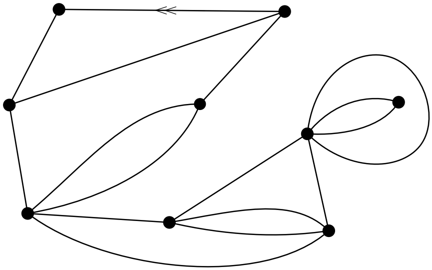
The six vertex model on random lattices using Jacobi theta functions

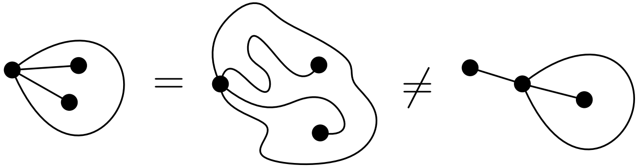
Andrew Elvey Price
Joint work with Paul Zinn-Justin

CNRS and Université de Tours

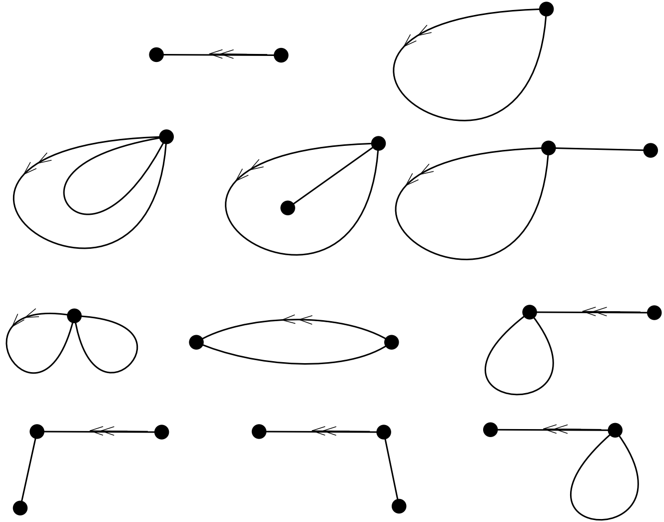
21/05/2021

Planar map: Drawing of (planar) graph on the sphere with a marked, directed *root* edge (up to orientation preserving homeomorphisms).

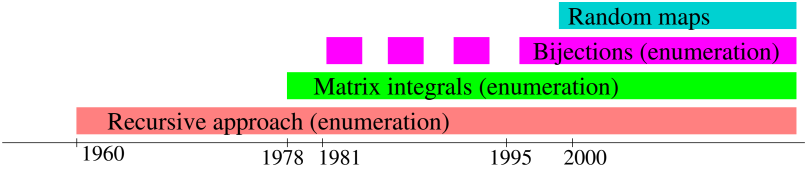




SMALL PLANAR MAPS



A chronology of planar maps

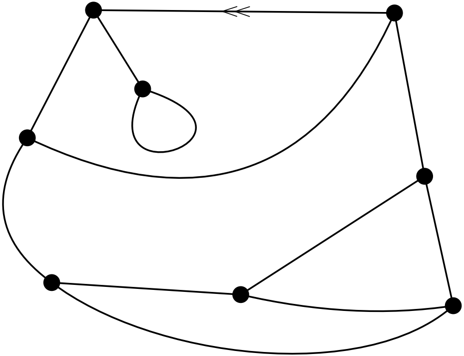


- **Recursive approach:** Tutte, Brown, Bender, Canfield, Richmond, Goulden, Jackson, Wormald, Walsh, Lehman, Gao, Wanless...
- **Matrix integrals:** Brézin, Itzykson, Parisi, Zuber, Bessis, Ginsparg, Kostov, Zinn-Justin, Boulatov, Kazakov, Mehta, Bouttier, Di Francesco, Guitter, Eynard...
- **Bijections:** Cori & Vauquelin, Schaeffer, Bouttier, Di Francesco & Guitter (BDG), Bernardi, Fusy, Poulalhon, Bousquet-Mélou, Chapuy...
- **Geometric properties of random maps:** Chassaing & Schaeffer, BDG, Marckert & Mokkadem, Jean-François Le Gall, Miermont, Curien, Albenque, Bettinelli, Ménard, Angel, Sheffield, Miller, Gwynne...

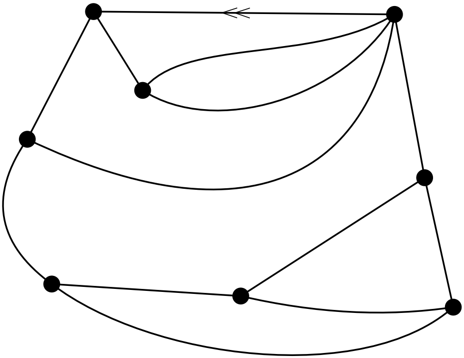
- **Introduction:** recursive method for (cubic) maps
- **Part 1:** Eulerian orientations - story time + bijections
- **Part 2:** Solving six vertex model
  - **Part 2a:** Functional equations
  - **Part 2b:** Solving functional equations (guess and check)
  - **Part 2c:** deriving the guesses
- **Part 3:** Modular properties and algebraicity for the 6-vertex model

Introduction: Counting cubic maps

(Tutte, 1962)



Definition: Quasi-cubic map: non-root vertices have degree 3
Definition: $c_{n,m}$ is the number of quasi-cubic maps with n edges where the root vertex has degree m and $C(t,x) = \sum_{n,m\geq 0} t^n x^m c_{n,m}$.



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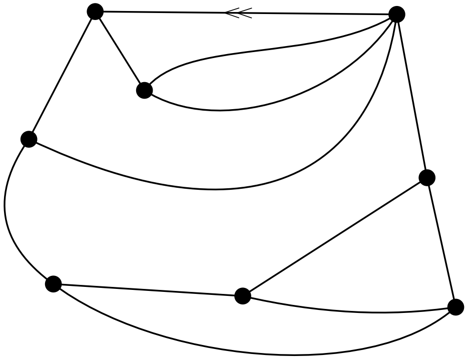
Contract the root edge



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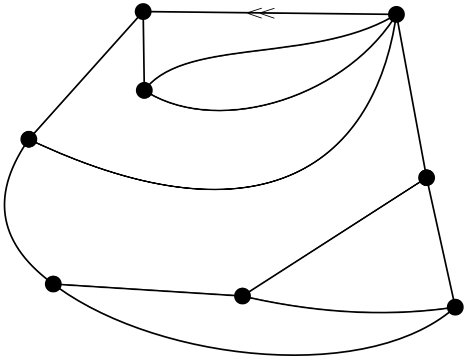

QUASI-CUBIC MAPS

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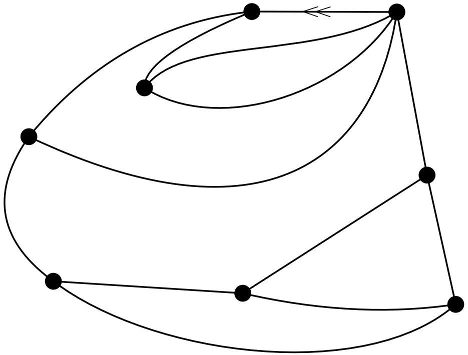

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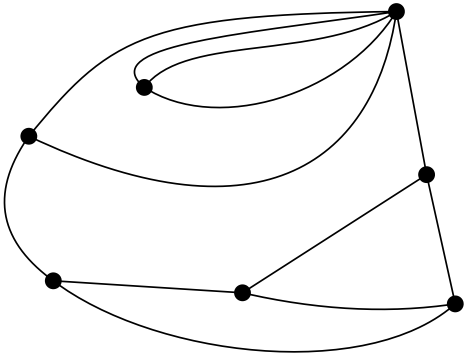

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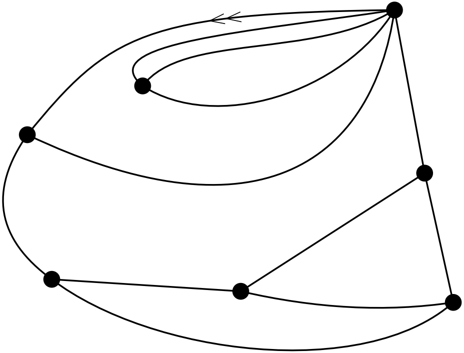

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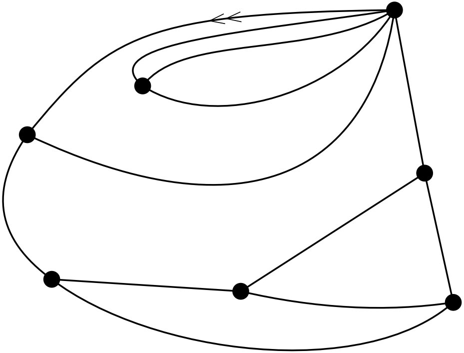
Contract the root edge → still quasi-cubic!



