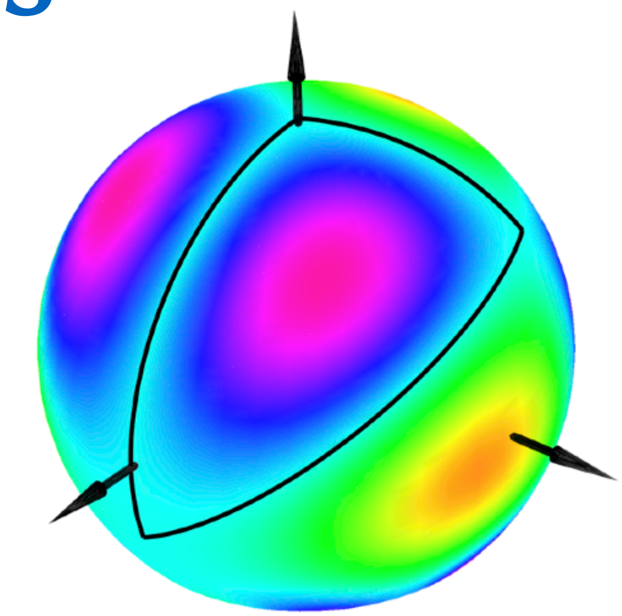
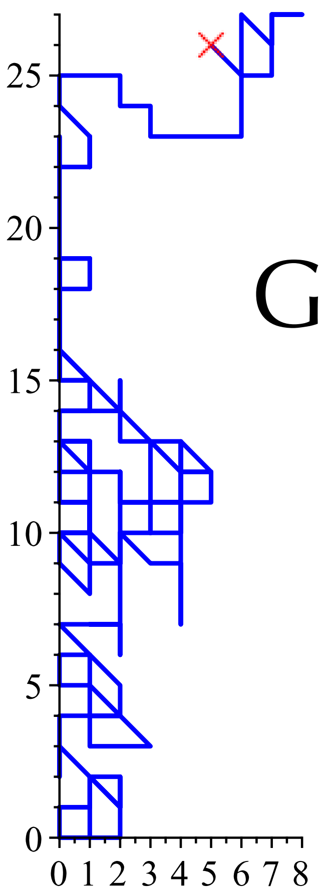
Walks in Cones and Tight Enclosures of Laplacian Eigenvalues



Bruno Salvy

AriC, Inria, ENS de Lyon



Groupe de Travail Transcendence et Combinatoire

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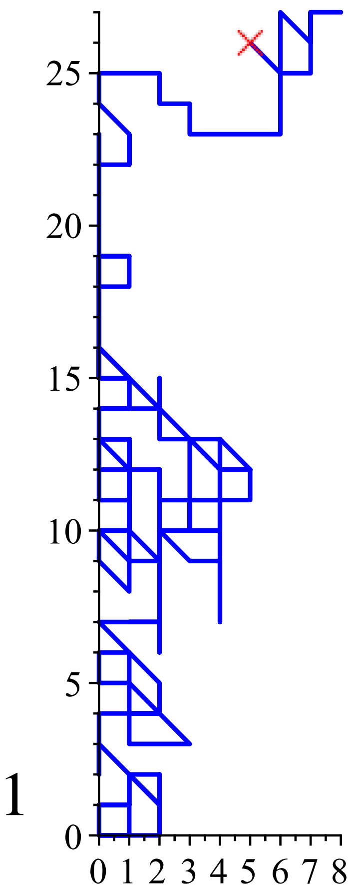
I. Long Introduction:
Walks in Cones

Lattice Walks: a Mine of Linear Recurrences Waiting for Tools



Walks from 0 to $P \in \mathbb{Z}^d$ staying in $K \subset \mathbb{R}^d$
using n steps in $\mathcal{S} \subset \mathbb{Z}^d$

