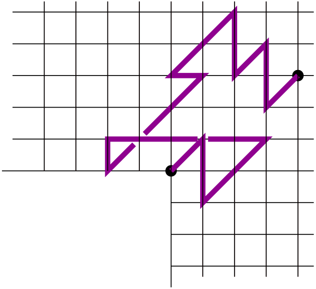
Invariants for three-quadrant walks

Mireille Bousquet-Mélou, CNRS, Université de Bordeaux



arXiv:2112.05776

Equations Equations Equations

- Quadrant walks

$$(1-t(x+\bar{x}+y+\bar{y}))xyQ(x,y)=xy-txQ(x,0)-tyQ(0,y)$$

- Three quadrants

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- Three quadrants, split in two:

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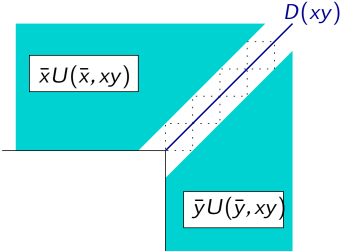
Notation: $\bar{x}:=1/x$, $\bar{y}:=1/y$.



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Notation: $\bar{x} := 1/x$, $\bar{y} := 1/y$. Rings of polynomials and series:

$A[t]$	$A(t)$	$A[[t]]$	$A((t))$
polynomials	rat. functions	formal power series	Laurent series

For instance, $C(x,y) \equiv C(x,y;t) \in \mathbb{Q}[x,\bar{x},y,\bar{y}][[t]]$.

Outline

I. Tutte's invariants

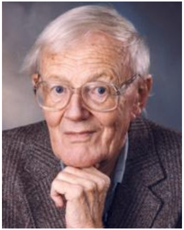
II. Invariants for small step walks

III. Invariants for quadrant walks

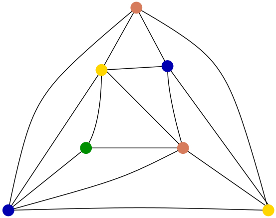
IV. Invariants for (some) three-quadrant walks

I. Tutte's invariants

Properly q-coloured planar triangulations (1973-1984)



William Tutte (1917-2002)



An equation in two catalytic variables [Tutte 1973]

For q-coloured planar triangulations, series $T(x,y;t) \equiv T(x,y) \in \mathbb{Q}[q,x,y][[t]]$:

$$T(x,y) = x(q-1) + xyt\,T(1,y)T(x,y) + xt\,\frac{T(x,y)-T(x,0)}{y} - x^2yt\,\frac{T(x,y)-T(1,y)}{x-1}.$$

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$$K(x,y)T(x,y) = \mathrm{RHS}(x,y)$$

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- The kernel $K(x,y)$ has two roots X_0 and X_1 in $\mathbb{Q}(q,y)((t))$.

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• The kernel $K(x,y)$ has two roots X_0 and X_1 in $\mathbb{Q}(q,y)((t))$.

• If (...) both X_0 and X_1 can be substituted for x in the equation $\Rightarrow$ four polynomial eqs. between X_0, X_1, $T(X_0,0)$, $T(X_1,0)$, y and $T(1,y)$:

$$K(X_0,y) = K(X_1,y) = 0, \qquad \text{RHS}(X_0,y) = \text{RHS}(X_1,y) = 0.$$

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$$K(X_0,y)=K(X_1,y)=0, \qquad \text{RHS}(X_0,y)=\text{RHS}(X_1,y)=0.$$

• Eliminate y and $T(1,y)$: two equations between X_0, X_1, $T(X_0,0)$, and $T(X_1,0)$ (with coeffs. in $\mathbb{Q}(q,t)$).

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Definition

An invariant is a series $I(x) \in \mathbb{Q}(q,x)((t))$ such that $I(X_0) = I(X_1)$.

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The invariant lemma

An invariant $I(x) = \sum_n I_n(x)t^n$ that has no pole at $x = 1$ in its coefficients $I_n(x)$ is independent of x (that is, lies in $\mathbb{Q}(q)((t))$).

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- Tutte's strategy: construct an invariant $I(x)$ with no pole at $x = 1$ (involving t, x and the series $T(x,0)$): it must be independent of x, and this gives an equation $I(x) = I(1)$ in only one catalytic variable, x.

An equation in two catalytic variables [Tutte 1973]

- From the two equations between X_0, X_1, $T(X_0,0)$, and $T(X_1,0)$, Tutte derives in fact two invariants:

$$I_0(x) = \frac{xt^2}{x-1} + 1 - \bar{x} + \bar{x}^2 + t^2 T(x,0)$$

and (when $q=3$)

$$I_1(x) = \bar{x}^6 - 2\bar{x}^4 I_0(x) + \bar{x}^2 I_0(x)^2.$$

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• He then eliminates the pole at $x = 1$ by considering the combination

$$I(x) := I_1(x) - I_0(x)^2 + 2(1+t^2) I_0(x),$$

which has no pole at $x = 1$.