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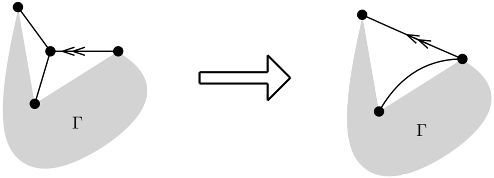

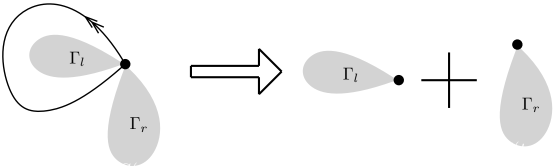

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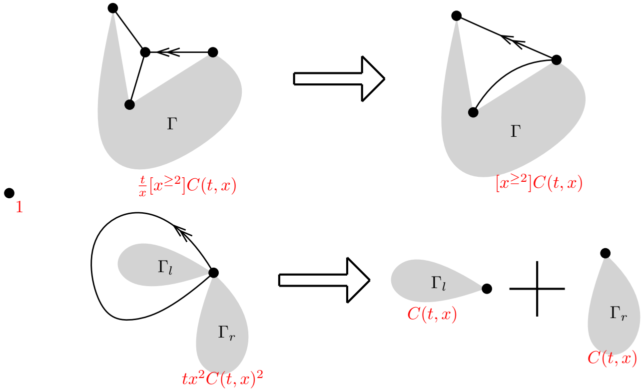

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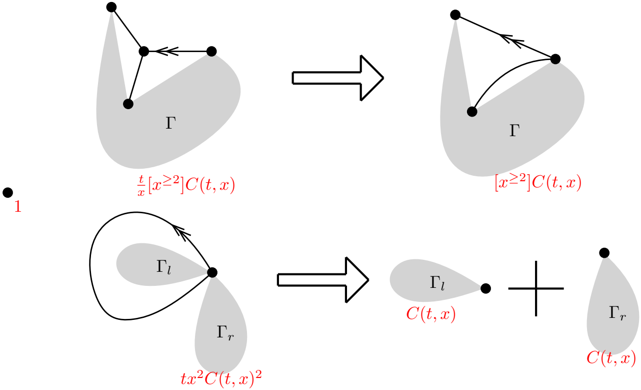
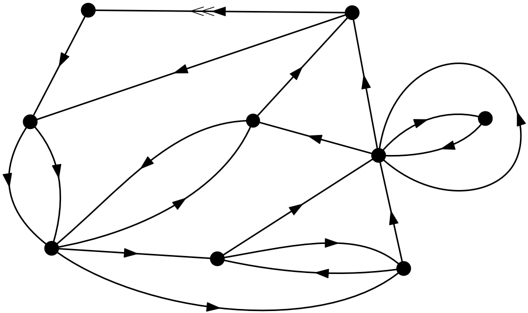


$$C(t,x) = 1 + \frac{t}{x}[x^{\geq 2}]C(t,x) + tx^2C(t,x)^2$$

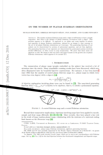
Part 1:
Eulerian orientations

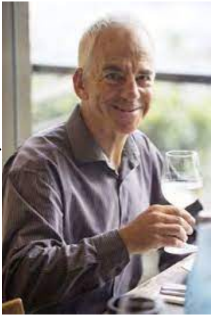


STORY TIME: BACK IN 2017...

This problem seems hard. Can you compute anything?

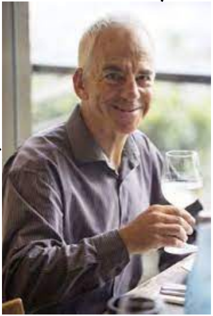
Ooh, looks fun!

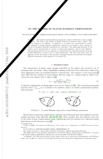

ON THE NUMBER OF PLANAR EULERIAN ORIENTATIONS



Ok, sounds fun!

Hey, this problem looks fun. wanna try it?





ON THE NUMBER OF PLANAR EULERIAN ORIENTATIONS

NICOLAS BONICHON, MIREILLE BOUSQUET-MÉLOU, PAUL DORBEC, AND CLAIRE PENNARUN

Abstract. The number of planar Eulerian maps with n edges is well-known to have a simple expression. But what is the number of planar Eulerian *orientations* with n edges? This problem appears to be difficult. To approach it, we define and count families of subsets and supersets of planar Eulerian orientations, indexed by an integer k, that converge to the set of all planar Eulerian orientations as k increases. The generating functions of our subsets can be characterized by systems of polynomial equations, and are thus algebraic. The generating functions of our supersets are characterized by polynomial systems involving divided differences, as often occurs in map enumeration. We prove that these series are algebraic as well. We obtain in this way lower and upper bounds on the growth rate of planar Eulerian orientations, which appears to be around 12.5.

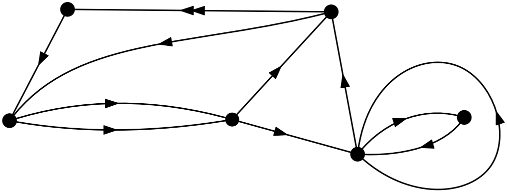
1. Introduction

The enumeration of planar maps (graphs embedded on the sphere) has received a lot of attention since the sixties. Many remarkable counting results have been discovered, which were often illuminated later by beautiful bijective constructions. For instance, it has been known[1] since 1963 that the number of rooted planar *Eulerian* maps (i.e., planar maps in which every vertex has even degree) with n edges is [54]:

$$m_n = \frac{3 \cdot 2^{n-1}}{(n+1)(n+2)} \binom{2n}{n}. \qquad (1)$$

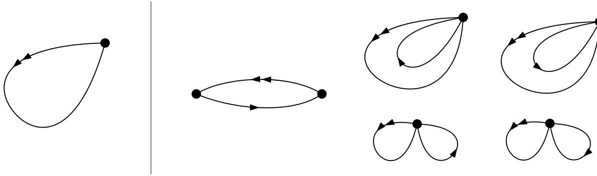
A bijective explanation involving plane trees can be found in [15]. The associated generating function $M(t) = \sum_{n \geq 0} m_n t^n$ is known to be *algebraic*, that is, to satisfy a polynomial equation. More precisely:

$$t^2 + 11t - 1 - (8t^2 + 12t - 1)M(t) + 16t^2 M(t)^2 = 0.$$



Each vertex has equally many incoming as outgoing edges.

Aim: Determine the number g_n of (rooted planar) Eulerian orientations with n edges



The generating function $\mathsf{G}(t) = \sum_{t=1}^{\infty} g_n t^n = t + 5t^2 + \dots$

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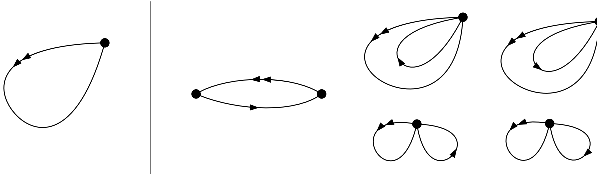

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In 2017, E.P. and Guttmann:

- Computed the number g_n of Eulerian orientations for $n < 100$.
- Predicted that
$$g_n \sim \kappa_g \frac{(4\pi)^n}{n^2(\log n)^2}.$$

EULERIAN ORIENTATIONS

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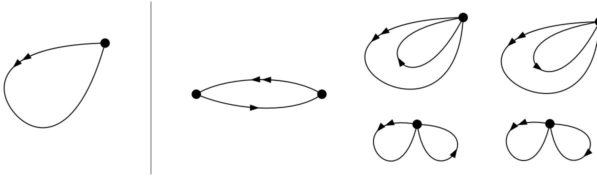

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Eulerian orientations

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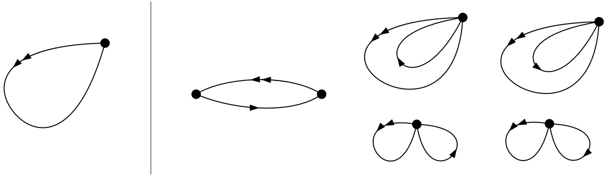

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Eulerian orientations exact solution

Let $\mathsf{R}_0(t)$ be the unique power series with constant term 0 satisfying

$$t = \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n}^2 \mathsf{R}_0(t)^{n+1}.$$

The generating function $\mathsf{G}(t) = \sum_{n=0}^{\infty} g_n t^n$ of rooted planar Eulerian orientations counted by edges is given by

$$\mathsf{G}(t) = \frac{1}{4t^2}(t - 2t^2 - \mathsf{R}_0(t)).$$

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As predicted,

$$g_n \sim \kappa_g \frac{(4\pi)^n}{n^2(\log n)^2}.$$

Eulerian orientations

[E.P. and Guttmann]

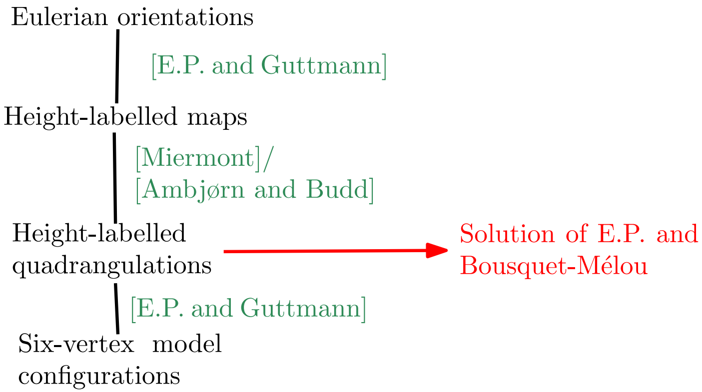
Height-labelled maps

[Miermont]/
[Ambjørn and Budd]

Height-labelled quadrangulations

[E.P. and Guttmann]

Six-vertex model configurations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Solution of E.P. and Bousquet-Mélou
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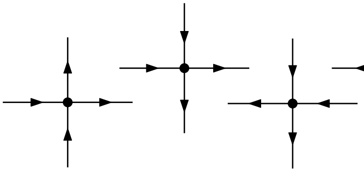
Six-vertex model configurations

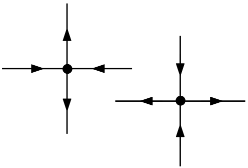
Part 1a: Eulerian orientations $\rightarrow$ Six vertex model

"Six" vertex model

Each vertex has 2 incoming and 2 outgoing edges.



Non-alternating
(weight t)



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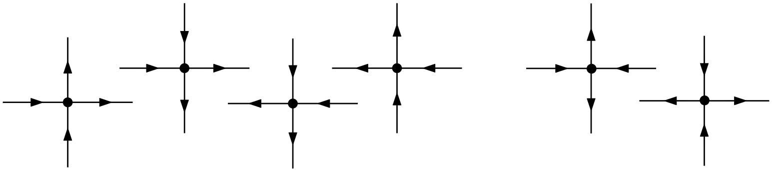
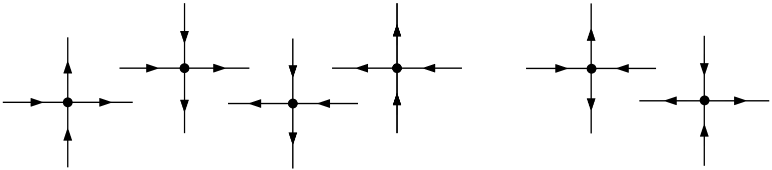


Non-alternating (weight t)

Alternating (weight $t\gamma$)

Definition: $q_{n,k}$ = number of maps with n vertices, k alternating and $n - k$ non-alternating.
Definition: *weight* of a map = product of weights of its vertices.