Ex.: $d = 2$, $\mathcal{S} = \{\uparrow, \downarrow, \rightarrow, \leftarrow, \nwarrow\}$, $K = \mathbb{N}^2$

$$u_{i,j,n} = u_{i-1,j,n-1} + u_{i,j-1,n-1} + u_{i+1,j,n-1} + u_{i,j+1,n-1} + u_{i+1,j+1,n-1}$$

$$u_{i,j,n} = 0 \quad \text{for} \quad (i,j) \notin K$$

Generating functions:

excursions

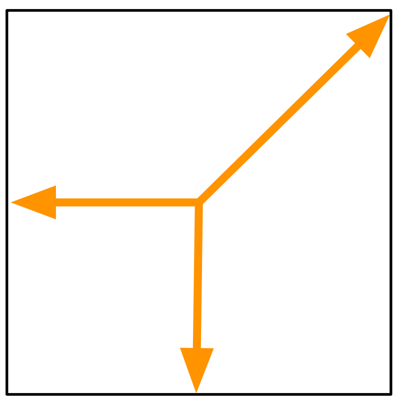
total number

$$U_K(x,y;z) := \sum_{i,j,n} u_{i,j,n} x^i y^j z^n, \quad U_K(0,0;z) = \sum_{n \geq 0} e_n z^n, \quad U_K(1,1;z) = \sum_{n \geq 0} u_n z^n.$$

Applications: queuing theory, statistical physics, combinatorics,…

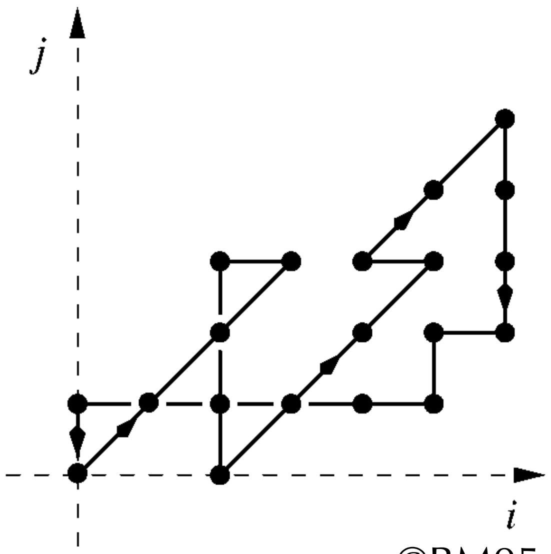
Questions: $\mathcal{S}, K \rightarrow$ asymptotics? nature of these series?

Example: Kreweras Walks



$$\mathcal{S} := \left\{ \binom{-1}{0}, \binom{0}{-1}, \binom{1}{1} \right\}$$

$$K = \mathbb{N}^2$$



Excursions:

$$e_{3n} = \frac{4^n\,(3n)!}{(n+1)!(2n+1)!} \sim C\,\frac{3^{3n}}{n^{5/2}}$$

Total number: $u_n \sim C' \dfrac{3^n}{n^{3/2}}$

$U_{\mathbb{N}^2}(x, y; z)$ is algebraic

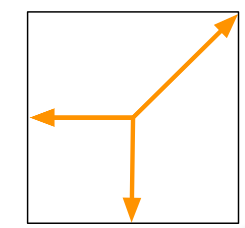
[Kreweras65, Niederhausen83, Gessel86, BousquetMelou05, Bernardi07, BernardiBousquet-MélouRaschel21]

Main Character: Generating Polynomial

$$\chi_{\mathcal{S}}(x_1, \dots, x_d) := \sum_{s \in \mathcal{S}} x_1^{s_1} \cdots x_d^{s_d}$$

Ex. Kreweras:

$$\chi := \frac{1}{x} + \frac{1}{y} + xy$$



For $k \in \mathbb{N}$, $\chi_{\mathcal{S}}^k = \sum_{m \in \mathbb{Z}^d} c_{k,m} x_1^{m_1} \cdots x_d^{m_d}$

num. walks from 0
to m in $\mathbb{Z}^d$ in k steps

Summing over k, $\dfrac{1}{1 - z\chi_{\mathcal{S}}} = \sum_{k,m} c_{k,m} x^m z^k = U_{\mathbb{Z}^d}(x; z)$

