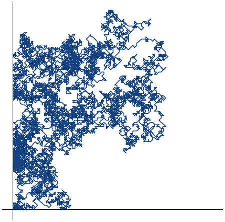
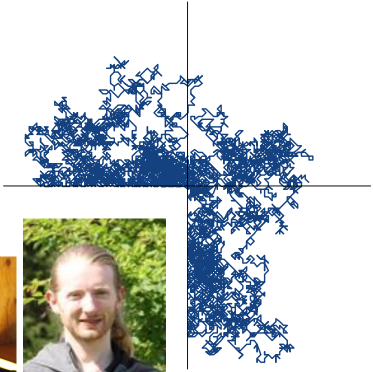
Les voeux 2022 de Bordeaux métropole

Three-quadrant walks: an introduction

Mireille Bousquet-Mélou, CNRS, Université de Bordeaux





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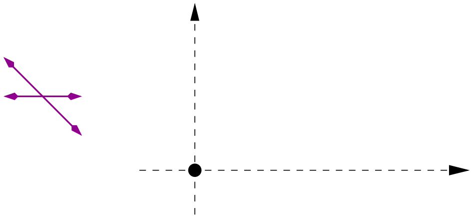
Counting walks in a cone

Let $\mathcal{S}$ be a finite subset of $\mathbb{Z}^d$ (set of steps) and $p_0 \in \mathbb{Z}^d$ (starting point).

Example. $\mathcal{S} = \{10, \bar{1}0, 1\bar{1}, \bar{1}1\}$, $p_0 = (0,0)$



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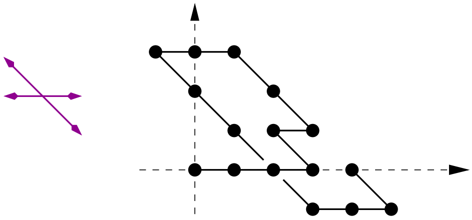

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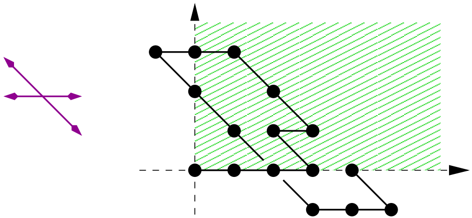
Let C be a cone of $\mathbb{R}^d$.

Example. $\mathcal{S} = \{10, \bar{1}0, 1\bar{1}, \bar{1}1\}$, $p_0 = (0,0)$ and $C = \mathbb{R}_+^2$.



A typical question

Questions

- What is the number $a(n)$ of n-step walks starting at p_0 and contained in C?
- For $\boldsymbol{i} = (i_1, \ldots, i_d) \in C$, what is the number $a(\boldsymbol{i}; n)$ of such walks that end at $\boldsymbol{i}$?

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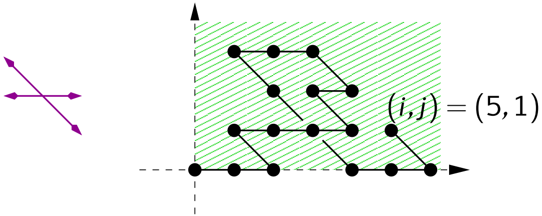

Why count walks in cones?

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- Many discrete objects can be encoded in that way:
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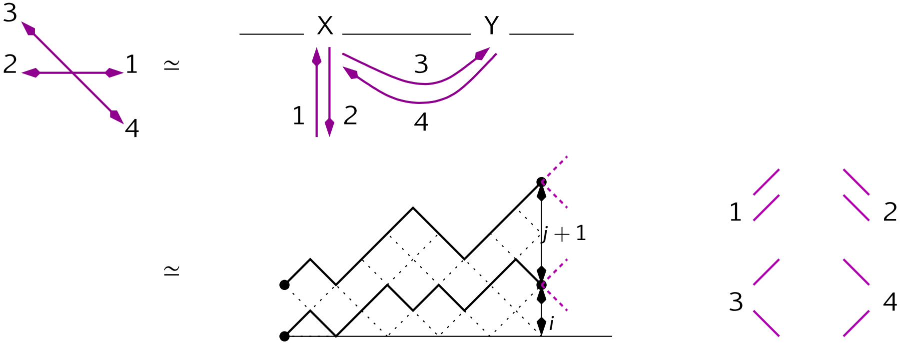

[Krattenthaler et al. 00]

$+$ Young tableaux of height 4 [Gouyou-Beauchamps 89]

Why count walks in cones?

- Many discrete objects can be encoded in that way:
  - in combinatorics, statistical physics...
  - in (discrete) probability theory: random walks, queuing theory...

- To reach a better understanding of functional equations with divided differences/discrete derivatives:

$$Q(x,y) = 1 + txyQ(x,y) + t\,\frac{Q(x,y) - Q(0,y)}{x} + t\,\frac{Q(x,y) - Q(x,0)}{y}$$

Generating functions

- Our original question:

$$a(n) = ? \qquad a(\boldsymbol{i}; n) = ?$$

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- Generating functions:

$$A(t) = \sum_{n \geq 0} a(n) t^n, \qquad A(x_1, \ldots, x_d; t) = \sum_{\boldsymbol{i}, n} a(\boldsymbol{i}; n) \boldsymbol{x}^{\boldsymbol{i}} t^n$$

$$= \sum_{w \text{ walk}} t^{|w|} \qquad\qquad = \sum_{w \text{ walk}} \boldsymbol{x}^{\boldsymbol{i}(w)} t^{|w|}$$

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Remarks

- $A(1, \ldots, 1; t) = A(t)$
- if $C \subset \mathbb{R}_+^d$, then $A(0, \ldots, 0; t)$ counts walks ending at $(0, \ldots, 0)$
- $A(0, x_2, \ldots, x_d; t)$ counts walks ending on the hyperplane $i_1 = 0$

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Can one express these series? What is their *nature*?

A hierarchy of formal power series

- Rational series

$$A(t) = \frac{1-t}{1-t-t^2}$$

- Algebraic series

$$1 - A(t) + tA(t)^2 = 0$$

- Differentially finite series (D-finite)

$$t(1-16t)A''(t) + (1-32t)A'(t) - 4A(t) = 0$$

- D-algebraic series

$$(2t + 5A(t) - 3tA'(t))A''(t) = 48t$$

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Multi-variate series: one DE per variable

I. Two dimensional cones

A (very) basic cone: the full space

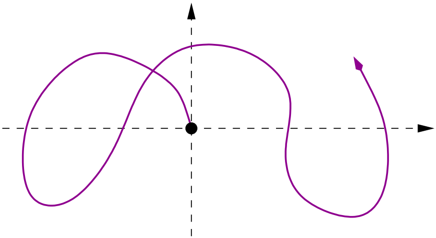
Rational series

If $\mathcal{S} \subset \mathbb{Z}^d$ is finite and $C = \mathbb{R}^d$, then $A(x;t)$ is rational:

$$a(n) = |\mathcal{S}|^n \quad \Leftrightarrow \quad A(t) = \sum_{n \geq 0} |\mathcal{S}|^n \, t^n = \frac{1}{1 - |\mathcal{S}|\, t}$$

More generally:

$$A(x;t) = \frac{1}{1 - t \sum_{s \in \mathcal{S}} x^s}.$$



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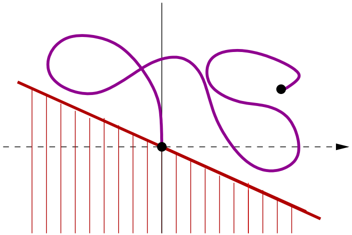
The step polynomial:

$$S(x) = \sum_{s \in \mathcal{S}} x^s$$

Also well-known: a (rational) half-space

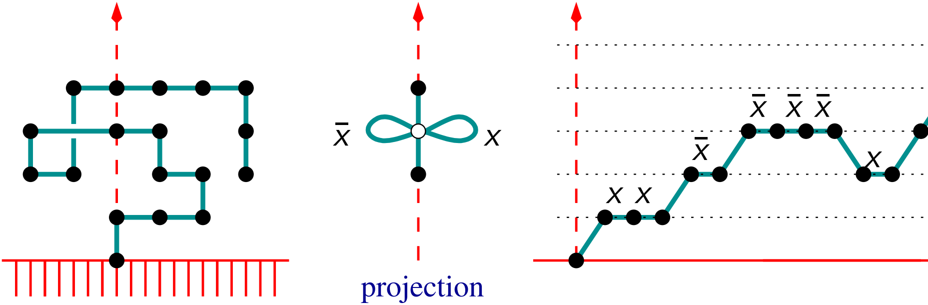
Algebraic series

If $\mathcal{S} \subset \mathbb{Z}^d$ is finite and C is a rational half-space, then $A(x;t)$ is algebraic, given by an explicit system of polynomial equations.