Each vertex has 2 incoming and 2 outgoing edges.



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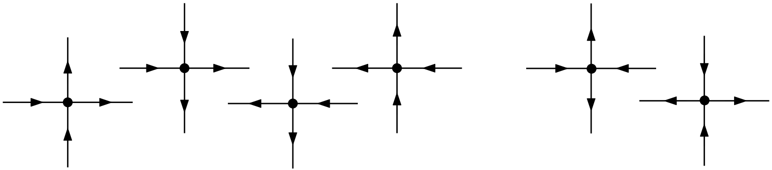

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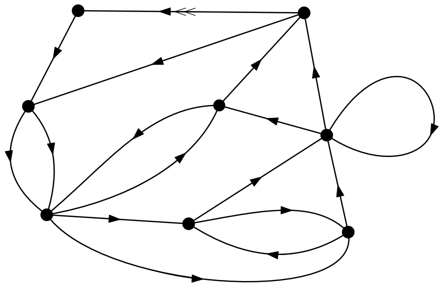
Aim: Determine $\mathsf{Q}(t,\gamma)$.

Eulerian orientations → height-labelled maps



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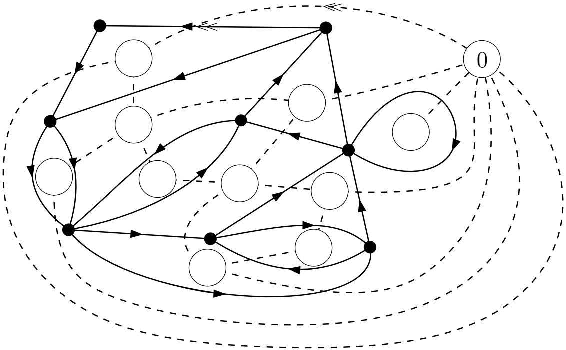
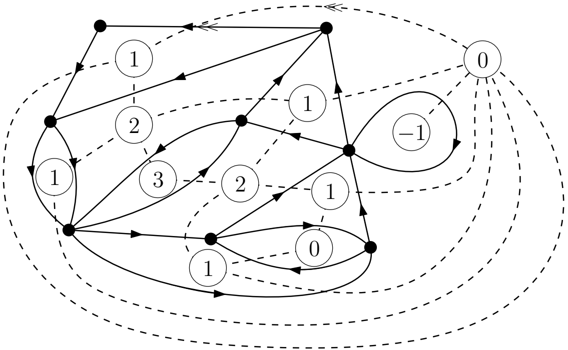
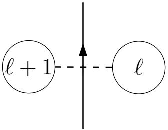


Bijection to height-labelled maps

Eulerian orientations $\rightarrow$ height-labelled maps





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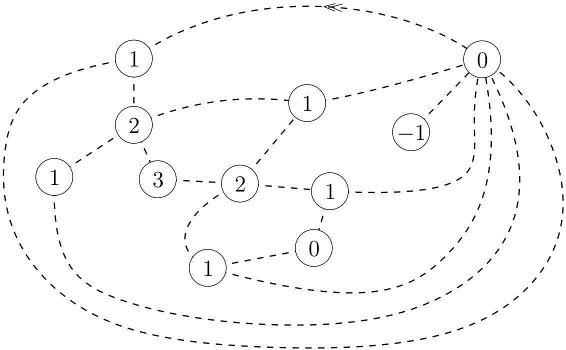
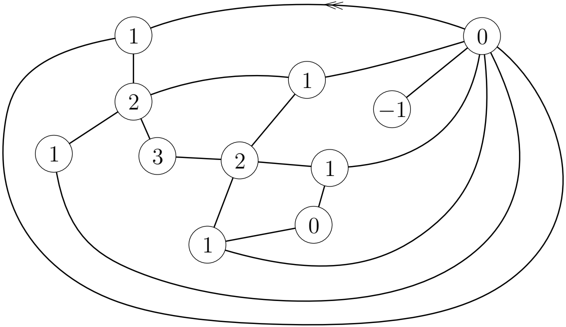
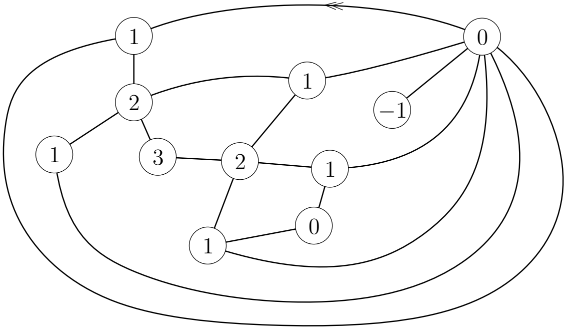


Eulerian orientations $\rightarrow$ height-labelled maps



BIJECTION TO HEIGHT-LABELLED MAPS

Eulerian orientations $\rightarrow$ height-labelled maps

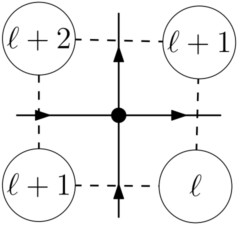


Labelled map:

- Adjacent labels differ by 1
- root edge from 0 to 1

Labelled quadrangulations

- adjacent labels differ by 1.
- By our bijection, $\mathsf{Q}(t,\gamma)$ counts labelled *quadrangulations* by faces (t) and *alternating* faces (γ).



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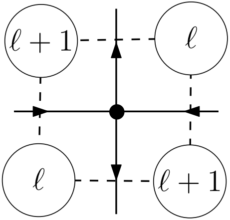

Alternating
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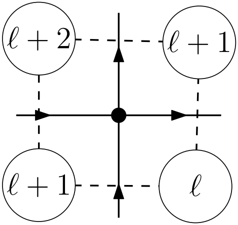

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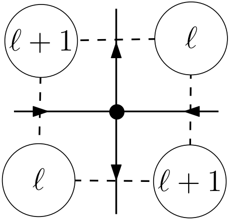
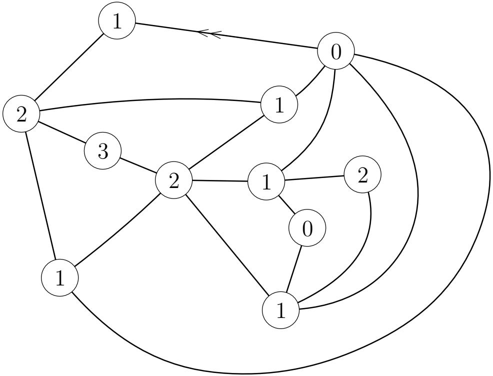


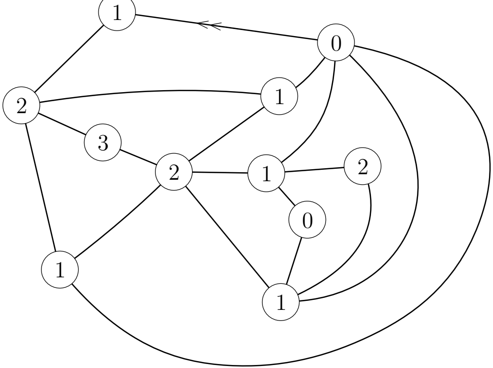
Alternating
(weight γt)

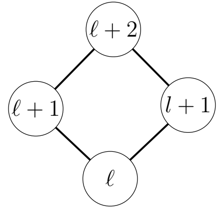
Next step: Bijection between labelled quadrangulations with no alternating faces and labelled maps (Miermont / Ambjørn and Budd).

Start with a quadrangulation with no alternating faces.

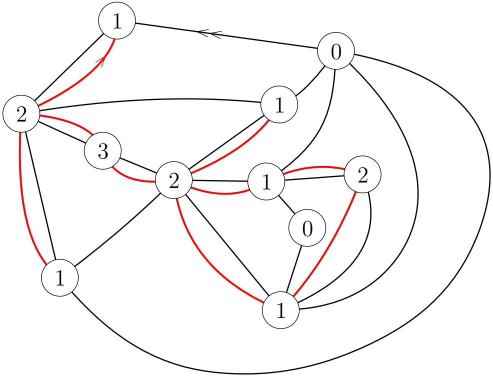


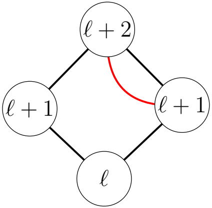
Start with a quadrangulation with no alternating faces.



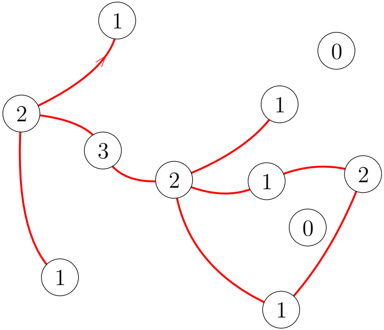


Draw a red edge in each face according to the rule.



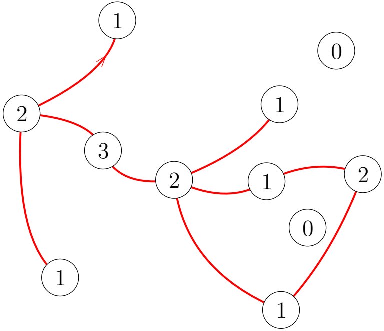


Remove all of the original edges.

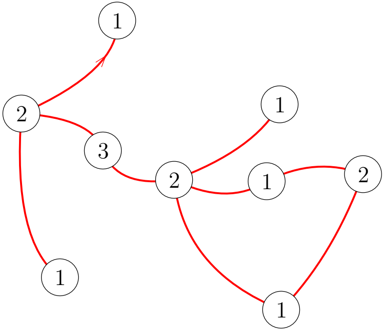


Labelled quadrangulations to labelled maps

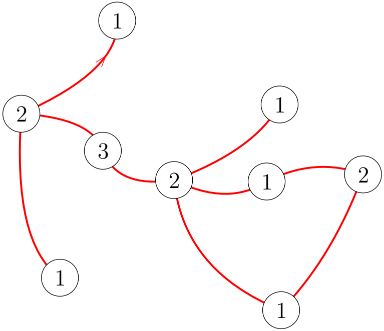
Remove any isolated vertices.