For an arbitrary cone K,

Kernel equation

$$(1 - z\chi_S) U_K(x; z) = 1 + \text{correcting terms encoding } \partial K$$

A Mysterious Secondary Character: Group of the Walk

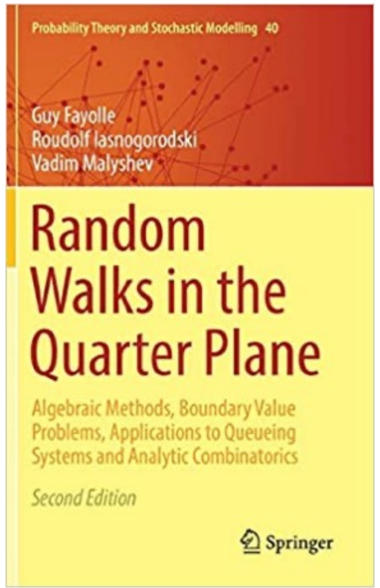
For *small-step* walks ($\max_i |s_i| = 1$, for all $s \in \mathcal{S}$)

For all $i \in \{1,\ldots,d\}$, $\chi_{\mathcal{S}} = A_i^- x_i^{-1} + A_i^0 + A_i^+ x_i$,

$$A_i^-, A_i^0, A_i^+ \in \mathbb{Z}[x_1^{\pm 1}, \ldots, x_{i-1}^{\pm 1}, x_{i+1}^{\pm 1}, \ldots, x_d^{\pm 1}]$$

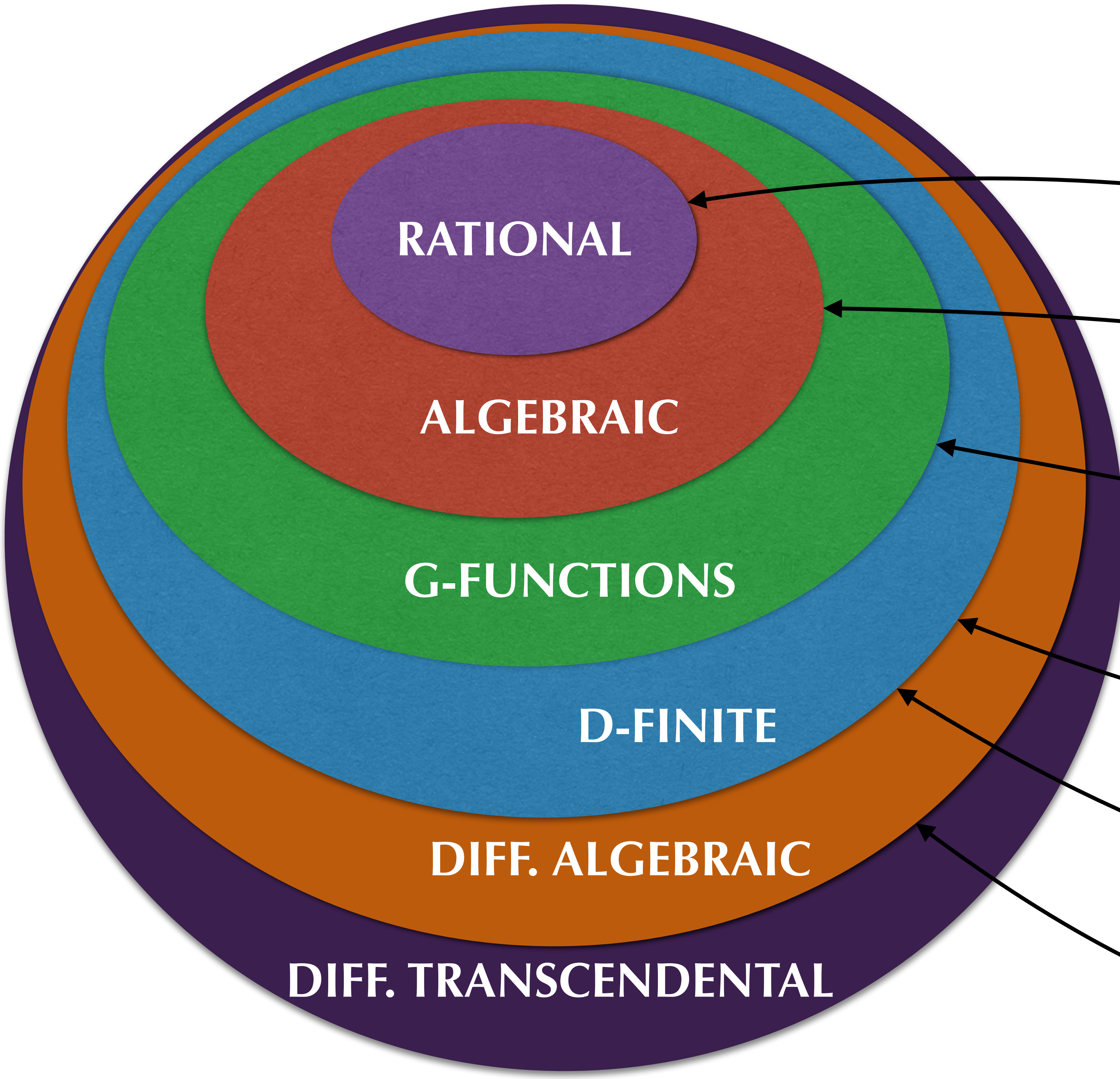
$$\psi_i : \left(x_j \mapsto x_j \text{ for } j \neq i, x_i \mapsto \frac{A_i^-}{A_i^+} \frac{1}{x_i} \right) \text{ fixes } \chi_{\mathcal{S}}$$