Also well-known: a (rational) half-space

By projection: Equivalent to weighted walks in 1D confined to a half-line



Notation: $\bar{x} := 1/x$

Walks on a half-line: main approaches

- Grammars [Labelle-Yeh 90, Merlini et al. 99, Duchon 00...]

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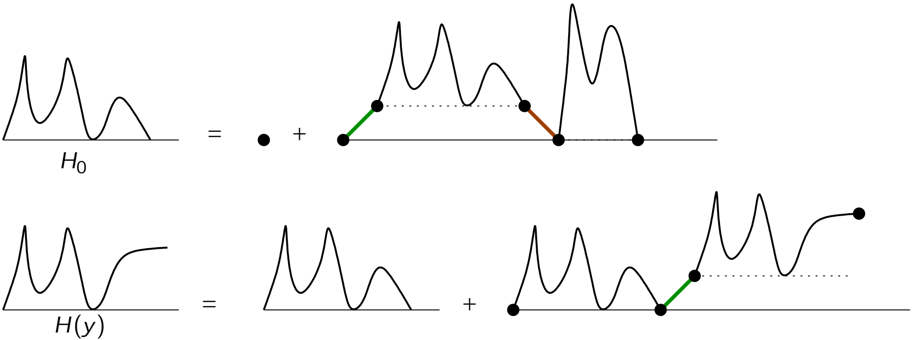
Example: $\mathcal{S} = \{-1, +1\}$

$$H_0 = 1 + t^2 H_0^2$$
$$H(y) = H_0 + tyH_0H(y)$$



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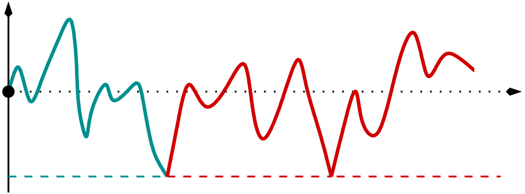
$$\frac{1}{1-tS(y)} = H_<(\bar{y})\, H(y)$$



where $H_<(\bar{y}) \in 1 + \bar{y}\mathbb{Q}[\bar{y}][[t]]$ and $H(y) \in \mathbb{Q}[y][[t]]$, with $\bar{y} = 1/y$.

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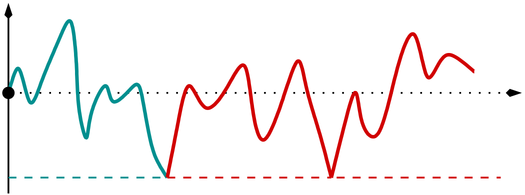

where $H_<(\bar{y}) \in 1 + \bar{y}\mathbb{Q}[\bar{y}][[t]]$ and $H(y) \in \mathbb{Q}[y][[t]]$, with $\bar{y} = 1/y$.
Factor $1 - tS(y)$ (as a polynomial in y)

$$\Rightarrow H(y) = -\frac{1}{t}\prod_{i=1}^{M} \frac{1}{y - Y_i},$$

where $M = \max \mathcal{S}$ and $Y_1, \ldots, Y_M$ are the roots of $1 - tS(y)$ that are infinite at $t = 0$.

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Example: $\mathcal{S} = \{-2, 1\}$. An equation in one catalytic variable y

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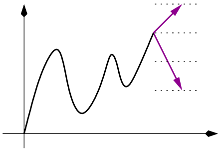
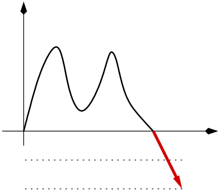
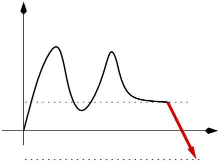
- $\oplus$  $\ominus$  $\ominus$ 

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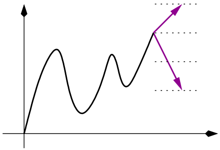
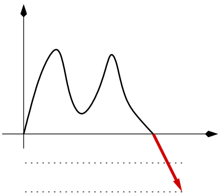
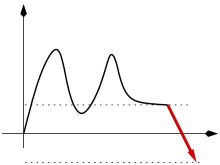
- $\oplus$  $\ominus$  $\ominus$ 

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In general,

$$(1 - tS(y))H(y) = \mathrm{Pol}(\bar{y}),$$

where Pol is a polynomial of degree $m = -\min \mathcal{S}$.

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In general,

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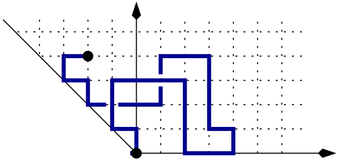
where Pol is a polynomial of degree $m = -\min \mathcal{S}$.

Factor $1 - tS(y) \Rightarrow$ Same expression as Gessel's factorization:

$$H(y) = -\frac{1}{t} \prod_{i=1}^{d} \frac{1}{y - Y_i}$$

The “next” case: two bounding hyperplanes

- Convex cone: walks in a quadrant



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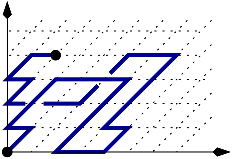
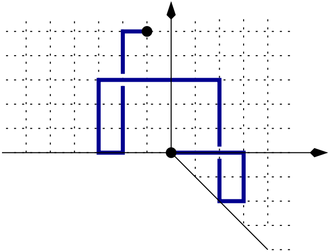


- Non-convex cone: walks avoiding a quadrant



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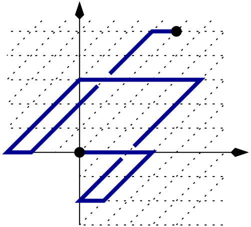

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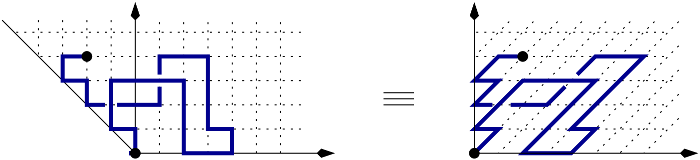

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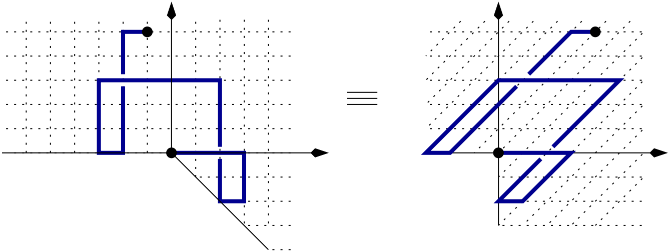
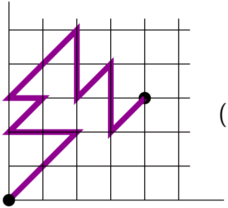


- Walks in the slit plane [mbm-Schaeffer 02, mbm 01, Rubey 04]

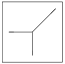
II. Quadrant walks



$(i,j) = (4,3)$

Quadrant walks: a small chronology

- Pre-2000: sporadic examples, with various motivations

Kreweras (ALG)		[Kreweras 65, Gessel 86, Niederhausen 83]