Remove any isolated vertices.

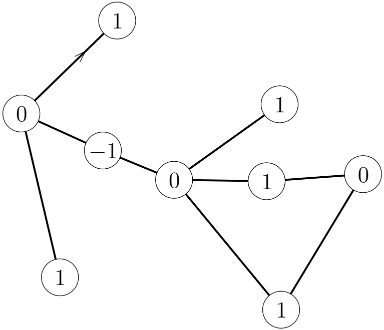


This is now a labelled map!



Labelled quadrangulations to labelled maps

This is now a labelled map! (after changing labels ℓ to $2-\ell$).



This bijection shows that $\mathsf{Q}(t,0) = \mathsf{G}(t)$, the generating function for rooted Eulerian orientations where t counts edges.

Eulerian orientations

[E.P. and Guttmann]

Height-labelled maps

[Miermont]/
[Ambjørn and Budd]

Height-labelled quadrangulations

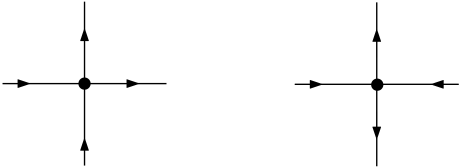
[E.P. and Guttmann]

Six-vertex model configurations

Part 2:
Solving the six vertex model

(on planar maps)

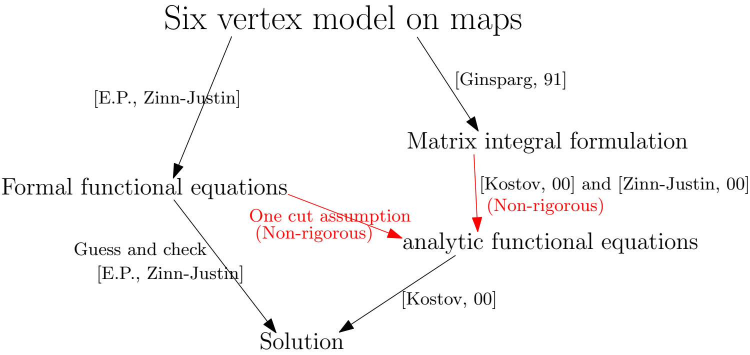
Each vertex has 2 incoming and 2 outgoing edges.



Definition: $q_{n,k}$ = number of maps with n vertices, k alternating and $n-k$ non-alternating.
Definition: *weight* of a map = product of weights of its vertices.
Definition: $\mathsf{Q}(t,\gamma) = \sum_{n,k} q_{n,k} t^n \gamma^k$ = sum of weights of all maps.
Aim: Determine $\mathsf{Q}(t,\gamma)$.

Background on the six vertex model

- "Solved" by Kostov in 2000 using matrix integral techniques.
- Solution was not rigorous.
- We made this argument rigorous and simplified the form of the solutions.



Solutions at $\gamma = 0, 1$ [E.P. and Bousquet-Mélou]

The generating function $\mathsf{Q}(t,0)$ is given by

$$t = \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n}^2 \mathsf{R}_0(t)^{n+1},$$

$$\mathsf{Q}(t,0) = \frac{1}{2t^2}(t - 2t^2 - \mathsf{R}_0(t)).$$

The generating function $\mathsf{Q}(t,1)$ is given by

$$t = \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n} \binom{3n}{n} \mathsf{R}_1(t)^{n+1},$$

$$\mathsf{Q}(t,1) = \frac{1}{3t^2}(t - 3t^2 - \mathsf{R}_1(t)).$$

Preview: Solution for $\mathsf{Q}(t,\gamma)$

Define

$$\vartheta(z,q) = \sum_{n=0}^{\infty} (-1)^n \left(e^{(2n+1)iz} - e^{-(2n+1)iz}\right) q^{(2n+1)^2/8}.$$

Let $q = q(t,\alpha)$ be the unique series satisfying

$$t = \frac{\cos\alpha}{64\sin^3\alpha}\left(-\frac{\vartheta(\alpha,q)\vartheta'''(\alpha,q)}{\vartheta'(\alpha,q)^2} + \frac{\vartheta''(\alpha,q)}{\vartheta'(\alpha,q)}\right).$$

Define $\mathsf{R}(t,\gamma)$ by

$$\mathsf{R}(t,-2\cos(2\alpha)) = \frac{\cos^2\alpha}{96\sin^4\alpha}\frac{\vartheta(\alpha,q)^2}{\vartheta'(\alpha,q)^2}\left(-\frac{\vartheta'''(\alpha,q)}{\vartheta'(\alpha,q)} + \frac{\vartheta'''(0,q)}{\vartheta'(0,q)}\right).$$

Then

$$\mathsf{Q}(t,\gamma) = \frac{1}{(\gamma+2)t^2}\left(t - (\gamma+2)t^2 - \mathsf{R}(t,\gamma)\right).$$

Part 2a: Deriving functional equations

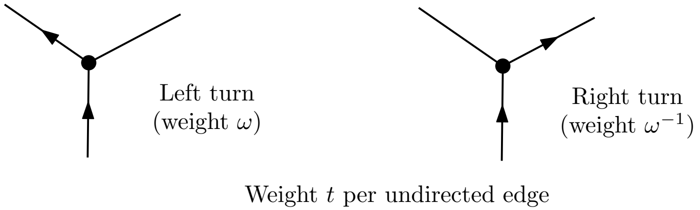
Recall: $\mathsf{Q}(t,\gamma)$ counts six-vertex model configurations with a weight t per vertex and γ per alternating vertex.

CUBIC EULERIAN PARTIAL ORIENTATIONS (CEPOS)

Recall: $\mathsf{Q}(t, \gamma)$ counts six-vertex model configurations with a weight t per vertex and γ per alternating vertex.

Definition: CEPO: a map using vertices of the types below

Vertex types:



Cubic Eulerian partial orientations (CEPOs)

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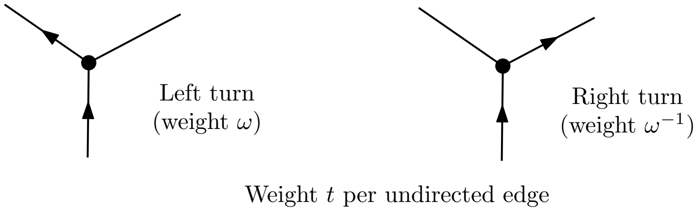

$\mathsf{C}(t,\omega) = \sum_{n,k} c_{n,k} t^n \omega^k$ counts CEPOs (using weights above).

$c_{n,2k}$: number of CEPOs with $n+k$ left turns and $n-k$ right turns

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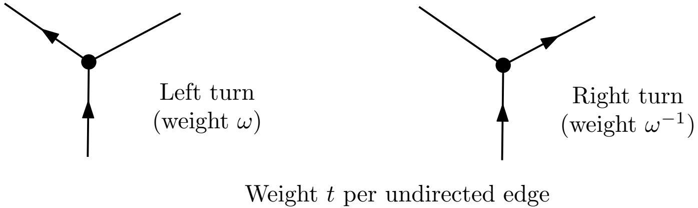

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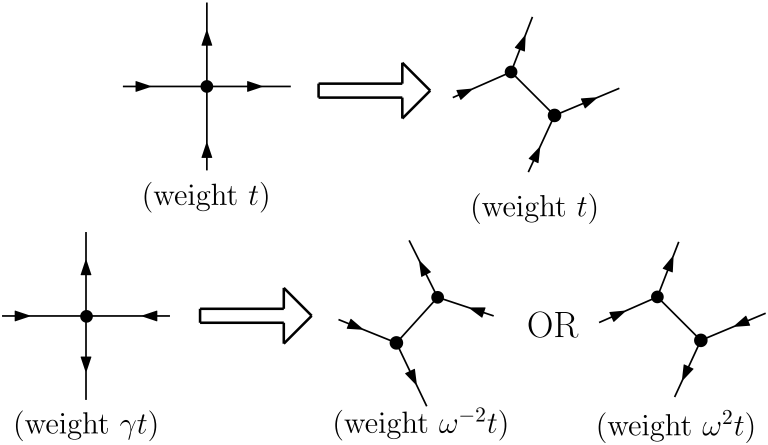
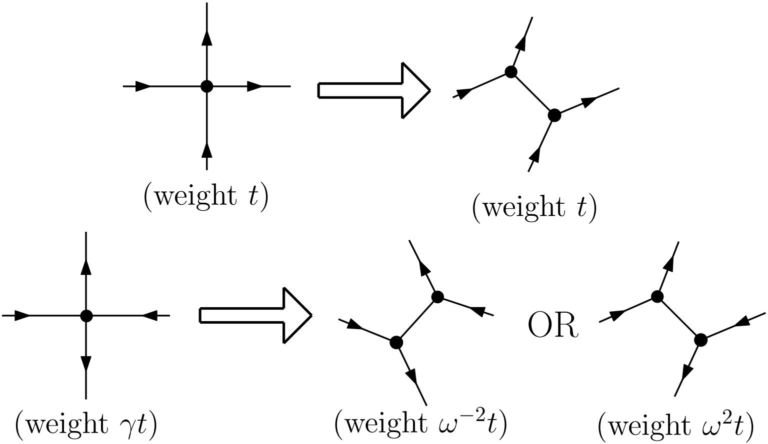
Theorem: $\mathsf{Q}(t,\omega^2+\omega^{-2}) = \mathsf{C}(t,\omega)$.

SIX VERTEX MODEL → CEPOS

Theorem: $\mathsf{Q}(t, \omega^2 + \omega^{-2}) = \mathsf{C}(t, \omega)$



Theorem: $\mathsf{Q}(t, \omega^2 + \omega^{-2}) = \mathsf{C}(t, \omega)$



Reverse direction: contract undirected edges.

Definition: quasi-CEPO: non-root vertices are left turns or right turns.