Group: $\mathcal{G}_{\mathcal{S}} := \langle \psi_1, \ldots, \psi_d \rangle$ generated by the ψ_i.



[Malyshev, Fayolle, Iasnogordski, Mishna, Bousquet-Mélou]

Classes of Univariate Power Series



$$U(z) := \sum_{n=0}^{\infty} u_n z^n$$

linear recurrent sequences

$P(z, U(z)) = 0$

words in context-free languages

In particular: D-finite power series with integer coefficients and radius of convergence in $(0,\infty)$

(u_n) satisfies a linear recurrence with polynomial coefficients

$U(z)$ satisfies a linear diff. eqn. with polynomial coefficients

$P(z, U(z), U'(z), \ldots) = 0$

Knowing where $U(z)$ fits helps deduce properties of (u_n) .

Generating Functions and Asymptotics

$$\text{For } U(z) := \sum_{n=0}^{\infty} u_n z^n \in \mathbb{Q}[[z]]$$

$$\text{if } u_n \sim C\rho^n n^{\alpha}, \quad n \to \infty$$

$$U \text{ rational} \implies \rho \text{ algebraic}, \alpha \in \mathbb{N},$$

Fibonacci

$$\left.\begin{array}{r} U \text{ algebraic} \\ U \text{ G-function} \end{array}\right\} \implies \rho \text{ algebraic}, \alpha \in \mathbb{Q},$$

Catalan

$$U \text{ D-finite} \implies \rho \text{ algebraic}, \alpha \in \overline{\mathbb{Q}}.$$

Conversely, asymptotics help classify.

Walks in $\mathbb{N}^2$: Recent Progress

79 models
with small steps

$\mathcal{G}_{\mathcal{S}}$ finite

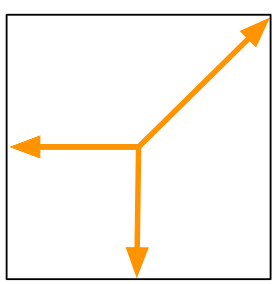
$\mathcal{G}_{\mathcal{S}}$ infinite

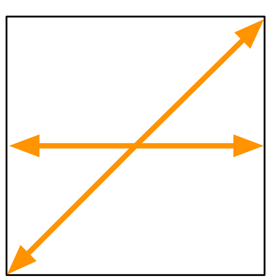
Starting point for this work

Guess & Prove (4)
algebraic
(1965,1985,2010)

Creative telescoping (19)
D-finite
(2010)

Probabilities & number theory (56)
non-D-finite
(2014)


Kreweras


Gessel

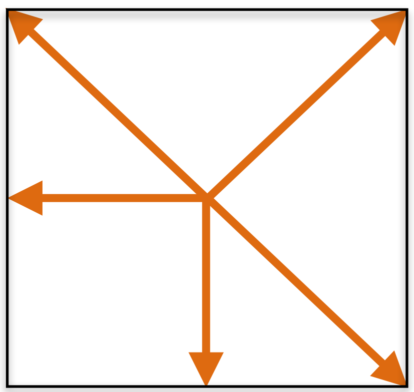
Tutte invariants
(2021)