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Invariants provide equations in one catalytic variable only.

($\Rightarrow$ algebraicity [mbm-Jehanne 06])

A computer algebra problem: separation of variables

- Given an ideal $\mathcal{I}$ of polynomials in X_0, X_1, T_0, T_1, with coefficients in some field $\mathbb{K}$, describe/construct some/all rational functions $R(X,T) \in \mathbb{K}(X,T)$ such that

$$R(X_0,T_0) = R(X_1,T_1) \quad \mathrm{mod}\, \mathcal{I}.$$

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- The question can be generalized to more variables $X_0, T_0, U_0, V_0 \ldots$ and $X_1, T_1, U_1, V_1 \ldots$

- For polynomials R (and, mostly, only two variables X_0 and X_1), see [Buchacher, Kauers, Pogudin 20(a)]

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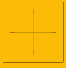

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$$2(1-t(x+\bar{x}+xy+\bar{x}\bar{y}))xyU(x,y) = y+R(x)+(a(y)x+b(y))S(y)$$

II. Invariants constructed from the kernel

The kernel is:

$$K(x,y) = 1 - tS(x,y), \qquad \text{with} \qquad S(x,y) = \sum_{(i,j)\in\mathcal{S}} x^i y^j.$$

When solved for x, it has two roots:

$$X_0 = a_0 t + \mathcal{O}(t^2), \qquad X_1 = \frac{a_1}{t} + \mathcal{O}(1)$$

Can we derive from

$$K(X_0, y) = K(X_1, y) = 0$$

an equation of the form

$$\boxed{I(X_0) = I(X_1),}$$

where $I(x) \in \mathbb{Q}(x)((t))$?

Invariants from the kernel

The kernel $K(\cdot, y)$ has two roots:

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Tutte would call invariant any series $I(x) \in \mathbb{Q}(x)((t))$ such that $I(X_0) = I(X_1)$. Define the series $J(y) \in \mathbb{Q}(y)((t))$ by

$$J(y) := I(X_0) = I(X_1) = \frac{1}{2}\left(I(X_0) + I(X_1)\right).$$

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Roughly speaking: $I(x) - J(y)$ is a "multiple" of $K(x, y)$. We also expect that

$$J(Y_0) = J(Y_1) = I(x)$$

for Y_0 and Y_1 the two roots (in y) of $K(x, \cdot)$.

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Invariants go by pairs $(I(x), J(y))$.

Divisibility by $K(x,y)$

The series $1/K(x,y)$ is well-defined in $\mathbb{Q}[x,\bar{x},y,\bar{y}][[t]]$:

$$\frac{1}{K(x,y)} = \frac{1}{1-tS(x,y)} = \sum_{n\geq 0} t^n S(x,y)^n.$$

A series of $\mathbb{Q}(x,y)((t))$ of the form

$$A(x,y) := \sum_n \frac{p_n(x,y)}{d_n(x)d_n'(y)} t^n$$

is divisible by $K(x,y)$ if the coefficients (of t^n, $n \in \mathbb{Z}$) in the series $A(x,y)/K(x,y)$ have poles of bounded order at $x=0$ and $y=0$. That is, there exists i,j such that the coefficients of $x^i y^j A(x,y)/K(x,y)$ have no pole at $x=0$ nor $y=0$.

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Equivalently, $A(x,y)$ has poles of bounded order at 0,

$$A(X_0,y) = 0, \quad \text{and} \quad A(x,Y_0) = 0,$$

where X_0 is the root of $K(\cdot,y)$ that is $\mathcal{O}(t)$, and analogously for Y_0.

Today's notion of invariants

A congruence

$A(x,y) \equiv B(x,y) \mod K(x,y)$ if $A(x,y) - B(x,y)$ is divisible by $K(x,y)$.

Definition

A pair of series $(I(x), J(y))$ in t with coefficients in $\mathbb{Q}(x)$ and $\mathbb{Q}(y)$ (respectively) is a pair of invariants if $I(x) \equiv J(y) \mod K(x,y)$.

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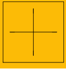
That is to say: the coefficients (of t^n, $n \geq 0$) in the ratio

$$H(x,y) = \frac{I(x) - J(y)}{K(x,y)},$$

which are rational functions of the form $p(x,y)/(d(x)d'(y))$, have poles of bounded order at $x = 0$ and $y = 0$.