Functional equations for quasi-CEPOs

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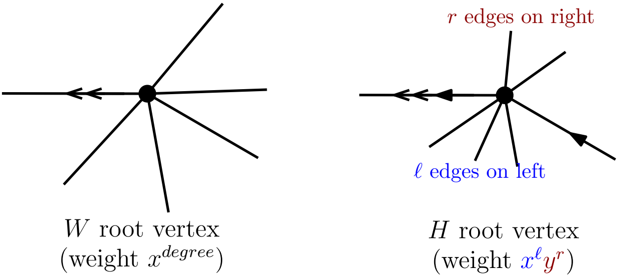
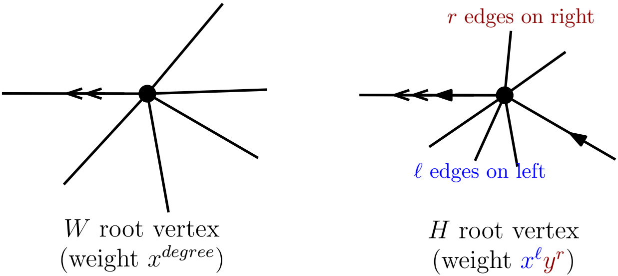


$\mathsf{W}(x) \equiv \mathsf{W}(t, \omega, x)$: root vertex is a W root vertex
$\mathsf{H}(x, y) \equiv \mathsf{H}(t, \omega, x, y)$: root vertex is a H root vertex

Functional equations for quasi-CEPOs

Definition: quasi-CEPO: non-root vertices are left turns or right turns.



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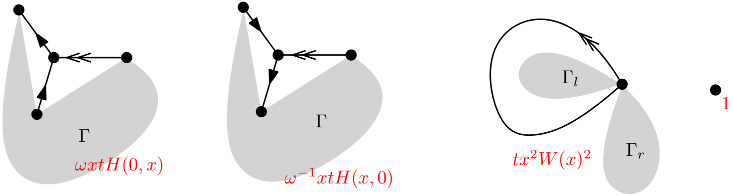
$$\mathsf{C}(t, \omega) = \mathsf{H}(t, \omega, 0, 0)$$

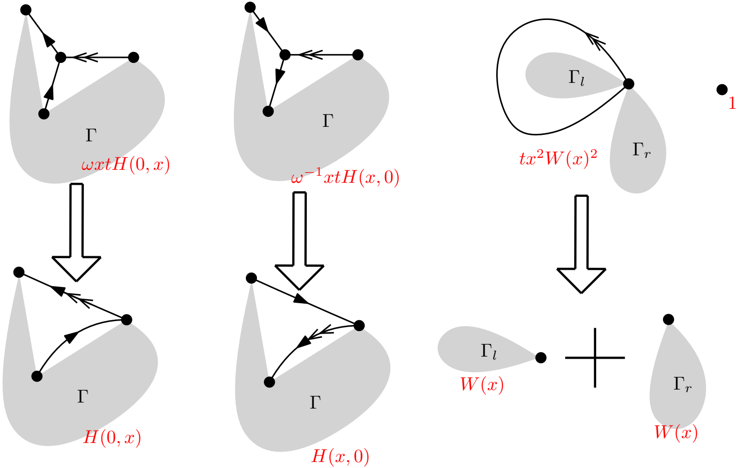
For functional equations: Contract the root edge.

Cubic Eulerian partial orientations



$$\mathsf{W}(x) = \omega xt\mathsf{H}(0,x) + \omega^{-1}xt\mathsf{H}(x,0) + x^2t\mathsf{W}(x)^2 + 1$$

CUBIC EULERIAN PARTIAL ORIENTATIONS



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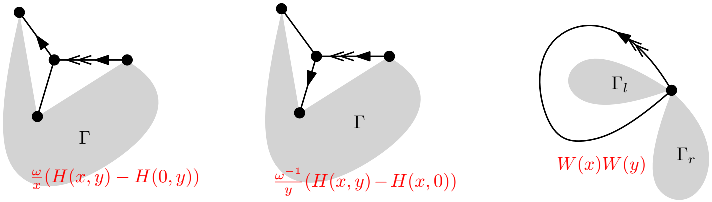
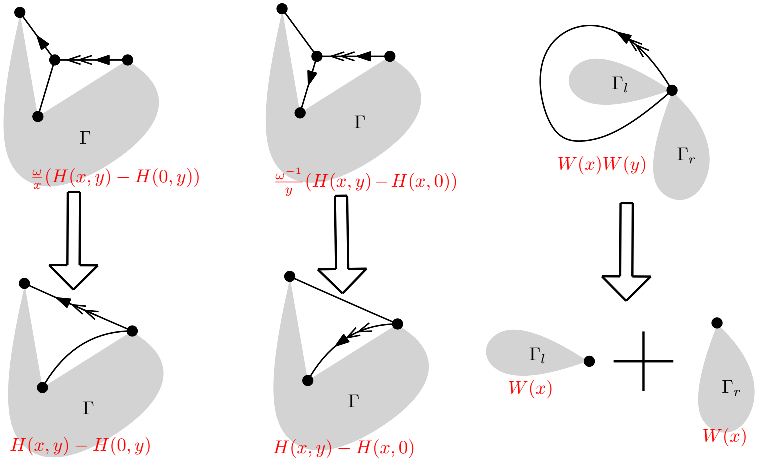


$$\mathsf{H}(x,y)=\frac{\omega}{x}\left(\mathsf{H}(x,y)-\mathsf{H}(0,y)\right)+\frac{\omega^{-1}}{y}\left(\mathsf{H}(x,y)-\mathsf{H}(x,0)\right)+\mathsf{W}(x)\mathsf{W}(y).$$

Cubic Eulerian partial orientations



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Part 2b:
Solving functional equations

Functional equations for the six vertex model

Recall: $\mathsf{C}(t,\omega) = \mathsf{Q}(t,\omega^2+\omega^{-2}) = \mathsf{H}(t,\omega,0,0) \equiv \mathsf{H}(0,0)$ is characterised by:

$$\mathsf{W}(x) = x^2 t\mathsf{W}(x)^2 + \omega x t\mathsf{H}(0,x) + \omega^{-1} x t\mathsf{H}(x,0) + 1$$

$$\mathsf{H}(x,y) = \mathsf{W}(x)\mathsf{W}(y) + \frac{\omega}{y}\left(\mathsf{H}(x,y) - \mathsf{H}(x,0)\right) + \frac{\omega^{-1}}{x}\left(\mathsf{H}(x,y) - \mathsf{H}(0,y)\right).$$

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So, we just need to guess and check

EXPRESSIONS FOR $\mathsf{W}(x)$ AND $\mathsf{H}(x, y)$

$$\vartheta(z) = \sum_{n=0}^{\infty}(-1)^n(e^{(2n+1)iz} + e^{-(2n+1)iz})e^{(n+1/2)^2 i\pi\tau}$$

$$\omega = ie^{-i\alpha}, \qquad t = \frac{\cos\alpha}{64\sin^3\alpha}\left(-\frac{\vartheta(\alpha)\vartheta'''(\alpha)}{\vartheta'(\alpha)^2} + \frac{\vartheta''(\alpha)}{\vartheta'(\alpha)}\right),$$

$$b = \frac{1}{16t}\frac{\cos(\alpha)}{\sin^3(\alpha)}\frac{\vartheta(\alpha)^2}{\vartheta'(\alpha)^2}, \qquad x_0 = \frac{\cos\alpha}{2\sin\alpha}\frac{\vartheta'(0)}{\vartheta'(\alpha)}, \qquad c = -\omega - \omega^{-1},$$

$$V(z) = b\left(\frac{\vartheta'(z)^2}{\vartheta(z)^2} - \frac{\vartheta''(z)}{\vartheta(z)} + \frac{\vartheta'''(0)}{3\vartheta'(0)}\right), \qquad x(z) = x_0\frac{\vartheta(z+\alpha)}{\vartheta(z)},$$

$$W^{(0)}(y) = -\frac{1}{2\pi}\int_0^{\pi}\frac{V(z-\frac{\pi\tau}{2})x'\left(z-\frac{\pi\tau}{2}\right)}{\left(y+c^{-1}-i\omega x(z-\frac{\pi\tau}{2})\right)x\left(z-\frac{\pi\tau}{2}\right)}dz,$$

$$\frac{1}{y}\mathsf{H}\left(0,\frac{1}{y}\right) = -\frac{\omega^{-1}}{2\pi}\int_0^{\pi}\frac{V(z-\frac{\pi\tau}{2})x'\left(z-\frac{\pi\tau}{2}\right)W^{(0)}\left(-c^{-1}-i\omega^{-1}x\left(z-\frac{\pi\tau}{2}\right)\right)}{(y+c^{-1}-i\omega x(z-\frac{\pi\tau}{2}))x\left(z-\frac{\pi\tau}{2}\right)}dz,$$

$$\frac{1}{y}\mathsf{H}\left(\frac{1}{y},0\right) = -\frac{\omega}{2\pi}\int_0^{\pi}\frac{V(z-\frac{\pi\tau}{2})x'\left(z-\frac{\pi\tau}{2}\right)W^{(0)}\left(-c^{-1}-i\omega^{3}x\left(z-\frac{\pi\tau}{2}\right)\right)}{(y+c^{-1}-i\omega x(z-\frac{\pi\tau}{2}))x\left(z-\frac{\pi\tau}{2}\right)}dz,$$

$$\mathsf{W}(x) = \frac{1}{x}W^{(0)}\left(\frac{1}{x}\right), \qquad \mathsf{H}(x,y) = \frac{xy\mathsf{W}(x)\mathsf{W}(y) - \omega x\mathsf{H}(x,0) - \omega^{-1}y\mathsf{H}(0,y)}{xy - \omega x - \omega^{-1}y}.$$

Expressions are formal series in t, ω, and (in some cases) x, y and e^{2iz}.