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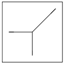
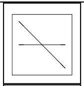
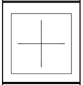
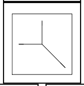
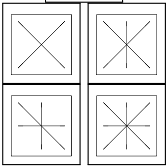
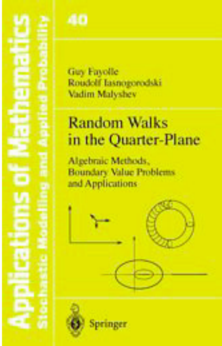
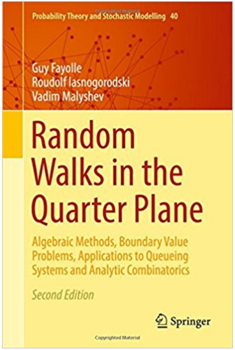
Kreweras (ALG)		[Kreweras 65, Gessel 86, Niederhausen 83]
Gouyou-Beauchamps (DF)		[Gouyou-Beauchamps 86
Simple walks (DF)		[Guy, Krattenthaler & Sagan 92]
Tandem walks (DF)		[Regev 81]
The reflection principle (DF)		[Gessel-Zeilberger 92]

Quadrant walks: a small chronology

Around 2000

- New ingredients
  - Functional equations with two "catalytic" variables
  - Functional equations in one catalytic variable have algebraic solutions [mbm-Jehanne 05]
  - Le petit livre jaune [Fayolle, Iasnogorodski & Malyshev 99], and the group of the walk





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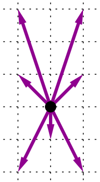
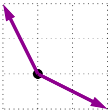
- Asking systematic questions
  - Why are some models (=Kreweras') algebraic? [mbm 02, 04]
  - Are all models D-finite? [mbm-Petkovšek 02]

DF



not DF

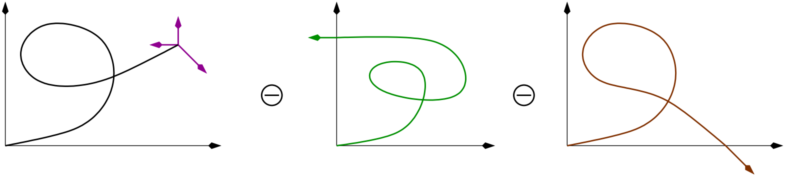


Functional equations for quadrant walks

A step-by-step construction

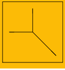
Example: $\mathcal{S} = \{01, \bar{1}0, 1\bar{1}\}$, with $\bar{x} := 1/x$ and $\bar{y} := 1/y$

$$Q(x,y) = 1 + t(y + \bar{x} + x\bar{y})Q(x,y) - t\bar{x}Q(0,y) - tx\bar{y}Q(x,0)$$



$$Q(x,y) \equiv Q(x,y;t) = \sum_{i,j,n \geq 0} q(i,j;n) x^i y^j t^n$$

Functional equations for quadrant walks



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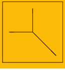
Functional equations for quadrant walks



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or

$$\boxed{\Big(1 - t(y + \bar{x} + x\bar{y})\Big)xyQ(x,y) = xy - tyQ(0,y) - tx^2Q(x,0)}$$

• The polynomial $1 - t(y + \bar{x} + x\bar{y})$ is the kernel of this equation

• The equation is linear, with two catalytic variables x and y (tautological at $x = 0$ or $y = 0$) [Zeilberger 00]

The group of the model

Example. Take $\mathcal{S} = \{\bar{1}0, 01, 1\bar{1}\}$, with step polynomial

$$S(x,y) = \frac{1}{x} + y + \frac{x}{y} = \bar{x} + y + x\bar{y}$$

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$$\Phi : (x,y) \mapsto (\,\cdot\,, y) \quad \text{and} \quad \Psi : (x,y) \mapsto (x, \,\cdot\,)$$

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They are involutions, and generate a finite dihedral group G:

$$\begin{tikzcd}
 & (\bar{x}y, y) \arrow[r, dash, "\Psi"] & (\bar{x}y, \bar{x}) \arrow[dr, dash, "\Phi"] & \\
(x,y) \arrow[ur, dash, "\Phi"] \arrow[dr, dash, "\Psi"'] & & & (\bar{y}, \bar{x}) \\
 & (x, x\bar{y}) \arrow[r, dash, "\Phi"'] & (\bar{y}, x\bar{y}) \arrow[ur, dash, "\Psi"'] &
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Remark. G can be defined for any model with small steps

The group is not always finite

- If $\mathcal{S} = \{0\bar{1}, \bar{1}\bar{1}, \bar{1}0, 11\}$, then $S(x,y) = \bar{x}(1+\bar{y}) + \bar{y} + xy$ and

$$\Phi : (x,y) \mapsto (\bar{x}\bar{y}(1+\bar{y}), y) \quad \text{and} \quad \Psi : (x,y) \mapsto (x, \bar{x}\bar{y}(1+\bar{x}))$$

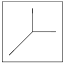
generate an infinite group:

$$\begin{tikzcd}
 & (\bar{x}\bar{y}(1+\bar{y}),y) \arrow[r, dash, "\Psi"] & \cdots \arrow[r, dash, "\Phi"] & \cdots \arrow[r, dash, "\Psi"] & \cdots \\
(x,y) \arrow[ur, dash, "\Phi"] \arrow[dr, dash, "\Psi"'] & & & & \\
 & (x,\bar{x}\bar{y}(1+\bar{x})) \arrow[r, dash, "\Phi"'] & \cdots \arrow[r, dash, "\Psi"'] & \cdots \arrow[r, dash, "\Phi"'] & \cdots
\end{tikzcd}$$

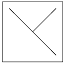
Marni à Bordeaux (2003-2004)

- [Mishna 06a, Mishna-Rechnitzer 07a]

Allowing three small steps:
three more models solved



alg



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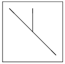

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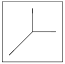

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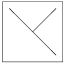

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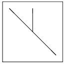
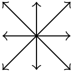


not DF

- [mbm-Mishna 08a] "Walks with small steps in the quarter plane"

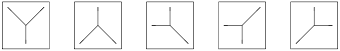
Small steps: $\mathcal{S} \subset \{-1, 0, 1\}^2 \setminus \{(0, 0)\}$. Only 2^8 models.



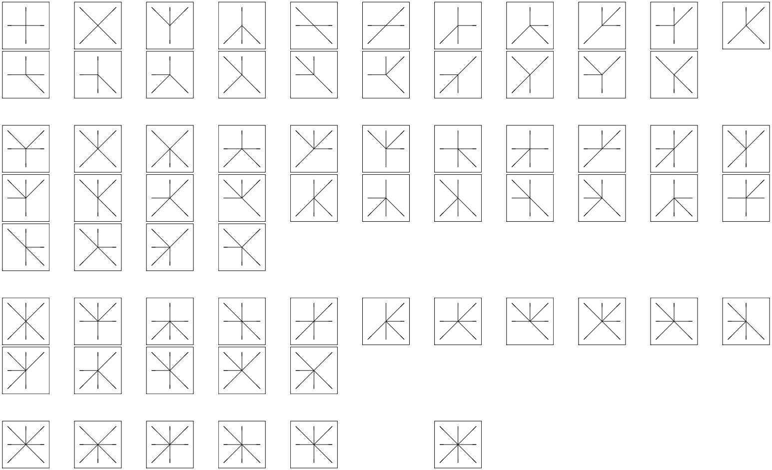
Some models are trivial, or equivalent to a half plane problem

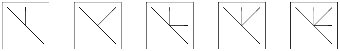
$\Rightarrow$ 79 really interesting and distinct models

Quadrant walks with small steps



Non-singular





Singular

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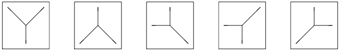

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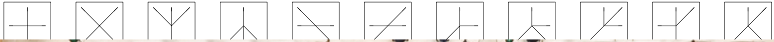
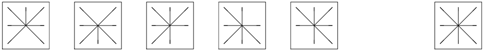
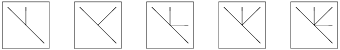






Singular

Classification of quadrant walks: the early days

(small steps)