Example 

- Simple walk

$$K(x,y) = 1 - t(x + \bar{x} + y + \bar{y}) = \left(\frac{1}{2} - t(x + \bar{x})\right) - \left(-\frac{1}{2} + t(y + \bar{y})\right)$$

Hence

$$I(x) := \frac{1}{2} - t(x + \bar{x}) \quad \text{and} \quad J(y) := -I(y) = -\frac{1}{2} + t(y + \bar{y})$$

form a pair of invariants, since

$$\frac{I(x) - J(y)}{K(x,y)} = 1.$$

An invariant lemma

Lemma [mbm 21(a)]

Let $(I(x), J(y))$ be a pair of invariants, and let

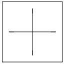
$$H(x,y) = \frac{I(x) - J(y)}{K(x,y)}.$$

If the coefficients of $H(x,y)$ (in t) vanish at $x = 0$ and at $y = 0$, then $I(x)$ and $J(y)$ are trivial:

$$I(x) = J(y) \in \mathbb{Q}((t)) \qquad \text{and} \qquad H(x,y) = 0.$$

Proof: expansion of all coefficients as series in x and y, plus ordering of the monomials.

Rational invariants

• Simple walk 

$$K(x,y) = 1 - t(x+\bar{x}+y+\bar{y}) = \left(\frac{1}{2} - t(x+\bar{x})\right) - \left(-\frac{1}{2} + t(y+\bar{y})\right)$$

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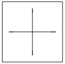
Existence of rational invariants [Bernardi, mbm, Raschel 17(a)]

The small step models that admit rational invariants are exactly those with a finite group (23 models).

• Simple walk 

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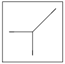
- Kreweras walk 

$$K(x,y) = 1 - t(xy + \bar{x} + \bar{y})$$

Then

$$I_0(x) := \bar{x}^2 - \bar{x}/t - x \quad \text{and} \quad J_0(y) = I_0(y)$$

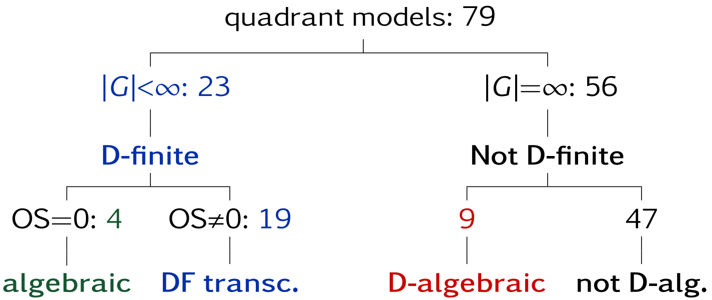
form a pair of invariants, since

$$\frac{I_0(x) - J_0(y)}{K(x,y)} = \frac{x-y}{x^2y^2} \cdot \frac{1}{t}$$

Rational invariants

Existence of rational invariants [Bernardi, mbm, Raschel 17(a)]

The small step models that admit rational invariants are exactly those with a finite group (23 models).



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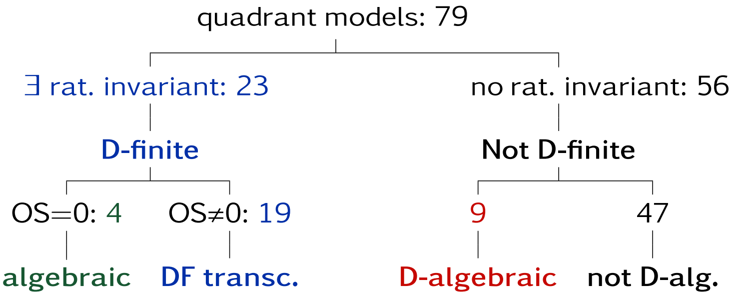

Weak invariants: an analytic notion

Let X_0 and X_1 be the roots of $K(\cdot, y)$ (for t a small real), of the form

$$X_{0,1}(y) = \frac{-b(y) \pm \sqrt{\Delta(y)}}{2a(y)}.$$

The discriminant $\Delta(y)$ is negative on two real intervals (y_1, y_2) and (y_3, y_4), with $|y_{1,2}| < 1$ and $|y_{3,4}| > 1$.

Weak invariants

The function $I(x)$ is a weak invariant if for $y \in [y_1, y_2]$,

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Existence of a weak invariant [Raschel 12, Bernardi, MBM & Raschel 17(a)]

For the 74 non-singular models with small steps, there exists an explicit weak invariant, which is D-algebraic in t, x and y.