Tutte invariants (9)
Diff. algebraic
(2021)

Differential Galois theory (47)
Diff. transcendental
(2018)

[Bernardi,Bostan,Bousquet-Mélou,Gessel,Gouyou-Beauchamps,Hardouin,Kauers, Kreweras,Melczer,Mishna,Raschel,Rechnitzer,Roques,Salvy,Singer]

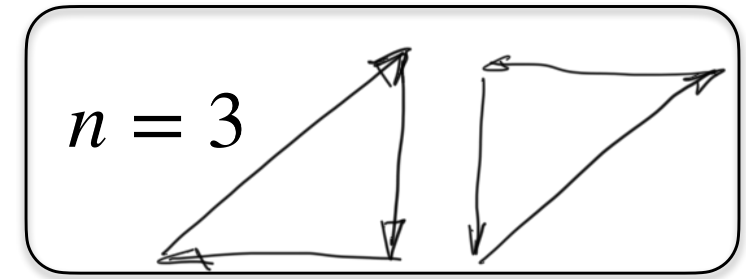
Example in $\mathbb{N}^2$

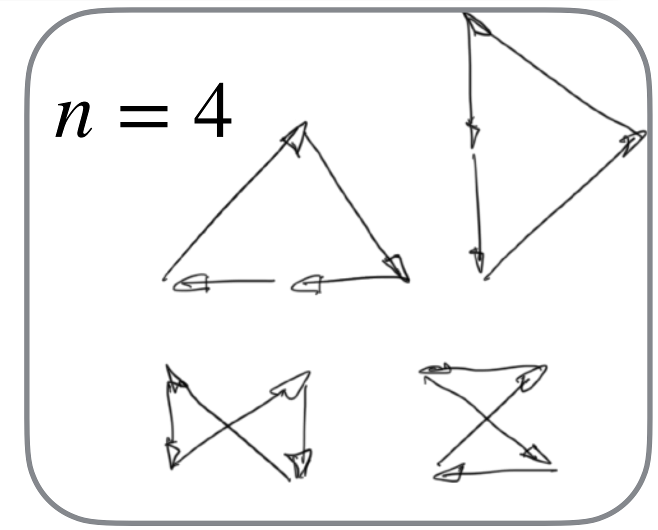


$\mathcal{S}$

Excursions:

$$(e_n) = 1,0,0,2,4,8,28,108,\dots$$





Asymptotics: $e_n \sim C\, 5^n n^\alpha$ with $\alpha = -1 - \dfrac{\pi}{\arccos(1/4)}$

Next 2 slides

$\alpha \notin \mathbb{Q}$

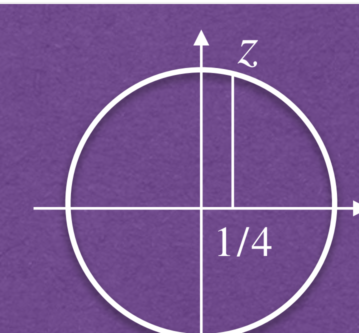
$\Rightarrow U(0,0,z)$ not D-finite

$\Rightarrow U(x,y,z)$ not D-finite

Proof: $\alpha \in \mathbb{Q}$

$\Rightarrow z$ root of unity, with

$$\frac{z+1/z}{2} = \frac{1}{4}$$



$\Rightarrow 2z^2 - z + 2$ divisible by a cyclotomic pol.

Contradiction.

[BostanRaschelSalvy2014]

Asymptotics from Probabilities (1/2)

Hyp.

$\mathcal{S} := \{s_1, \ldots, s_N\} \subset \mathbb{Z}^d$

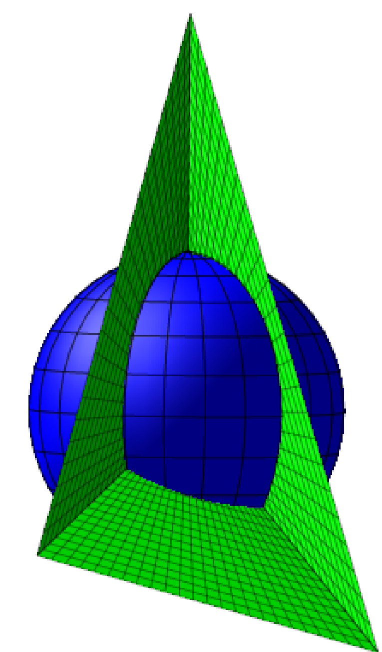
+ non-degeneracy conditions

K a cone with apex at 0

+ regularity conditions on ∂K

No drift: $\sum_i s_i = 0$

Normalized: $\left(\sum_{k=1}^{d} s_i^{(k)} s_j^{(k)} \right)_{i,j} = \mathbf{Id}$