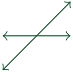
quadrant models: 79

$|G| < \infty$: 23 — **D-finite?**

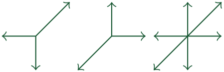
- 1 — **DF?**
- 3 — **algebraic**
- 19 — **DF**

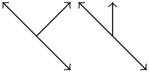
$|G| = \infty$: 56 — **Not D-finite?**

- 2 — **not DF**
- 54 — **?**



Gessel





2000-2020: more authors, techniques, and results

- quadrant models: 79
  - $|G| < \infty$: 23
    - **D-finite**
      - 4
        - **algebraic**
      - 19
        - **DF transc.**
  - $|G| = \infty$: 56
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      - 9
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Formal power series algebra

Complex analysis

Differential Galois theory

Computer algebra

D-finite series
- G-functions
- arithmetic properties
- asymptotics

Random walks in probability

2000-2020: more authors, techniques, and results

Bostan, Kauers
Koutschan, Zeilberger
Pech, Chyzak
van Hoeij

Rechnitzer
Bernardi
Melczer

Kurkova
Raschel

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Dreyfus, Hardouin
Roques, Singer

Salvy

Denisov, Wachtel

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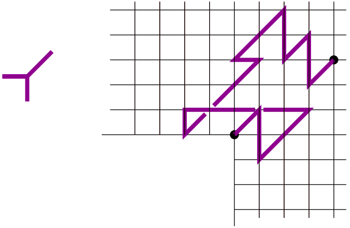
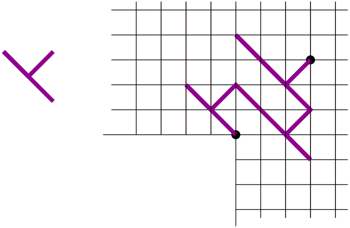
III. Three-quadrant problems with small steps

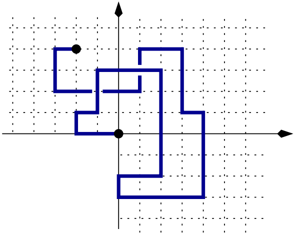


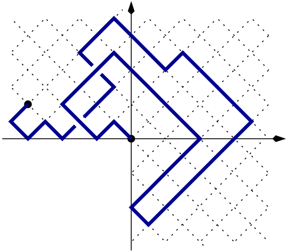


The 5 singular models become trivial (rational GF) $\Rightarrow$ 74 interesting models

A shorter history

Around 2006 (?)



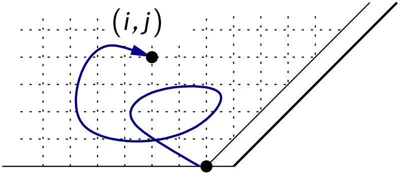


Motivation: a new approach to Gessel's walks

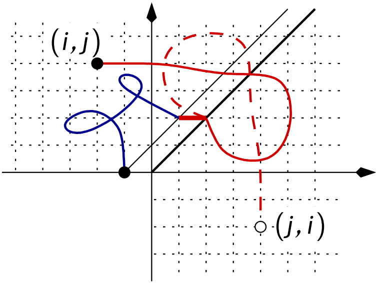
Idea: count Gessel's walks by the reflection principle

If $c_{i,j}(n)$ counts walks starting at $(-1,0)$ and avoiding the negative quadrant, then for $j \geq 0$ and $i < j$,

$$c_{i,j}(n) - c_{j,i}(n) = g_{i+1,j}(n).$$



A Gessel walk



An *a posteriori* motivation

A problem with nice numbers!

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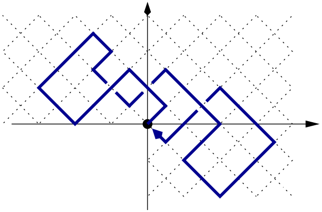
A problem with nice numbers!

Example: the number of walks of length $2n$ on the diagonal square lattice starting and ending at $(0,0)$ and avoiding the negative quadrant is

$$c(0,0;2n) = \frac{16^n}{9}\left(3\,\frac{(1/2)_n^2}{(2)_n^2} + 8\,\frac{(1/2)_n(7/6)_n}{(2)_n(4/3)_n} - 2\,\frac{(1/2)_n(5/6)_n}{(2)_n(5/3)_n}\right)$$

with $(a)_n = a(a+1)\cdots(a+n-1)$.



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Two known components: Quadrant walks (left) and Gessel walks (right). Conjectured before 2007...

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[mbm 15] "Square lattice walks avoiding a quadrant"

A shorter history... but still a few papers

[BM15] MBM. Square lattice walks avoiding a quadrant

[S16] S. Mustapha. Non-D-finite walks in a three-quadrant cone

[B17] T. Budd. Winding of simple walks on the square lattice

[RT18] K. Raschel, A. Trotignon. On walks avoiding a quadrant

[T19] A. Trotignon. Discrete harmonic functions in the three-quarter plane

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[BKT20] M. Buchacher, M. Kauers, A. Trotignon. Quadrant walks starting outside the quadrant

[EP20] A. Elvey Price. Counting lattice walks by winding angle

[BMW21] mbm, M. Wallner. Walks avoiding a quadrant and the reflection principle

[BM21] MBM. Enumeration of three-quadrant walks via invariants: some diagonally symmetric models

[EP22] A. Elvey Price. Enumeration of three-quadrant walks with small steps and walks on other M-quadrant cones

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IV. Some ideas and guiding lines

- Asymptotics

- A functional equation for the series $C(x, y; t)$ in $\mathbb{Q}[x, \bar{x}, y, \bar{y}][[t]]$
- Back to convex cones and series in $\mathbb{Q}[x, y][[t]]$

- Cancel, or not cancel? (the kernel)
- Algebraic vs. analytic approaches

Asymptotics for walks ending at (i,j) (excursions)

Fix a step set $\mathcal{S} \subset \mathbb{Z}^2$, not contained in a half-plane, with step polynomial $S(x,y)$. Then for $i,j \geq 0$,

$$q(i,j;n) \simeq \kappa \mu^n n^{-1-\pi/\theta}$$

and for $i \geq 0$ or $j \geq 0$,

$$c(i,j;n) \simeq \kappa' \mu^n n^{-1-\frac{\pi/\theta}{2\pi/\theta-1}}$$

with

$$\mu = S(x_c,y_c), \qquad \theta = \arccos\left(-\frac{S''_{1,2}(x_c,y_c)}{\sqrt{S''_{1,1}(x_c,y_c)S''_{2,2}(x_c,y_c)}}\right)$$

where $S'_1(x_c,y_c) = S'_2(x_c,y_c) = 0$, with $x_c, y_c > 0$.

[Denisov & Wachtel 15, Bostan, Raschel & Salvy 14, Mustapha 19]

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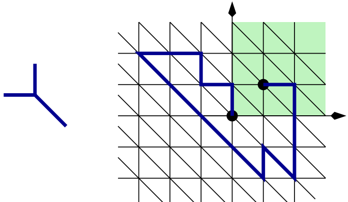
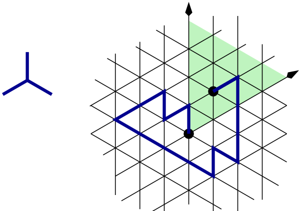
Three steps:

- change the weights of the steps so as to get a walk with no drift
- decorrelate vertical and horizontal moves to find the "true cone" where the walk leaves
- compare with Brownian trajectories





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where $S'_1(x_c,y_c) = S'_2(x_c,y_c) = 0$, with $x_c, y_c > 0$.

Proposition [Bostan, Raschel & Salvy 14]

If θ/π is not a rational number, then the series $Q_{ij} := \sum_n q(i,j;n)t^n$ and $C_{ij} := \sum_n c(i,j;n)t^n$ are not D-finite. Neither are $Q(x,y)$ and $C(x,y)$.

Application: non-D-finite three-quadrant models

three-quadrant models: 74

$|G| < \infty$: 23

$|G| < \infty$: 51

?

?