III. Invariants for quadrant walks

Denote $\overline{K}(x,y) := xyK(x,y)$ (a polynomial).
The quadrant functional equation reads

$$\begin{aligned}\overline{K}(x,y)Q(x,y) &= xy + \overline{K}(x,0)Q(x,0) + \overline{K}(0,y)Q(0,y) - \overline{K}(0,0)Q(0,0)\\ &= xy + R(x) + S(y).\end{aligned}$$

In particular,

$$\boxed{xy + R(x) + S(y) \equiv 0 \mod K(x,y).}$$

Invariants from quadrant equations

Generic form of a quadrant equation:

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$$xy \equiv f(x) + g(y) \mod K(x,y),$$

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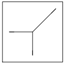
$$f(x) + R(x) + g(y) + S(y) \equiv 0 \mod K(x,y)$$

so that

$$I_1(x) = f(x) + R(x) \qquad \text{and} \qquad J_1(y) = -g(y) - S(y)$$

form a pair of invariants – involving $Q(x,0)$ and $Q(0,y)$.

Decoupling of xy modulo $K(x,y)$

- Kreweras walk 

$$K(x,y) = 1 - t(xy + \bar{x} + \bar{y})$$

Then

$$xy = \frac{1}{t} - \bar{x} - \bar{y} - \frac{K(x,y)}{t} \equiv f(x) + g(y) \mod K(x,y),$$

with

$$f(x) = \frac{1}{2t} - \bar{x}, \qquad g(y) = f(y) = \frac{1}{2t} - \bar{y}.$$

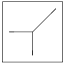
This gives a new pair of invariants:

$$I_1(x) = \frac{1}{2t} - \bar{x} - txQ(x,0), \qquad J_1(y) = -I_1(y).$$

Decoupling of xy modulo $K(x,y)$

Lemma [Bernardi, mbm & Raschel 17(a)]

The monomial xy decouples as $f(x)+g(y)$ modulo $K(x,y)$ for exactly $13=4+9$ of the 79 interesting quadrant models.

- Kreweras walk 

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- $\exists$ rat. invariant: 23
  - **D-finite**
    - $\exists$ Q-inv. 4
      - **algebraic**
    - no Q-inv.: 19
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Combine the rational invariant and the Q-invariant to form trivial invariants: uniform proofs of algebraicity.

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IV.1. Invariants for three-quadrant walks: first attempt

Denote $\overline{K}(x,y) := xyK(x,y)$ (a polynomial).

The three-quadrant functional equation reads

$$\begin{aligned}\overline{K}(x,y)C(x,y) &= xy + \overline{K}(x,0)C_{-,0}(\bar{x}) + \overline{K}(0,y)C_{0,-}(\bar{y}) + \overline{K}(0,0)C_{0,0} \\ &= xy + R(\bar{x}) + S(\bar{y}).\end{aligned}$$

But $C(x,y) \in \mathbb{Q}[x,\bar{x},y,\bar{y}][[t]]$ has poles of *unbounded* order at 0, and we *cannot* say that

$$xy + R(\bar{x}) + S(\bar{y}) \equiv 0 \mod K(x,y).$$

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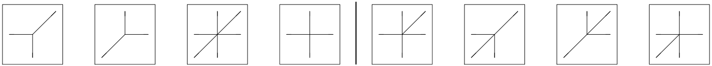
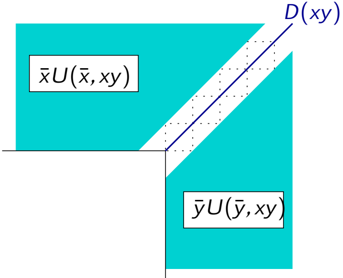


Let $\mathcal{S}$ be a small step model that is x/y-symmetric and does not contain ↖ (nor ↘), and write

$$C(x,y) = \bar{x}U(\bar{x},xy) + D(xy) + \bar{y}U(\bar{y},xy)$$

where $U(x,y) \in \mathbb{Q}[x,y][[t]]$, $D(y) \in \mathbb{Q}[y][[t]]$.



The “split in two parts” equation

- Define the companion model of $\mathcal{S}$:

$$\mathscr{S} := \{(j-i, j) : (i,j) \in \mathcal{S}\},$$

with associated kernel $\mathscr{K}(x,y) = 1 - t\mathscr{S}(x,y) = 1 - tS(\bar{x}, xy)$. Write

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- Then

$$2\mathscr{K}(x,y)xyU(x,y) = y + 2\overline{\mathscr{K}}(x,0)U(x,0) + \overline{K}(0,0)D(0) + (t\mathscr{V}_0(y) + 2tx\mathscr{V}_+(y) - 1)\, yD(y)$$

so that

$$\boxed{y + R(x) + \mathscr{N}(x,y)S(y) \equiv 0 \mod \mathscr{K}(x,y)}$$

where $\mathscr{N}(x,y) = t\mathscr{V}_0(y) + 2tx\mathscr{V}_+(y) - 1$, with $R(x) \sim U(x,0)$ and $S(y) \sim D(y)$.

