q}(x, y)[z]/\langle m \rangle$. A basis of its solution space corresponds to a basis of the vector space of section-free orbit equations.

The summands

$$p_3(\alpha)F_i(p_1(\alpha), p_2(\alpha))$$

of an orbit equation are represented by polynomials

$$p_1, p_2, p_3 \in \mathbb{Q}(x, y)[\alpha]$$

and the minimal polynomials

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How can we determine the positive part

$$[x^> y^>] p_3(\alpha)F_i(p_1(\alpha), p_2(\alpha))?$$

An example.

What is the positive part of the solution F to the equation

$$(1-x)F - 1 = 0?$$

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The equation has two series solutions

$$F = \sum_{n=0}^{\infty} x^n \quad \text{and} \quad F = -\sum_{n=1}^{\infty} \bar{x}^n.$$

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The positive part depends on over which field,

$$\mathbb{Q}((x)) \quad \text{or} \quad \mathbb{Q}((\overline{x})),$$

the equation is solved.

Laurent series in several variables

Proposition

Let $C \subseteq \mathbb{R}^n$ be a strongly convex cone and let $k \in \mathbb{N}$. Then

$$\mathbb{C}_C[[x]] := \left\{ \phi = \sum_I \phi_I x^I \mid \phi_I \in \mathbb{C} \text{ and } \operatorname{supp}(\phi) \subseteq C \cap \frac{1}{k}\mathbb{Z}^n \right\}$$

is a ring with respect to addition and multiplication.

Proposition

Let $\preceq$ be an additive order on $\mathbb{Q}^n$, let $\mathcal{C}$ be the set of all cones whose smallest element with respect to it is 0 and define

$$\mathbb{C}_{\preceq}[[x]] := \bigcup_{C \in \mathcal{C}} \mathbb{C}_C[[x]] \quad \text{and} \quad \mathbb{C}_{\preceq}((x)) := \bigcup_{e \in \mathbb{Z}^n} x^e \mathbb{C}_{\preceq}[[x]].$$

Then $\mathbb{C}_{\preceq}[[x]]$ is a ring, and $\mathbb{C}_{\preceq}((x))$ is a field.

Problem

Given a simple field extension $\mathbb{C}(x, y, t)[\alpha]$ of $\mathbb{C}(x, y, t)$, find an embedding of it into a field $\mathbb{C}_{\preceq}((x, y, t))$ of Laurent series.

Algorithm [Generalized Newton-Puiseux Algorithm]

Input: A polynomial $p(x_1, \ldots, x_{n+1}) \in \mathbb{C}[x_1, \ldots, x_{n+1}]$, square-free and non-constant, an admissible edge e of its Newton polytope, an element $w \in C(e)^*$ defining a total order on $\mathbb{Q}^n$, and an integer k.

Output: A list of $|\Pi_{n+1}(e)|$ many pairs $(c_{\alpha_1} x^{\alpha_1} + \cdots + c_{\alpha_N} x^{\alpha_N}, C)$ with $c_{\alpha_1} x^{\alpha_1}, \ldots, c_{\alpha_N} x^{\alpha_N}$ the first N terms of a series solution ϕ, ordered with respect to w, a strictly convex cone C such that $\mathrm{supp}(\phi) \subseteq \{\alpha_1, \ldots, \alpha_{N-1}\} \cup \alpha_N + C$, and $N \geq k$ minimal such that the series solutions are distinguished by their first N terms.

To give meaning to

$$[x^{>}y^{>}]p_3(\alpha)F_i(p_1(\alpha), p_2(\alpha)),$$

we determine series expansions ϕ_1, ϕ_2, ϕ_3 of

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and find a description of their support using the Newton-Puiseux algorithm. A description of the support of

$$F_i(\phi_1, \phi_2)$$

is then found using...

Theorem

Let $C \subseteq \mathbb{R}^n$ be a strongly convex cone and $F \in \mathbb{C}_C[[x]]$, and let $\preceq$ be an additive order on $\mathbb{Z}^m$ and $g_1, \ldots, g_n \in \mathbb{C}_\preceq((x)) \setminus \{0\}$. Let $M \in \mathbb{Z}^{m \times n}$ be the matrix whose i-th column consists of the leading exponent $\operatorname{lexp}(g_i(x))$, and let C' be a cone that contains MC and $\operatorname{supp}(g_i/\operatorname{lt}(g_i))$ for $i = 1, \ldots, n$. If $C \cap \ker M = \{0\}$ and if C' is strongly convex, then $F(g_1, \ldots, g_n)$ is well-defined and belongs to $\mathbb{C}_{C'}[[x]]$.

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