Application: non-D-finite three-quadrant models

three-quadrant models: 74

$|G| < \infty$: 23 $\quad$ $|G| < \infty$: 51

? $\quad$ **Not D-finite**

[Mustapha 19]

[Bostan, Raschel, Salvy 14]

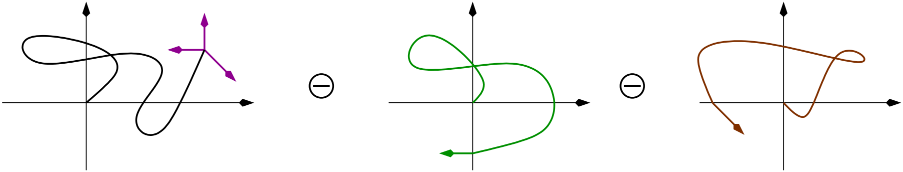
A functional equation

Step by step construction:

$$C(x,y;t) \equiv C(x,y) = 1 + t(y + \bar{x} + x\bar{y})C(x,y) - t\bar{x}C_{0,-}(\bar{y}) - tx\bar{y}C_{-,0}(\bar{x})$$

with

$$C_{0,-}(\bar{y}) = \sum_{j<0,n\geq 0} c(0,j;n)y^j t^n, \qquad C_{-,0}(\bar{x}) = \sum_{i<0,n\geq 0} c(i,0;n)x^i t^n.$$



$$C(x,y;t) = \sum_{i,j,n} c(i,j;n)x^i y^j t^n$$

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Step by step construction:

$$C(x,y;t) \equiv C(x,y) = 1 + t(y+\bar{x}+x\bar{y})C(x,y) - t\bar{x}C_{0,-}(\bar{y}) - tx\bar{y}C_{-,0}(\bar{x})$$

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or

$$\Big(1 - t(y+\bar{x}+x\bar{y})\Big)C(x,y) = 1 - t\bar{x}C_{0,-}(\bar{y}) - tx\bar{y}C_{-,0}(\bar{x}),$$

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or

$$\boxed{\Big(1 - t(y + \bar{x} + x\bar{y})\Big)xyC(x,y) = xy - tyC_{0,-}(\bar{y}) - tx^2C_{-,0}(\bar{x}).}$$

A comparison

- First quadrant:

$$\big(1 - t(y + \bar{x} + x\bar{y})\big)xyQ(x,y) = xy - tyQ(0,y) - tx^2Q(x,0)$$

- three-quadrants:

$$\big(1 - t(y + \bar{x} + x\bar{y})\big)xyC(x,y) = xy - tyC_{0,-}(\bar{y}) - tx^2C_{-,0}(\bar{x})$$

with

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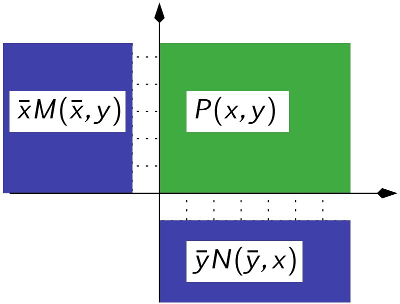
- A similar form... but $C(x,y)$ involves negative powers of x and y (Laurent polynomials)

Back to convex cones

- Split $C(x,y)$ into two or three parts:

$$C(x,y) = P(x,y) + \bar{x}M(\bar{x},y) + \bar{y}N(\bar{y},x)$$



The series $P(x,y), M(x,y)$ and $N(x,y)$ have coefficients in $\mathbb{Q}[x,y]$.

Back to convex cones

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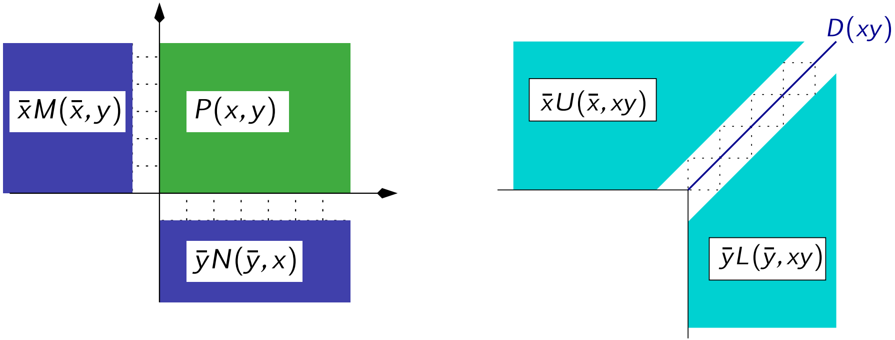
$$\begin{aligned} C(x,y) &= P(x,y) + \bar{x}M(\bar{x},y) + \bar{y}N(\bar{y},x) \\ &= \bar{x}U(\bar{x},xy) + D(xy) + \bar{y}L(\bar{y},xy) \end{aligned}$$



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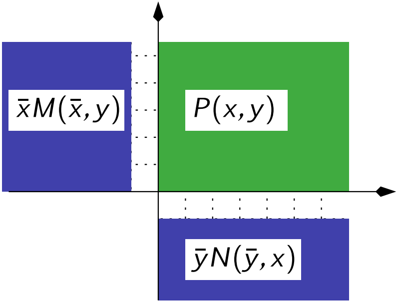
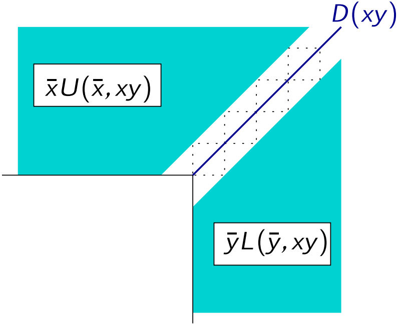
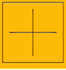




The series $P(x,y), M(x,y)$ and $N(x,y)$ have coefficients in $\mathbb{Q}[x,y]$. The same holds for $U(x,y)$ and $L(x,y)$.

- x/y-Symmetric models: $M(x,y) = N(x,y)$ and $U(x,y) = L(x,y)$.

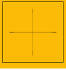
Convex cones give new types of equations 

- Quadrant

$$xyK(x,y)Q(x,y) = xy - R(x) - S(y)$$

with $K(x,y) = 1 - t(x + \bar{x} + y + \bar{y})$ and $R(x) = txQ(x,0) = S(x)$

Convex cones give new types of equations



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- Three-quadrants, split in two:

$$2xy\mathscr{K}(x,y)U(x,y) = y - R(x) - (1 - 2tx(1+y))\,S(y)$$

with $\mathscr{K}(x,y) = 1 - t(x + \bar{x} + xy + \bar{x}\bar{y})$, $R(x) = 2tU(x,0)$ and $S(y) = yD(y)$.

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- Three-quadrants, split in three:

$$yK(x,y)(2M(x,y) - M(0,y)) = \\ ty - 2tM(x,0) + t(x - \bar{x})yM(0,y) + tM(y,0)$$

Cancel (the kernel), or not cancel?

The equation $K(x,y) = 0$ has two roots, Y_0 and Y_1, with

$$Y_0 = a_0 t + \mathcal{O}(t^2), \qquad Y_1 = \frac{a_1}{t} + \mathcal{O}(1)$$

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If (...) both Y_i can be substituted for y,

$$0 = xy - R(x) - S(Y_0) = xy - R(x) - S(Y_1)$$

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- Three-quadrants, split in two:

$$2xy\mathscr{K}(x,y)U(x,y) = y - R(x) - (1 - x\mathscr{V}(y))S(y)$$

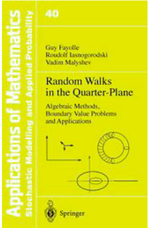
$$\boxed{\sqrt{\Delta(Y_0)}S(Y_0) - \sqrt{\Delta(Y_1)}S(Y_1) = Y_0 - Y_1}$$

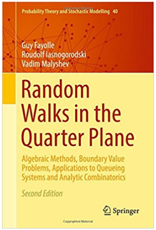
with $\Delta(y)$ rational.