A new type of decoupling

$$\boxed{y + R(x) + \mathcal{N}(x,y)S(y) \equiv 0 \mod \mathcal{K}(x,y)}$$

Good news: the square of the "nasty" factor $\mathcal{N}(x,y)$ is "nice":

$$\mathcal{N}(x,y)^2 \equiv \Delta(y) \mod \mathcal{K}(x,y)$$

where $\Delta(y)$ is the discriminant (in x) of $x\mathcal{K}(x,y)$.

$$\Delta(y) = (1 - t\mathcal{V}_0(y))^2 - 4t^2\mathcal{V}_-(y)\mathcal{V}_+(y).$$

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We have a new pair of $\mathcal{S}$-invariants – involving $U(x,0)$ and $D(y)$:

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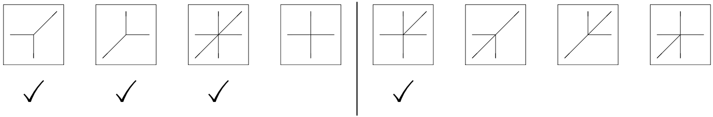
Decoupling of y modulo $\mathscr{K}(x,y)$, new style

Lemma [mbm 21 (a)]

The monomial y decouples as $F(x) + \mathscr{N}(x,y)G(y)$ modulo $\mathscr{K}(x,y)$ for exactly 4 of the 8 symmetric models $\mathcal{S}$ under consideration.



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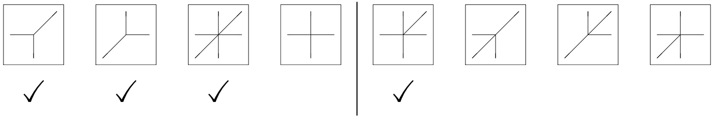

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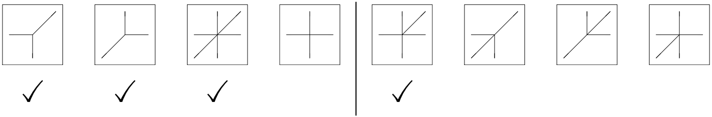

By combining the new $\mathscr{S}$-invariants $(\mathscr{I}_2(x), \mathscr{J}_2(y))$ with known $\mathscr{S}$-invariants (rational, or weak, or involving the $\mathscr{S}$-quadrant series $\mathscr{Q}(x,0)$ and $\mathscr{Q}(0,y)$), one can prove (D-)algebraicity of $U(x,0)$, $D(y)$ and $C(x,y)$.

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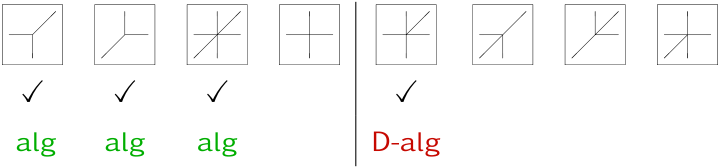

By combining the new $\mathscr{S}$-invariants $(\mathscr{J}_2(x), \mathscr{J}_2(y))$ with known $\mathscr{S}$-invariants (rational, or weak, or involving the $\mathscr{S}$-quadrant series $\mathscr{Q}(x,0)$ and $\mathscr{Q}(0,y)$), one can prove (D-)algebraicity of $U(x,0)$, $D(y)$ and $C(x,y)$.

Example: Kreweras' walks

Take $\mathcal{S} = \{\nearrow, \leftarrow, \downarrow\}$, so that $\mathscr{S} = \{\uparrow, \rightarrow, \searrow\}$. Start from

$$y + R(x) + \mathscr{N}(x,y)S(y) \equiv 0$$

with $\mathscr{N}(x,y) = ty + 2tx - 1$, $R(x) = -2tU(x,0)$, $S(y) = yD(y)$.

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- Decoupling, new style: since

$$y = -2x + 1/t + \mathcal{N}(x,y)/t = F(x) + \mathcal{N}(x,y)G(y),$$

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$$\boxed{\mathscr{I}_2(x) = \left(2tU(x,0) + 2x - \frac{1}{t}\right)^2, \qquad \mathscr{J}_2(y) = \Delta(y)\left(yD(y) + \frac{1}{t}\right)^2}$$

Construction of trivial invariants

- Two known pairs of $\mathscr{S}$-invariants:

$$\mathscr{I}_0(x) = \bar{x} + x/t - x^2, \qquad \mathscr{J}_0(x) = \mathscr{I}_0(y),$$

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- Define

$$\mathscr{I}(x) = \mathscr{I}_2(x) - 4\mathscr{I}_1(x), \qquad \mathscr{J}(y) = \mathscr{J}_2(y) - 4\mathscr{J}_1(y).$$

Then $(\mathscr{I}(x), \mathscr{J}(y))$ is a pair of $\mathscr{S}$-invariants with no pole at 0.