Solution for $\mathsf{Q}(t,\gamma)$

Define

$$\vartheta(z,q) = \sum_{n=0}^{\infty}(-1)^n(e^{(2n+1)iz} - e^{-(2n+1)iz})q^{(2n+1)^2/8}.$$

Let $q = q(t,\alpha)$ be the unique series satisfying

$$t = \frac{\cos\alpha}{64\sin^3\alpha}\left(-\frac{\vartheta(\alpha,q)\vartheta'''(\alpha,q)}{\vartheta'(\alpha,q)^2} + \frac{\vartheta''(\alpha,q)}{\vartheta'(\alpha,q)}\right).$$

Define $\mathsf{R}(t,\gamma)$ by

$$\mathsf{R}(t,-2\cos(2\alpha)) = \frac{\cos^2\alpha}{96\sin^4\alpha}\frac{\vartheta(\alpha,q)^2}{\vartheta'(\alpha,q)^2}\left(-\frac{\vartheta'''(\alpha,q)}{\vartheta'(\alpha,q)} + \frac{\vartheta'''(0,q)}{\vartheta'(0,q)}\right).$$

Then

$$\mathsf{Q}(t,\gamma) = \frac{1}{(\gamma+2)t^2}\left(t - (\gamma+2)t^2 - \mathsf{R}(t,\gamma)\right).$$

Part 2c: Non-rigorously deriving expressions for $\mathsf{W}(x)$ and $\mathsf{H}(x, y)$

(by copying Kostov)

Recall: Equations to solve:

$$\mathsf{W}(x) = x^2 t \mathsf{W}(x)^2 + \omega x t \mathsf{H}(0, x) + \omega^{-1} x t \mathsf{H}(x, 0) + 1$$

$$\mathsf{H}(x, y) = \mathsf{W}(x)\mathsf{W}(y) + \frac{\omega}{y}\left(\mathsf{H}(x, y) - \mathsf{H}(x, 0)\right) + \frac{\omega^{-1}}{x}\left(\mathsf{H}(x, y) - \mathsf{H}(0, y)\right).$$

“SOLVING” FUNCTIONAL EQUATIONS

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Step 1: Write $\omega = e^{i\alpha}$ for $\alpha \in \mathbb{R}$ and choose $t \in \mathbb{R}$ small
$\rightarrow$ series converge for $|x|, |y| < 1$

"SOLVING" FUNCTIONAL EQUATIONS

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Step 2: (One cut assumption) Assume $\mathsf{W}(x)$ and $\mathsf{H}(x, 0)$ have extensions that are analytic on $\mathbb{C} \setminus [r_1, r_2]$, for some $r_1, r_2 \in \mathbb{R}$

"Solving" functional equations

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Step 3: (Kernel method) write $X(v) = \frac{\omega+\omega^{-1}}{1-iv(\omega^2+1)}$ and $Y(v) = \frac{\omega+\omega^{-1}}{1-iv(\omega^{-2}+1)}$ to parameterise $\left(1 - \frac{\omega}{y} - \frac{\omega^{-1}}{x}\right) = 0.$
The second equation becomes:

$$0 = \mathsf{W}(X(v))\mathsf{W}(Y(v)) - \frac{\omega}{Y(v)}\mathsf{H}(X(v),0) - \frac{\omega^{-1}}{X(v)}\mathsf{H}(0,Y(v))$$

New equations:

$$\mathsf{W}(x) = x^2 t\mathsf{W}(x)^2 + \omega xt\mathsf{H}(0,x) + \omega^{-1}xt\mathsf{H}(x,0) + 1$$

$$0 = \mathsf{W}(X(v))\mathsf{W}(Y(v)) - \frac{\omega}{Y(v)}\mathsf{H}(X(v),0) - \frac{\omega^{-1}}{X(v)}\mathsf{H}(0,Y(v))$$

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By analysing the cuts, we find that

$$U(v) := v\omega X(v)\mathsf{W}\left(X(v)\right) + v\omega^{-1}Y(v)\mathsf{W}\left(Y(v)\right) + \frac{iv^2}{t(\omega^2-\omega^{-2})} - \frac{v}{t(\omega+\omega^{-1})^2}$$

is analytic except on 2 cuts $i\omega[x_1', x_2']$ and $-i\omega^{-1}[x_1', x_2']$ and satisfies

$$U(i\omega(x+i0)) = U(-i\omega^{-1}(x-i0)),$$

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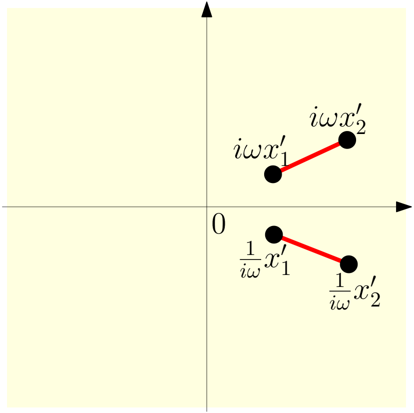
for $x \in [x_1', x_2']$.

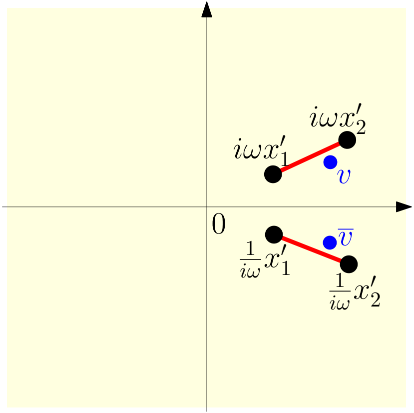
UNDERSTANDING $U(v)$



$U(v)$ analytic except on slits .

$U(i\omega(x \pm i0)) = U(-i\omega^{-1}(x \mp i0))$, for $x \in [x_1', x_2']$.
i.e., as $v \to$ slit, $U(v) - U(\overline{v}) \to 0$.

UNDERSTANDING $U(v)$



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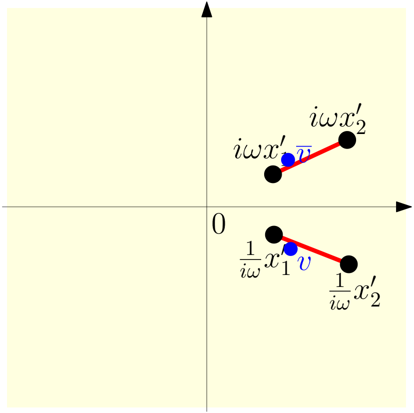
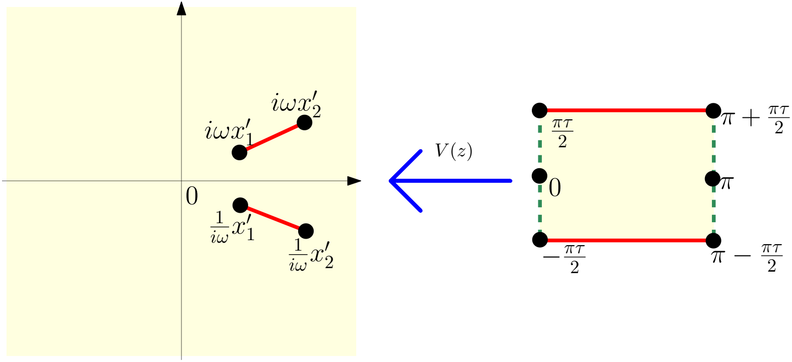


$U(v)$ analytic except on slits .

$U(i\omega(x \pm i0)) = U(-i\omega^{-1}(x \mp i0))$, for $x \in [x_1', x_2']$.
i.e., as $v \to$ slit, $U(v) - U(\overline{v}) \to 0$.

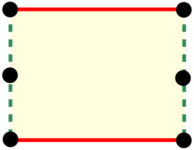
Solving for $U(v)$

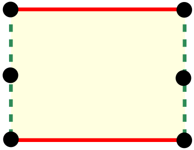


Left: $U(v)$ is analytic on yellow region.
There is a unique $\tau \in i\mathbb{R}_{>0}$ and conformal map $V(z)$ from the flat cylinder of height $\pi\tau$ onto this region ($V(z) = V(z+\pi)$).

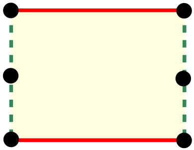
SOLVING FOR $U(v)$

Solving for $U(v)$

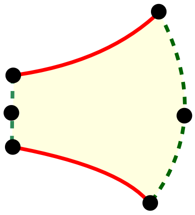




SOLVING FOR $U(v)$



Solving for $U(v)$



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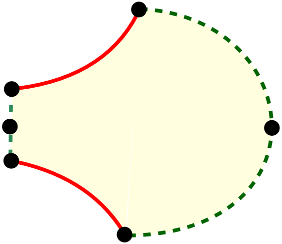
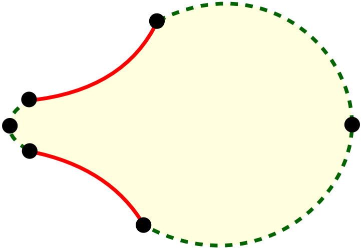
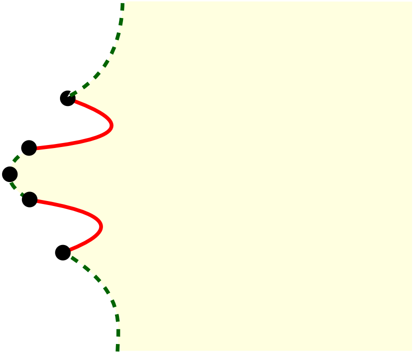