V. On analytic approaches

using cancellation of the kernel

[Raschel-Trotignon 18, Dreyfus-Trotignon 20]





Analytic approaches

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• The above equations yield boundary values problems of the Riemann-Carleman type. For instance, the quadrant equation

$$S(Y_0) - S(Y_1) = x(Y_0 - Y_1)$$

implies that for y on a certain curve $\mathcal{L}$ of the complex plane,

$$\boxed{S(y) - S(\bar{y}) = X(y)(y - \bar{y})}$$

where $\bar{y}$ is now the complex conjugate of y, and $X(y) = X(\bar{y})$ is the smallest root of $K(X, y) = 0$.

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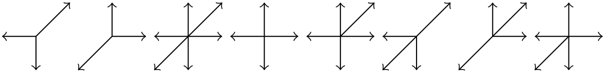
• Similarly, in the three-quadrant case, for 8 diagonally symmetric models,

$$\boxed{\sqrt{\Delta(y)}S(y) - \sqrt{\Delta(\bar{y})}S(\bar{y}) = y - \bar{y}}$$



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The equation

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$$S(y) = yD(y) = \frac{\kappa y w'(y)}{\sqrt{(w(y) - w_1)(w(y) - w_2)}} \int_{\mathcal{L}} \frac{z w'(z)}{\sqrt{w(z) - w_0}\,(w(z) - w(y))} dz$$

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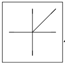
cf. in the quadrant case,

$$Q(0, y) = \mathrm{alg}(y) + \kappa \int_{\mathcal{L}} \mathrm{alg}(y) \frac{z w'(z)(w(y) - w_0)}{(w(z) - w_0)(w(z) - w(y))} dz$$

for explicit algebraic functions $\mathrm{alg}(y)$.

[Raschel-Trotignon 18, Trotignon]

Square roots in three-quadrant models

Example: for , one can also write an integral-free expression using invariants (talk 3):

$$\sqrt{\Delta(y)}\left(yD(y)+\frac{1-y}{t(1+y)}\right)=\frac{\alpha(w(y)-a)}{\sqrt{w(y)-w(0)}\,(w(y)-w(-1))},$$

with

$$\Delta(y):=(1-ty)^2-4t^2\bar{y}(1+y)^2.$$

[mbm 21]

Analytic vs. algebraic approaches

The equation

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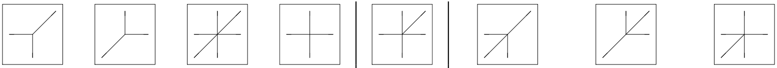
Classification of some x/y symmetric three-quadrant models



DF	DF	DF	DF	DA	not DA	not DA	not DA

[Raschel-Trotignon 18, Dreyfus-Trotignon 20]

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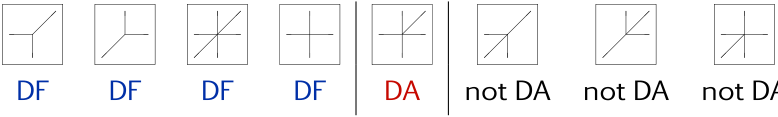

DF	DF	DF	DF	DA	not DA	not DA	not DA
alg	alg	alg					

[mbm, Raschel-Trotignon 18, Dreyfus-Trotignon 20]

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- application of Galois theory of difference equations

Two drawbacks

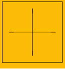
- misses simpler cases
- the algebraic/differential nature in terms of the variable t is not clear

VI. An elementary algebraic approach... for some quadrant models

An algebraic approach with no kernel cancellation, which generalizes the reflection principle.

$\Longrightarrow$ A reflection principle for three-quadrant models?

Some simple quadrant models



- The equation reads (with $K(x,y) = 1 - t(x + \bar{x} + y + \bar{y})$):

$$K(x,y)xyQ(x,y) = xy - txQ(x,0) - tyQ(0,y)$$

- The orbit of (x,y) under G is

$$(x,y) \overset{\Phi}{\longleftrightarrow} (\bar{x},y) \overset{\Psi}{\longleftrightarrow} (\bar{x},\bar{y}) \overset{\Phi}{\longleftrightarrow} (x,\bar{y}) \overset{\Psi}{\longleftrightarrow} (x,y).$$

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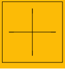

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- All transformations of G leave $K(x,y)$ invariant. Hence

$$\begin{array}{lcccccc}
K(x,y)\,xyQ(x,y) & = & xy & - & txQ(x,0) & - & tyQ(0,y) \\
K(x,y)\,\bar{x}yQ(\bar{x},y) & = & \bar{x}y & - & t\bar{x}Q(\bar{x},0) & - & tyQ(0,y) \\
K(x,y)\,\bar{x}\bar{y}Q(\bar{x},\bar{y}) & = & \bar{x}\bar{y} & - & t\bar{x}Q(\bar{x},0) & - & t\bar{y}Q(0,\bar{y}) \\
K(x,y)\,x\bar{y}Q(x,\bar{y}) & = & x\bar{y} & - & txQ(x,0) & - & t\bar{y}Q(0,\bar{y}).
\end{array}$$

Some simple quadrant models

- All transformations of G leave $K(x,y)$ invariant. Hence

$$\begin{aligned}
K(x,y)\, xyQ(x,y) &= xy - txQ(x,0) - tyQ(0,y)\\
K(x,y)\, \bar{x}yQ(\bar{x},y) &= \bar{x}y - t\bar{x}Q(\bar{x},0) - tyQ(0,y)\\
K(x,y)\, \bar{x}\bar{y}Q(\bar{x},\bar{y}) &= \bar{x}\bar{y} - t\bar{x}Q(\bar{x},0) - t\bar{y}Q(0,\bar{y})\\
K(x,y)\, x\bar{y}Q(x,\bar{y}) &= x\bar{y} - txQ(x,0) - t\bar{y}Q(0,\bar{y}).
\end{aligned}$$

$\Rightarrow$ Form the alternating sum of the equation over the orbit:

$$K(x,y)\Big(xyQ(x,y) - \bar{x}yQ(\bar{x},y) + \bar{x}\bar{y}Q(\bar{x},\bar{y}) - x\bar{y}Q(x,\bar{y})\Big) =$$

$$xy - \bar{x}y + \bar{x}\bar{y} - x\bar{y}$$

(the orbit sum).

Why is this interesting? 

$$xyQ(x,y) - \bar{x}yQ(\bar{x},y) + \bar{x}\bar{y}Q(\bar{x},\bar{y}) - x\bar{y}Q(x,\bar{y}) = \frac{xy - \bar{x}y + \bar{x}\bar{y} - x\bar{y}}{1 - t(x + \bar{x} + y + \bar{y})}$$

- Both sides are power series in t, with coefficients in $\mathbb{Q}[x,\bar{x},y,\bar{y}]$.

Why is this interesting?

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- Both sides are power series in t, with coefficients in $\mathbb{Q}[x, \bar{x}, y, \bar{y}]$.
- Extract the part with positive powers of x and y:

$$xyQ(x,y) = [x^{>0}y^{>0}]\,\frac{xy - \bar{x}y + \bar{x}\bar{y} - x\bar{y}}{1 - t(x + \bar{x} + y + \bar{y})}$$

is a D-finite series.
[Lipshitz 88]

The alternating orbit sum for finite groups

- For the 23 models with a finite group,

$$\sum_{g\in G} \mathrm{sign}(g) g(xyQ(x,y)) = \frac{1}{K(x,y)} \sum_{g\in G} \mathrm{sign}(g) g(xy) = \frac{OS}{K(x,y)},$$

where $g(A(x,y)) := A(g(x,y))$.

- The right-hand side is an explicit rational series.