Construction of trivial invariants

- Two known pairs of $\mathscr{S}$-invariants:

$$\mathscr{I}_0(x) = \bar{x} + x/t - x^2, \qquad \mathscr{J}_0(x) = \mathscr{J}_0(y),$$

$$\mathscr{I}_1(x) = t\mathscr{Q}(x,0) - x/t + x^2, \qquad \mathscr{J}_1(y) = -\bar{y} - t\mathscr{Q}(0,y) + t\mathscr{Q}(0,0).$$

- We have just found another pair:

$$\mathscr{I}_2(x) = \left(2tU(x,0) + 2x - \frac{1}{t}\right)^2 = \mathcal{O}(x^0),$$

$$\mathscr{J}_2(y) = \Delta(y)\left(yD(y) + \frac{1}{t}\right)^2 = -4\bar{y} + \mathcal{O}(y^0).$$

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$$\mathscr{K}(x,y)\mathscr{H}(x,y) = \mathscr{I}(x) - \mathscr{J}(y)$$

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$$\mathscr{K}(x,y)\mathscr{H}(x,y) = \mathscr{I}(x) - \mathscr{J}(y)$$

where $\mathscr{H}(x,y)$ vanishes at $x=0$ and $y=0 \Rightarrow \mathscr{I}_2(x)$ and $\mathscr{J}_2(y)$ are trivial.

Trivial invariants for reverse Kreweras' steps

Conclusion:

$$\mathscr{I}_2(x) = \left(2tU(x,0) + 2x - \frac{1}{t}\right)^2 = 4\left(t\mathscr{Q}(x,0) - x/t + x^2\right) + cst,$$

$$\mathscr{J}_2(y) = \Delta(y)\left(yD(y) + \frac{1}{t}\right)^2 = 4\left(-\bar{y} - t\mathscr{Q}(0,y) + t\mathscr{Q}(0,0)\right) + cst,$$

with $\Delta(y) = (1-ty)^2 - 4t\bar{y}$.

The constant can be determined in terms of $\mathscr{Q}$ by specializing y to the unique root of $\Delta(y)$ that is a power series in t.

But $\mathscr{Q}(x,0)$ and $\mathscr{Q}(0,y)$ are well known, and algebraic...

The GF of Kreweras walks in three quadrants [mbm 21(a)]

- Walks ending on the negative x-axis: series $U(x,0)$, with

$$\frac{1}{2}\left(2tU(x,0)+2x-\frac{1}{t}\right)^2=\frac{(1-Z^3)^{3/2}}{Z^2}+(1-xZ)^2\left(\frac{1}{Z^2}-\frac{1}{x}\right)$$
$$+\left(\bar{x}+Z-\frac{2x}{Z}\right)\sqrt{1-Z\frac{4+Z^3}{4}x+\frac{Z^2}{4}x^2}.$$

- Walks ending on the diagonal: series $D(x)$, with

$$\frac{\Delta(x)}{2}\left(xD(x)+\frac{1}{t}\right)^2=\frac{(1-Z^3)^{3/2}}{Z^2}+(1-xZ)^2\left(\frac{1}{Z^2}-\frac{1}{x}\right)$$
$$-\left(\bar{x}+Z-\frac{2x}{Z}\right)\sqrt{1-Z\frac{4+Z^3}{4}x+\frac{Z^2}{4}x^2}.$$

where $\Delta(x)=(1-tx)^2-4t\bar{x}$ and $Z=t(2+Z^3)$.

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- All walks in three quadrants:

$$xy(1-t(\bar{x}+\bar{y}+xy))C(x,y)=xy-tU(\bar{x},0)-tU(\bar{y},0).$$

(Algebraicity of excursions proved by [Elvey Price, FPSAC 20])

Asymptotics and harmonic function

[mbm 21(a)]

- Number of n-step walks ending at (i,j) in the three quadrant plane:

$$c_{i,j}(n) \sim -\frac{H_{i,j}}{\Gamma(-3/4)} 3^n n^{-7/4} \qquad (\text{for } n+i+j \equiv 0 \mod 3)$$

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- The function H is $\mathcal{S}$-harmonic, that is,

$$H_{i,j} = \frac{1}{3}\left(H_{i-1,j-1} + H_{i+1,j} + H_{i,j+1}\right),$$

where by convention $H_{i,j} = 0$ if $(i,j) \notin \mathcal{C}$. By symmetry, $H_{i,j} = H_{j,i}$.