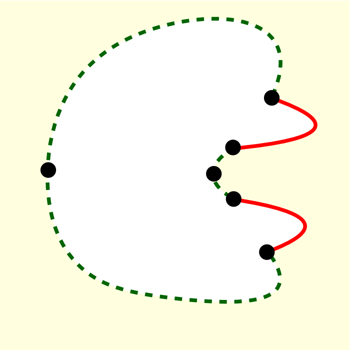
SOLVING FOR $U(v)$



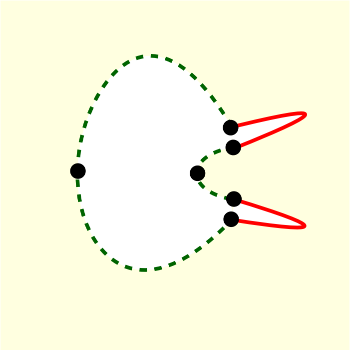
SOLVING FOR $U(v)$



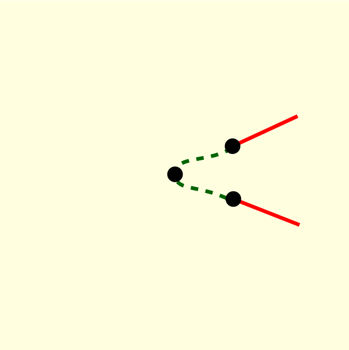
SOLVING FOR $U(v)$



SOLVING FOR $U(v)$



Solving for $U(v)$



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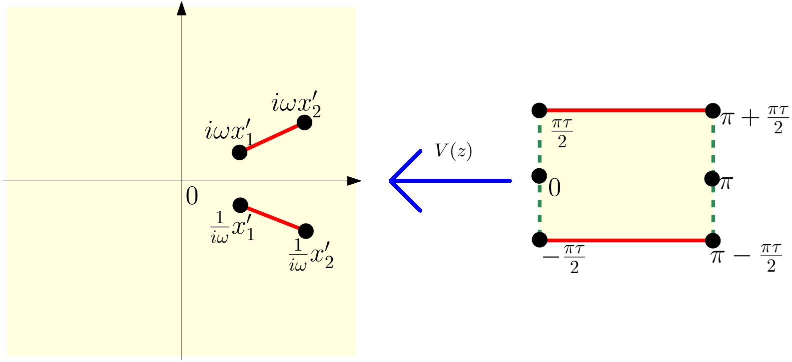
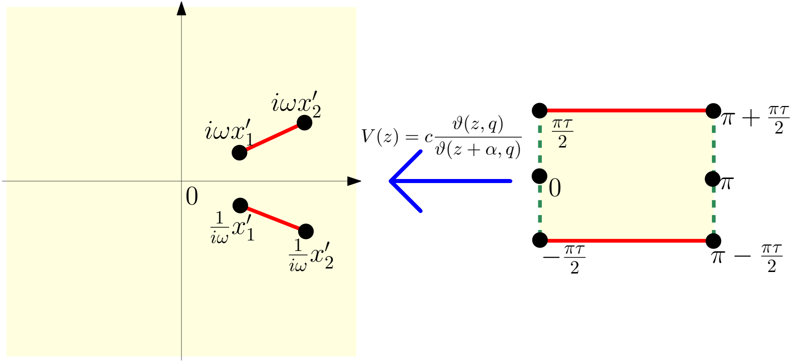


Left: $U(v)$ is analytic on yellow region.

SOLVING FOR $U(v)$

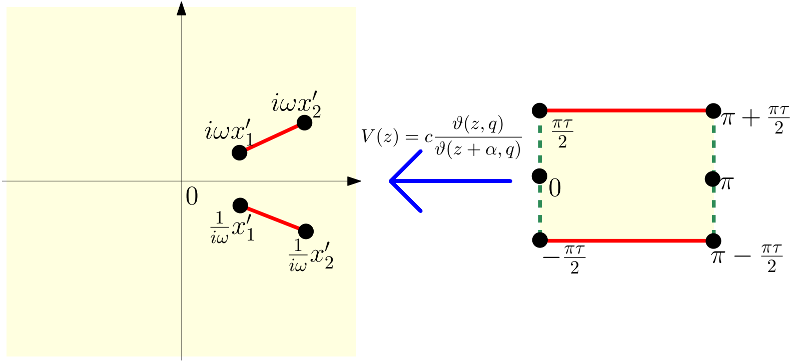


Left: $U(v)$ is analytic on yellow region.

$$V(z) = c\frac{\vartheta(z,q)}{\vartheta(z+\alpha,q)},$$

where $\omega = ie^{i\alpha}$ and $q = e^{2\pi i\tau}$.

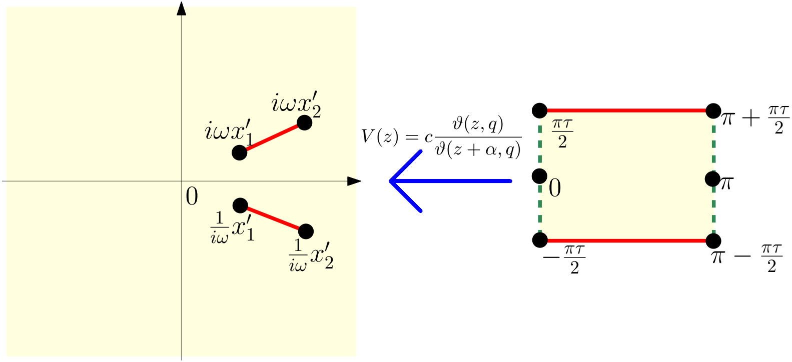
Solving for $U(v)$



$$U(i\omega(x \pm i0)) = U(-i\omega^{-1}(x \mp i0))$$

$$\Rightarrow \qquad U(V(z + \frac{\pi\tau}{2})) = U(V(z - \frac{\pi\tau}{2}))$$

Solving for $U(v)$



$$U(i\omega(x \pm i0)) = U(-i\omega^{-1}(x \mp i0))$$
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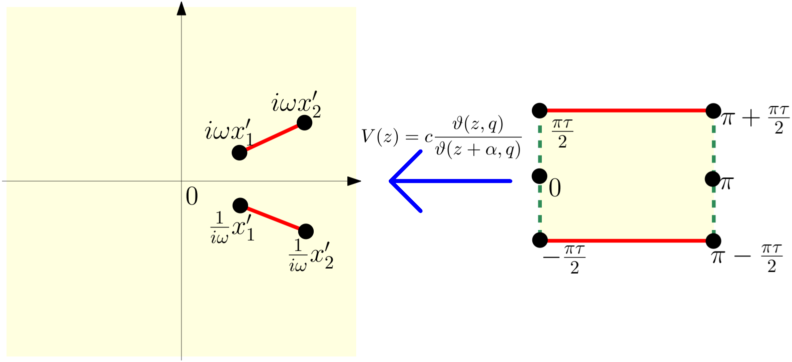
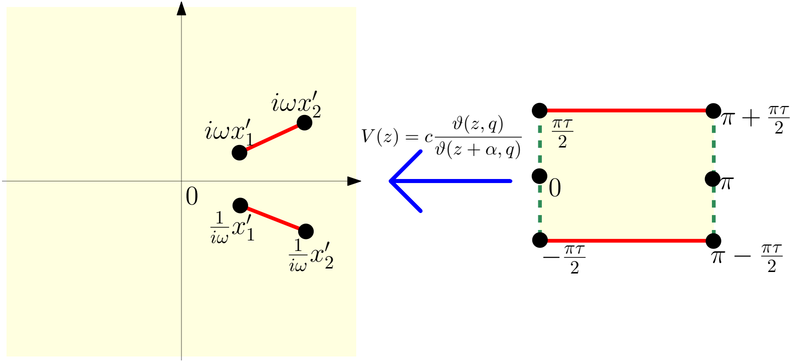
Solving for $U(v)$



$$U(i\omega(x \pm i0)) = U(-i\omega^{-1}(x \mp i0))$$

$$\Rightarrow \quad U(V(z+\pi\tau)) = U(V(z))$$

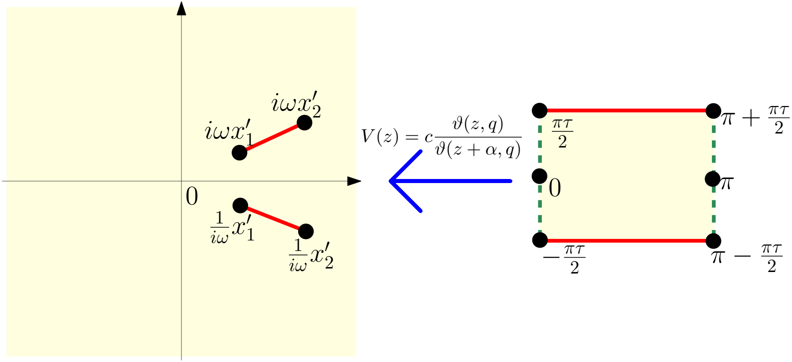
$$\Rightarrow \quad U(V(z)) = A + B\wp(z+\alpha,\tau)$$



$$U(i\omega(x \pm i0)) = U(-i\omega^{-1}(x \mp i0))$$
$$\Rightarrow \quad U(V(z+\pi\tau)) = U(V(z))$$
$$\Rightarrow \quad U(V(z)) = A + B\wp(z+\alpha,\tau)$$

Hooray, it's solved!

Solving for $U(v)$



$$
\begin{aligned}
& U(i\omega(x \pm i0)) = U(-i\omega^{-1}(x \mp i0)) \\
\Rightarrow \quad & U(V(z+\pi\tau)) = U(V(z)) \\
\Rightarrow \quad & U(V(z)) = A + B\wp(z+\alpha,\tau)
\end{aligned}
$$

$\rightarrow$ integral expression for $\mathsf{W}(x)$ and $\mathsf{H}(x,y) \rightarrow \mathsf{Q}(t,\gamma)$.

Solution for $\mathsf{Q}(t,\gamma)$

Define

$$\vartheta(z,q) = \sum_{n=0}^{\infty}(-1)^n\left(e^{(2n+1)iz} - e^{-(2n+1)iz}\right)q^{(2n+1)^2/8}.$$

Let $q = q(t,\alpha)$ be the unique series satisfying

$$t = \frac{\cos\alpha}{64\sin^3\alpha}\left(-\frac{\vartheta(\alpha,q)\vartheta'''(\alpha,q)}{\vartheta'(\alpha,q)^2} + \frac{\vartheta''(\alpha,q)}{\vartheta'(\alpha,q)}\right).$$

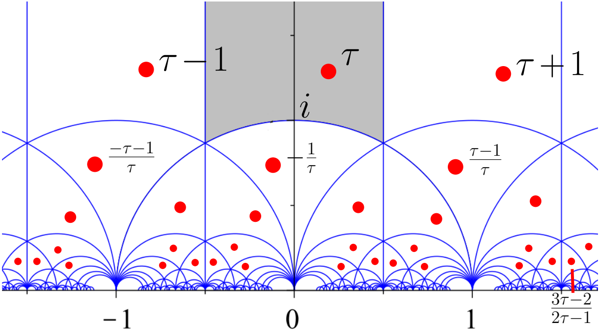
Define $\mathsf{R}(t,\gamma)$ by

$$\mathsf{R}(t,-2\cos(2\alpha)) = \frac{\cos^2\alpha}{96\sin^4\alpha}\frac{\vartheta(\alpha,q)^2}{\vartheta'(\alpha,q)^2}\left(-\frac{\vartheta'''(\alpha,q)}{\vartheta'(\alpha,q)} + \frac{\vartheta'''(0,q)}{\vartheta'(0,q)}\right).$$

Then

$$\mathsf{Q}(t,\gamma) = \frac{1}{(\gamma+2)t^2}\left(t - (\gamma+2)t^2 - \mathsf{R}(t,\gamma)\right).$$

Part 3:
Modular properties in special cases



Nice reference for modular properties of theta functions:

Elliptic Modular Forms and Their Applications, Zagier, 2008.