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- For the 19 models where the orbit sum is non-zero,

$$xyQ(x,y) = [x^{>0}y^{>0}] \frac{OS}{K(x,y)}$$

is a D-finite series. [mbm-Mishna 10]

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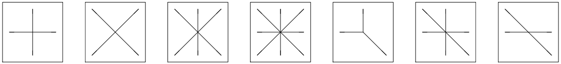
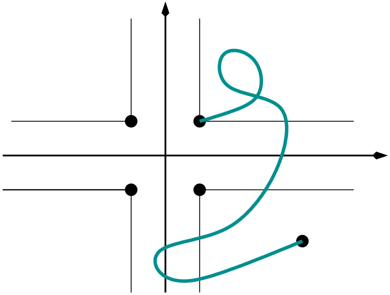
- In particular, for the 7 Weyl models, this is just the reflection principle of [Gessel & Zeilberger 92]



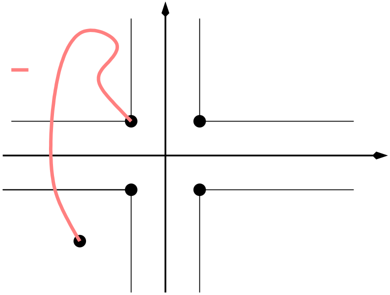
The reflection principle

$$\frac{xy}{1 - t(x + \bar{x} + y + \bar{y})}$$

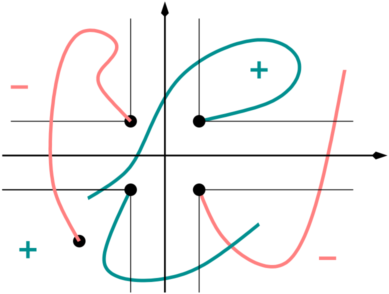


The reflection principle

$$\frac{-\bar{x}y}{1 - t(x + \bar{x} + y + \bar{y})}$$

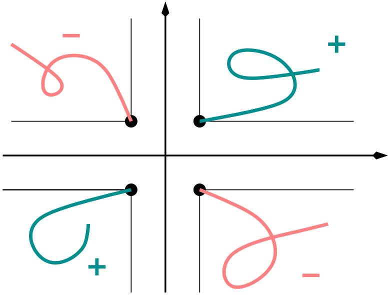


The reflection principle



$$\frac{xy - \bar{x}y + \bar{x}\bar{y} - x\bar{y}}{1 - t(x + \bar{x} + y + \bar{y})} =$$

The reflection principle



$$\frac{xy-\bar{x}y+\bar{x}\bar{y}-x\bar{y}}{1-t(x+\bar{x}+y+\bar{y})}=xyQ(x,y)-\bar{x}yQ(\bar{x},y)+\bar{x}\bar{y}Q(\bar{x},\bar{y})-x\bar{y}Q(x,\bar{y})$$

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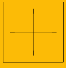
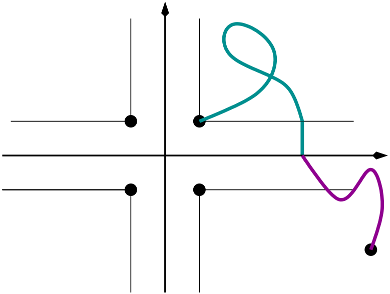
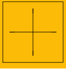


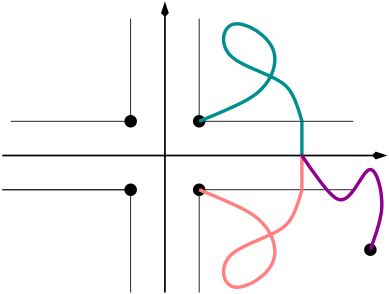


$$\frac{xy-\bar{x}y+\bar{x}\bar{y}-x\bar{y}}{1-t(x+\bar{x}+y+\bar{y})} = xyQ(x,y)-\bar{x}yQ(\bar{x},y)+\bar{x}\bar{y}Q(\bar{x},\bar{y})-x\bar{y}Q(x,\bar{y})$$

The reflection principle

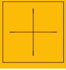


sign-reversing partial involution



$$\frac{xy - \bar{x}y + \bar{x}\bar{y} - x\bar{y}}{1 - t(x + \bar{x} + y + \bar{y})} = xyQ(x,y) - \bar{x}yQ(\bar{x},y) + \bar{x}\bar{y}Q(\bar{x},\bar{y}) - x\bar{y}Q(x,\bar{y})$$

The reflection principle



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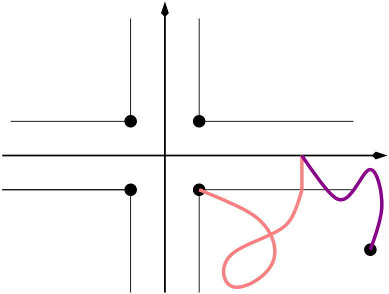
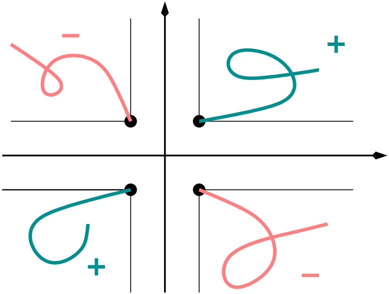


$$\frac{xy - \bar{x}y + \bar{x}\bar{y} - x\bar{y}}{1 - t(x + \bar{x} + y + \bar{y})} = xyQ(x,y) - \bar{x}yQ(\bar{x},y) + \bar{x}\bar{y}Q(\bar{x},\bar{y}) - x\bar{y}Q(x,\bar{y})$$

The reflection principle

sign-reversing partial involution



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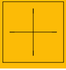

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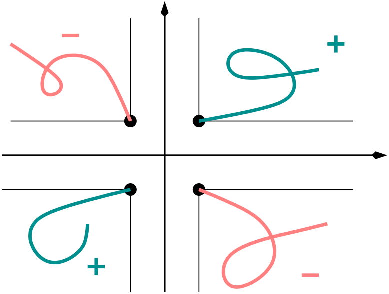
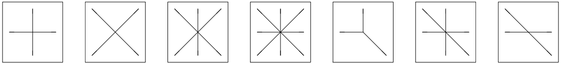


$$\frac{xy-\bar{x}y+\bar{x}\bar{y}-x\bar{y}}{1-t(x+\bar{x}+y+\bar{y})} = xyQ(x,y)-\bar{x}yQ(\bar{x},y)+\bar{x}\bar{y}Q(\bar{x},\bar{y})-x\bar{y}Q(x,\bar{y})$$

- Similar reflection argument for the 7 Weyl models.



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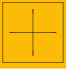

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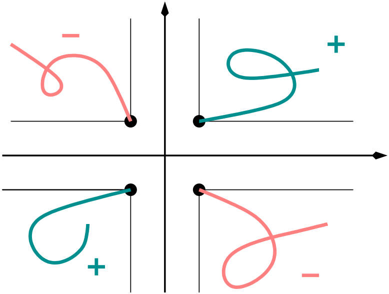
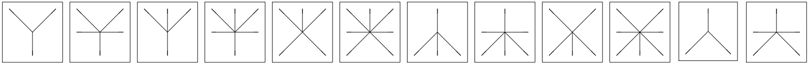


$$\frac{xy - \bar{x}y + \bar{x}\bar{y} - x\bar{y}}{1 - t(x + \bar{x} + y + \bar{y})} = xyQ(x,y) - \bar{x}yQ(\bar{x},y) + \bar{x}\bar{y}Q(\bar{x},\bar{y}) - x\bar{y}Q(x,\bar{y})$$

- The 12 non-Weyl models



The alternating orbit sum for finite groups

- For the 23 models with a finite group,

$$\sum_{g\in G} \mathrm{sign}(g) g(xyQ(x,y)) = \frac{1}{K(x,y)} \sum_{g\in G} \mathrm{sign}(g) g(xy) = \frac{OS}{K(x,y)},$$

where $g(A(x,y)) := A(g(x,y))$.

- The right-hand side is an explicit rational series.

- For the 19 models where the orbit sum is non-zero,

$$xyQ(x,y) = [x^{>0}y^{>0}] \frac{OS}{K(x,y)}$$

is a D-finite series.

- For the 4 models with vanishing orbit sum, $Q(x,y)$ is algebraic.

Classification of quadrant walks with small steps

- quadrant models: 79
  - $|G|<\infty$: 23
    - **D-finite**
      - OS=0: 4
        - **algebraic**
      - OS≠0: 19
        - **DF transc.**
  - $|G|=\infty$: 56
    - **Not D-finite**
      - decoupled: 9
        - **D-alg.**
      - not decoupled: 47
        - **not D-alg.**

A reflection principle for three-quadrant walks?