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$$\mathcal{H}(x,y) := \sum_{j \geq 0, i \leq j} H_{i,j} x^{j-i} y^j,$$

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[Trotignon 19(a)]

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From counting series to discrete harmonic functions

To prove:

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Let

$$D(y) = \sum_{i \geq 0} c_{i,i}(n) y^i t^n = \sum_{i \geq 0} D_i(t) y^i,$$

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A singularity analysis around $t = 1/3$ (i.e. $Z = 1$) of $D(y)$ (performed with care), gives the result.

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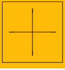
The simple walk 

Recall from Michael's lecture: it is a good idea to consider $A(x,y)$ given by

$$xyA(x,y) := xyC(x,y) - \frac{1}{3}\left(xyQ(x,y) - \bar{x}yQ(\bar{x},y) - x\bar{y}Q(x,\bar{y})\right),$$

which satisfies

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Split $A(x,y)$ in two parts, etc. The equation

$$\boxed{y + R(x) + \mathscr{N}(x,y)S(y) \equiv 0 \mod \mathscr{K}(x,y)}$$

becomes

$$\boxed{2y(1+x^2)/3 + R(x) + \mathscr{N}(x,y)S(y) \equiv 0 \mod \mathscr{K}(x,y)}$$

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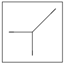
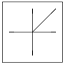
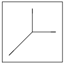
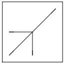
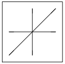
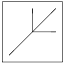
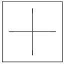
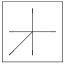
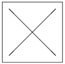
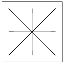
and the initial term $2y(1+x^2)/3$ decouples (new style).
$\Rightarrow$ Algebraicity of $A(x,y)$

Ten solved symmetric models

$\mathcal{S}$	Inv.	Refl.	Analysis	$\mathcal{S}$	Inv.	Refl.	Analysis + Galois
	alg				D-alg		
	alg						
	alg						
							
							
							

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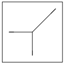
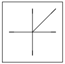
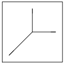
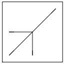
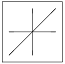
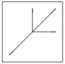
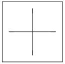
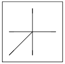
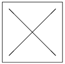
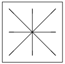
	alg				D-alg		
	alg						
	alg						
	DF						
							
							

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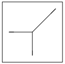
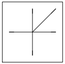
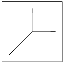
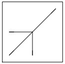
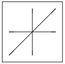
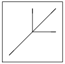
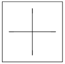
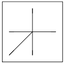
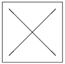
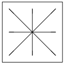
	alg				D-alg		
	alg						
	alg						
	DF						
	DF						
							

What the invariant approach gives: general picture

- Explicit rational expression of

$$\mathscr{I}_2(x) = (f(\bar{x}) + R(x))^2,$$

where

$$R(\bar{x}) = \overline{K}(x,0)C_-(\bar{x}) + \frac{1}{2}\overline{K}(0,0)C_{0,0}$$

in terms of the quadrant generating function $\mathscr{Q}(x,0)$ for the companion model $\mathscr{S}$. Same for $D(y)$ and $\mathscr{Q}(0,y)$.

What the invariant approach gives: general picture

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where

$$R(\bar{x}) = \overline{K}(x,0)C_-(\bar{x}) + \frac{1}{2}\overline{K}(0,0)C_{0,0}$$

in terms of the quadrant generating function $\mathscr{Q}(x,0)$ for the companion model $\mathscr{S}$. Same for $D(y)$ and $\mathscr{Q}(0,y)$.

$$\mathscr{I}_2(x) = \left(2t\bar{x}C_-(x) + 2x - \frac{1}{t}\right)^2 = 4\left(t\mathscr{Q}(x,0) - x/t + x^2\right) + cst,$$

$$\mathscr{J}_2(y) = \Delta(y)\left(yD(y) + \frac{1}{t}\right)^2 = 4\left(-\bar{y} - t\mathscr{Q}(0,y) + t\mathscr{Q}(0,0)\right) + cst,$$

What the invariant approach gives: general picture

- Explicit rational expression of

$$\mathscr{I}_2(x) = (f(\bar{x}) + R(x))^2,$$

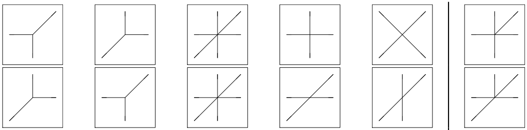
where

$$R(\bar{x}) = \overline{K}(x,0)C_{-}(\bar{x}) + \frac{1}{2}\overline{K}(0,0)C_{0,0}$$

in terms of the quadrant generating function $\mathscr{Q}(x,0)$ for the companion model $\mathscr{S}$. Same for $D(y)$ and $\mathscr{Q}(0,y)$.