Recall: $\vartheta(z|\tau) = \vartheta(z, e^{i\pi\tau}) = \sum_{n=0}^{\infty}(-1)^n(e^{(2n+1)iz} + e^{-(2n+1)iz})e^{(n+1/2)^2 i\pi\tau}$

Aim: relate $\vartheta(z|\tau)$ to other τ values

Natural transformations: $\tau \to \tau + 1$ and $\tau \to -\frac{1}{\tau}$

Equations:

- $\vartheta(z|\tau + 1) = e^{i\pi/4}\vartheta(z, \tau)$
- $\vartheta\left(\frac{z}{\tau}| - \frac{1}{\tau}\right) = -i(-i\tau)^{\frac{1}{2}} \exp\left(\frac{i}{\pi\tau}z^2\right)\vartheta(z|\tau)$

These transformations generate the group of transformations

$$\tau \to \frac{a\tau + b}{c\tau + d},$$

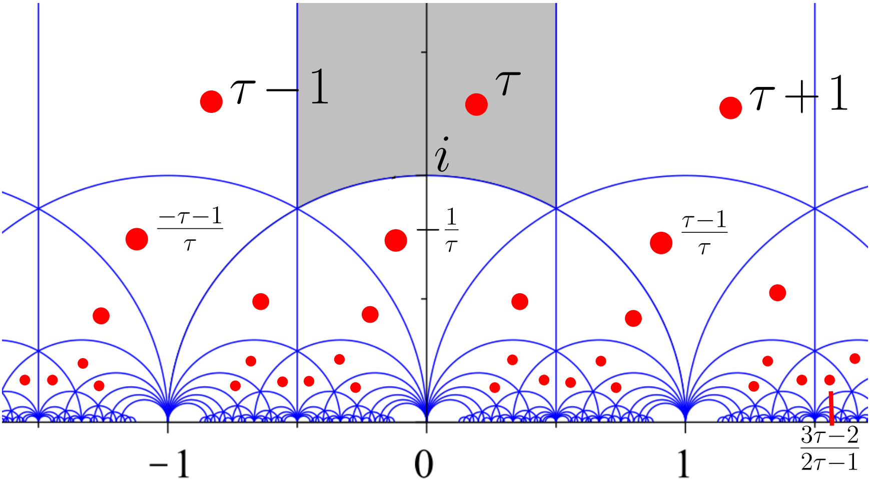
satisfying $ad - bc = 1$.

This is isomorphic to the group $SL_2(\mathbb{Z})$ of matrices

$$\begin{bmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{bmatrix},$$

with determinant 1.

Orbit of $SL_2(\mathbb{Z})$



Definition: $SL_2(\mathbb{Z})$ is the group of matrices $\begin{bmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{bmatrix}$ with determinant 1.

Action on upper half plane $\mathbb{H} = \{z \in \mathbb{C} | \mathbf{im}(z) > 0\}$**:**

$$\begin{bmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{bmatrix} \cdot \tau = \frac{a\tau + b}{c\tau + d}.$$

Modular functions

Definition: $SL_2(\mathbb{Z})$ is the group of matrices $\begin{bmatrix} \text{a} & \text{b} \\ \text{c} & \text{d} \end{bmatrix}$ with determinant 1.

Action on upper half plane $\mathbb{H} = \{z \in \mathbb{C} | \mathbf{im}(z) > 0\}$:

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Definition: Let Γ be a finite index subgroup of $SL_2(\mathbb{Z})$. A *modular function* is a meromorphic function $f : \mathbb{H} \to \mathbb{C}$ satisfying the following for all $\rho \in \Gamma$:

$$f(\rho \cdot \tau) := f\left(\frac{a\tau + b}{c\tau + d}\right) = f(\tau).$$

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Action on upper half plane $\mathbb{H} = \{z \in \mathbb{C} | \mathbf{im}(z) > 0\}$**:**

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$$f(\rho \cdot \tau) := f\left(\frac{a\tau + b}{c\tau + d}\right) = f(\tau).$$

Theorem: (classical) All modular functions are algebraically related

ALGEBRAICITY FOR $\mathsf{Q}(t,\gamma)$??

Recall:

$$t = \frac{\cos\alpha}{64\sin^3\alpha}\left(-\frac{\vartheta(\alpha,\tau)\vartheta'''(\alpha,\tau)}{\vartheta'(\alpha,\tau)^2}+\frac{\vartheta''(\alpha,\tau)}{\vartheta'(\alpha,\tau)}\right).$$

$$\mathsf{R}(t,-2\cos(2\alpha)) = \frac{\cos^2\alpha}{96\sin^4\alpha}\frac{\vartheta(\alpha,\tau)^2}{\vartheta'(\alpha,\tau)^2}\left(-\frac{\vartheta'''(\alpha,\tau)}{\vartheta'(\alpha,\tau)}+\frac{\vartheta'''(0,\tau)}{\vartheta'(0,\tau)}\right).$$

$$\mathsf{Q}(t,\gamma) = \frac{1}{(\gamma+2)t^2}\left(t-(\gamma+2)t^2-\mathsf{R}(t,\gamma)\right).$$

Algebraicity for $\mathsf{Q}(t,\gamma)$??

Recall:

$$t = \frac{\cos\alpha}{64\sin^3\alpha}\left(-\frac{\vartheta(\alpha,\tau)\vartheta'''(\alpha,\tau)}{\vartheta'(\alpha,\tau)^2}+\frac{\vartheta''(\alpha,\tau)}{\vartheta'(\alpha,\tau)}\right).$$

$$\mathsf{R}(t,-2\cos(2\alpha)) = \frac{\cos^2\alpha}{96\sin^4\alpha}\frac{\vartheta(\alpha,\tau)^2}{\vartheta'(\alpha,\tau)^2}\left(-\frac{\vartheta'''(\alpha,\tau)}{\vartheta'(\alpha,\tau)}+\frac{\vartheta'''(0,\tau)}{\vartheta'(0,\tau)}\right).$$

$$\mathsf{Q}(t,\gamma) = \frac{1}{(\gamma+2)t^2}\left(t-(\gamma+2)t^2-\mathsf{R}(t,\gamma)\right).$$

Theorem [E.P. and Zinn-Justin]: if $\alpha\in\pi\mathbb{Q}$, then R and

$$\mathsf{S} = \frac{1}{t}\frac{d^2t}{d\mathsf{R}^2}$$

are both modular functions (when written as functions of τ), so they are algebraically related.

$$Q(t,\gamma) = \frac{1}{(\gamma+2)t^2}\left(t - (\gamma+2)t^2 - R(t,\gamma)\right)$$

Specific cases:

- $\gamma = 0$: $\quad \dfrac{1}{t}\dfrac{d^2 t}{dR^2} = \dfrac{4}{R(1-16R)}$.
- $\gamma = 1$: $\quad \dfrac{1}{t}\dfrac{d^2 t}{dR^2} = \dfrac{6}{R(1-27R)}$.
- $\gamma = -1$: $\quad \dfrac{1}{t}\dfrac{d^2 t}{dR^2} = \dfrac{2}{h(1+h)(1+4h)(1-8h)}$, where $R = h(1+2h)$.
- $\gamma = \frac{1+\sqrt{5}}{2}$: $\quad R$ and $S = \dfrac{1}{t}\dfrac{d^2 t}{dR^2}$ are related by

$$R = h\left(1 - \frac{1+\sqrt{5}}{2}h\right) / \left(1 + (2+\sqrt{5})h\right)^3$$

$$S = (5+\sqrt{5})\left(1 + (2+\sqrt{5})h\right)^6 / \left(h\left(1 - \frac{11-5\sqrt{5}}{2}h\right)\left(1 - \frac{11+5\sqrt{5}}{2}h\right)^2\left(1 - \frac{\sqrt{5}-1}{2}h\right)\right)$$

THE CASES $\gamma = 0, 1$

for $\gamma = 0$: the equation $\frac{1}{t}\frac{d^2t}{d\mathsf{R}^2} = \frac{4}{\mathsf{R}(1-16\mathsf{R})}$ implies

$$t = \sum_{n=0}^{\infty} \frac{1}{n+1}\binom{2n}{n}^2 \mathsf{R}(t,0)^{n+1},$$

for $\gamma = 1$: the equation $\frac{1}{t}\frac{d^2t}{d\mathsf{R}^2} = \frac{6}{\mathsf{R}(1-27\mathsf{R})}$ implies

$$t = \sum_{n=0}^{\infty} \frac{1}{n+1}\binom{2n}{n}\binom{3n}{n} \mathsf{R}(t,1)^{n+1}$$

Recall: in each case

$$\mathsf{Q}(t,\gamma) = \frac{1}{(\gamma+2)t^2}\left(t - (\gamma+2)t^2 - \mathsf{R}(t,\gamma)\right).$$

- Are $\mathsf{W}(x)$ and $\mathsf{H}(x,y)$ D-algebraic?
- If not, the full generating functions that are used in the solution are non-D-algebraic, but the single parameter generating function is D-algebraic. How common is this? Are there cases where we can be confident it doesn't happen (e.g., quarter plane walks/excursions)?

Thank you!