- Equations for the quadrant and for three-quadrants:

$$\big(1 - t(x + \bar{x} + y + \bar{y})\big)xyQ(x,y) = xy - tyQ(0,y) - txQ(x,0)$$

$$\big(1 - t(x + \bar{x} + y + \bar{y})\big)xyC(x,y) = xy - tyC_{0,-}(\bar{y}) - txC_{-,0}(\bar{x})$$

- The group: $(x,y),(\bar{x},y),(\bar{x},\bar{y}),(x,\bar{y})$

A reflection principle for three-quadrant walks?

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- The group: $(x,y), (\bar{x},y), (\bar{x},\bar{y}), (x,\bar{y})$
- The alternating sums of $xyQ(x,y)$ and $xyC(x,y)$ are the same:

$$\Big(1 - t(x + \bar{x} + y + \bar{y})\Big)(xyC(x,y) - \bar{x}yC(\bar{x},y) + \bar{x}\bar{y}C(\bar{x},\bar{y}) - x\bar{y}C(x,\bar{y}))$$

$$= \Big(1 - t(x + \bar{x} + y + \bar{y})\Big)(xyQ(x,y) - \bar{x}yQ(\bar{x},y) + \bar{x}\bar{y}Q(\bar{x},\bar{y}) - x\bar{y}Q(x,\bar{y}))$$

$$= xy - \bar{x}y + \bar{x}\bar{y} - x\bar{y}.$$

A reflection principle for three-quadrant walks?

- The alternating sums of $xyQ(x,y)$ and $xyC(x,y)$ are the same:

$$\begin{aligned} xyC(x,y) - \bar{x}yC(\bar{x},y) + \bar{x}\bar{y}C(\bar{x},\bar{y}) - x\bar{y}C(x,\bar{y}) \\ = xyQ(x,y) - \bar{x}yQ(\bar{x},y) + \bar{x}\bar{y}Q(\bar{x},\bar{y}) - x\bar{y}Q(x,\bar{y}) \\ = \left(xy - \bar{x}y + \bar{x}\bar{y} - x\bar{y}\right)/\left(1 - t(x + \bar{x} + y + \bar{y})\right) \end{aligned}$$

A reflection principle for three-quadrant walks?

- The alternating sums of $xyQ(x,y)$ and $xyC(x,y)$ are the same:

$$\begin{aligned}xyC(x,y)-\bar{x}yC(\bar{x},y)+\bar{x}\bar{y}C(\bar{x},\bar{y})-x\bar{y}C(x,\bar{y})\\ =xyQ(x,y)-\bar{x}yQ(\bar{x},y)+\bar{x}\bar{y}Q(\bar{x},\bar{y})-x\bar{y}Q(x,\bar{y})\\ =\Big(xy-\bar{x}y+\bar{x}\bar{y}-x\bar{y}\Big)/\Big(1-t(x+\bar{x}+y+\bar{y})\Big)\end{aligned}$$

- The alternating sums of the following series are also the same:

$$-\bar{x}yQ(\bar{x},y),\quad \bar{x}\bar{y}Q(\bar{x},\bar{y}),\quad -x\bar{y}Q(x,\bar{y}).$$

A reflection principle for three-quadrant walks?

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- Define $A(x,y)$ by

$$xyC(x,y) := xyA(x,y) + \frac{1}{3}\left(xyQ(x,y) - \bar{x}yQ(\bar{x},y) - x\bar{y}Q(x,\bar{y})\right).$$

A reflection principle for three-quadrant walks?

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$$xyC(x,y) := xyA(x,y)+\frac{1}{3}\left(xyQ(x,y)-\bar{x}yQ(\bar{x},y)-x\bar{y}Q(x,\bar{y})\right).$$

- Then $A(x,y)$ has orbit sum zero and satisfies

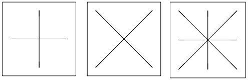
$$\boxed{(1-t(x+\bar{x}+y+\bar{y}))xyA(x,y)=\frac{2xy+\bar{x}y+x\bar{y}}{3}-tyA_-(\bar{y})-txA_-(\bar{x})}$$

Three-quadrant equations with vanishing orbit sum: algebraicity?

- The series $A(x,y)$ has orbit sum zero and satisfies

$$K(x,y)xyA(x,y) = (2xy + \bar{x}y + x\bar{y})/3 - tyA_{-}(\bar{y}) - txA_{-}(\bar{x})$$

Thm. The series $A(x,y)$ is algebraic for the three following models.



[mbm 16, mbm & Wallner 21], talk 2

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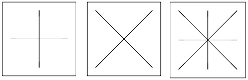

[mbm 16, mbm & Wallner 21], talk 2

Caveat: one can design slightly exotic quadrant-like equations with vanishing orbit sum and transcendental solutions [Buchacher, Kauers, Trotignon 20]

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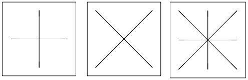


[mbm 16, mbm & Wallner 21], talk 2

Caveat: one can design slightly exotic quadrant-like equations with vanishing orbit sum and transcendental solutions [Buchacher, Kauers, Trotignon 20]

- The reflection principle also plays a role in the two papers on the winding number of lattice walks, with applications to tree-quadrant walks [Budd 17, Elvey Price 20]

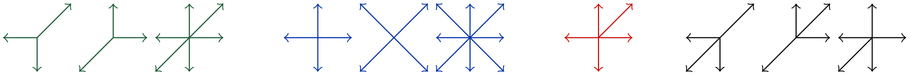
VII. The current state of affairs

Ten diagonally symmetric models

10 models

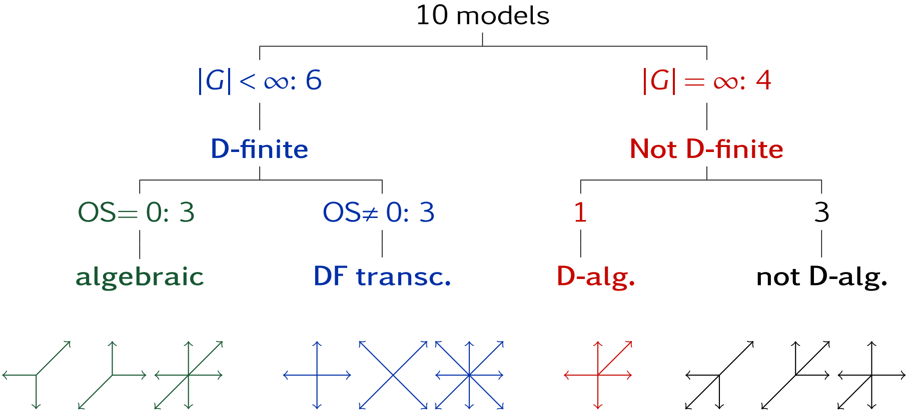
$|G| < \infty$: 6 — **D-finite**
- OS$=0$: 3 — **algebraic**
- OS$\neq 0$: 3 — **DF transc.**

$|G| = \infty$: 4 — **Not D-finite**
- 1 — **D-alg.**
- 3 — **not D-alg.**



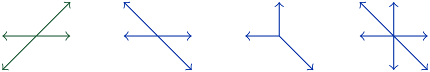
[mbm, Wallner, Raschel, Trotignon, Mustapha, Dreyfus]

Ten diagonally symmetric models



[mbm, Wallner, Raschel, Trotignon, Mustapha, Dreyfus]

$+$ D-finiteness results for some excursions [Budd 17, Elvey Price 20]



Le bouquet final: Andrew's talk (talk 4)

The series $C(x,y)$ and $Q(x,y)$ are of the same algebraic/differential nature, at least in x and y.

three-quadrant models: 74

$|G| < \infty$: 23 — $|G| = \infty$: 51

D-finite — **Not D-finite**

4 — 19 — 9 — 42

algebraic — **DF transc.** — **D-alg.** — **not D-alg.**