$\mathcal{S}$

$\mathscr{S}$



deg. $\mathscr{Q}(x,0)$	6	6	8	24	24	∞
deg. $C_{-}(x)$ or $A_{-}(x)$	24	24	64	24	24	∞

What the invariant approach gives: general picture

- Explicit rational expression of

$$\mathscr{I}_2(x) = (f(\bar{x}) + R(x))^2,$$

where

$$R(\bar{x}) = \overline{K}(x,0)C_-(\bar{x}) + \frac{1}{2}\overline{K}(0,0)C_{0,0}$$

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- Harmonic functions: explicit and algebraic in the 5 DF cases; in the DA case, a conjectured relation between the three-quadrant $\mathcal{S}$-harmonic function and the quadrant $\mathscr{S}$-harmonic function.

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$$\mathscr{I}_2(x) = (f(\bar{x}) + R(x))^2,$$

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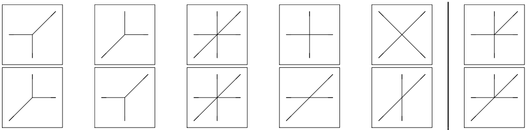
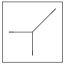
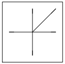
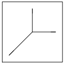
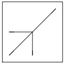
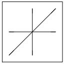
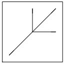
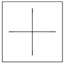
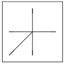
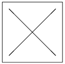
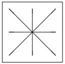
$\mathcal{S}$

$\mathscr{S}$

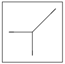
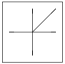
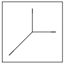
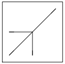
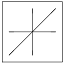
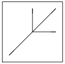
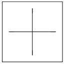
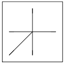
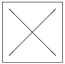
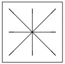


deg. $\mathscr{H}(x,0)$	2	2	2	3	3	
deg. $\mathcal{H}_-(x)$	4	4	4	3	3	

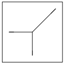
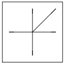
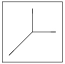
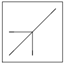
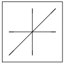
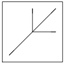
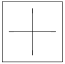
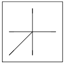
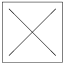
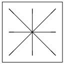
Ten solved symmetric models

$\mathcal{S}$	Inv.	Refl.	Analysis	$\mathcal{S}$	Inv.	Refl.	Analysis + Galois
	alg				D-alg		
	alg						
	alg						
	DF						
	DF						
							

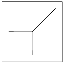
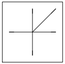
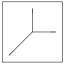
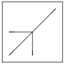
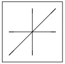
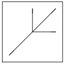
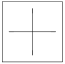
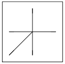
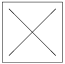
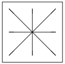
Ten solved symmetric models

$\mathcal{S}$	Inv.	Refl.	Analysis	$\mathcal{S}$	Inv.	Refl.	Analysis + Galois
	alg				D-alg		
	alg						
	alg						
	DF	DF					
	DF	DF					
		DF					

Ten solved symmetric models

$\mathcal{S}$	Inv.	Refl.	Analysis	$\mathcal{S}$	Inv.	Refl.	Analysis + Galois
	alg		DF		D-alg		
	alg		DF				
	alg		DF				
	DF	DF	DF				
	DF	DF					
		DF					

Ten solved symmetric models

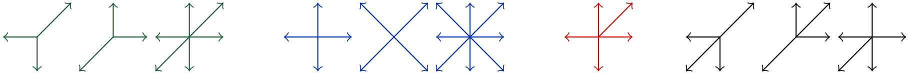
$\mathcal{S}$	Inv.	Refl.	Analysis	$\mathcal{S}$	Inv.	Refl.	Analysis + Galois
	alg		DF		D-alg		D-alg
	alg		DF				not DA
	alg		DF				not DA
	DF	DF	DF				not DA
	DF	DF					
		DF					

Ten solved symmetric models

10 models

rat. invariants: 6 — **D-finite**
- decoupling: 3 — **algebraic**
- no decoupling: 3 — **DF transc.**

no rat. inv.: 4 — **Not D-finite**
- decoupling: 1 — **D-alg.**
- no decoupling: 3 — **not D-alg.**



[mbm, Wallner, Raschel, Trotignon, Mustapha, Dreyfus]

Ten solved symmetric models

- 10 models
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    - **Not D-finite**
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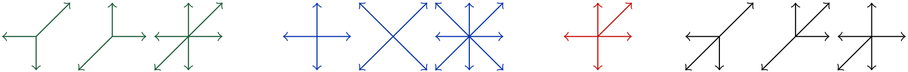

[mbm, Wallner, Raschel, Trotignon, Mustapha, Dreyfus]

Andrew Elvey Price: same nature as the quadrant series, at least in x and y (next talk)