Conclusion $\boxed{\#\text{excursions} \sim C \, |\, \mathcal{S} \,|^n n^{-\alpha_{\mathcal{S}}},}$

$\alpha_{\mathcal{S}} := \sqrt{\lambda_1 + (d/2 - 1)^2} + 1,$

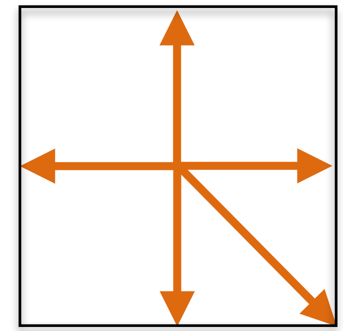
λ_1: fundamental eigenvalue of $\Delta_{\mathbb{S}^{d-1}}$ on $\mathbb{S}^{d-1} \cap K$.

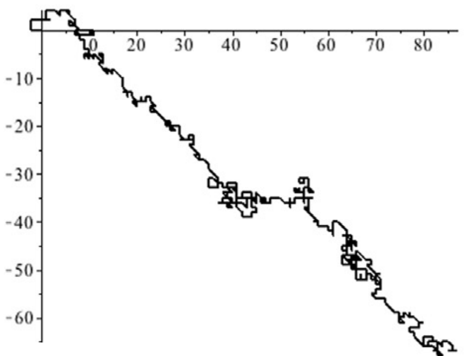
Principle: reduce to Brownian motion in K

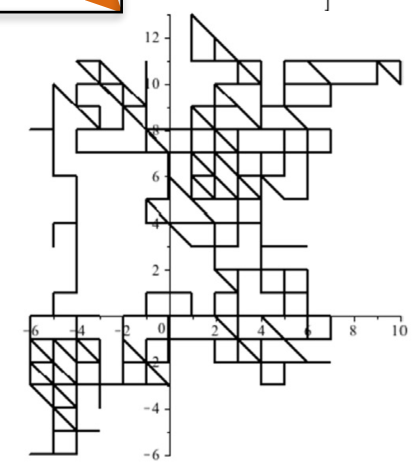
def soon

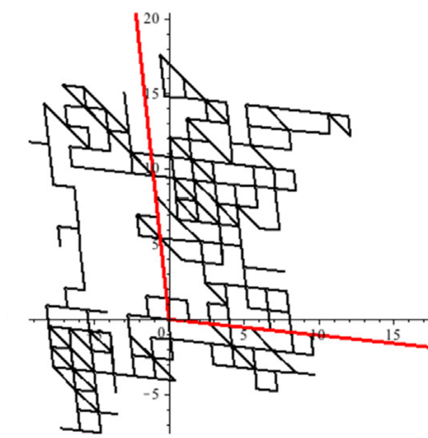
[DenisovWachtel2015]

Asymptotics from Probabilities (2/2)









Reduce cases with drifts and covariance by

1. Adding weights $1/w_i, w_i$ to the steps in direction i, for $i = 1,\ldots,d$

2. A linear change M of coordinates

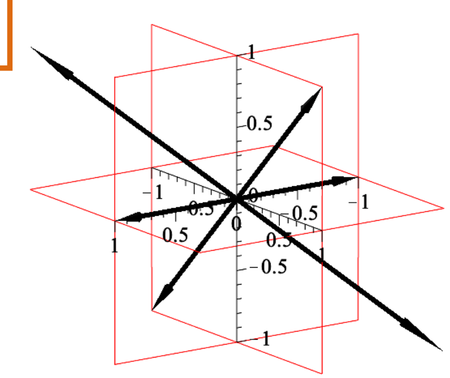
Conclusion

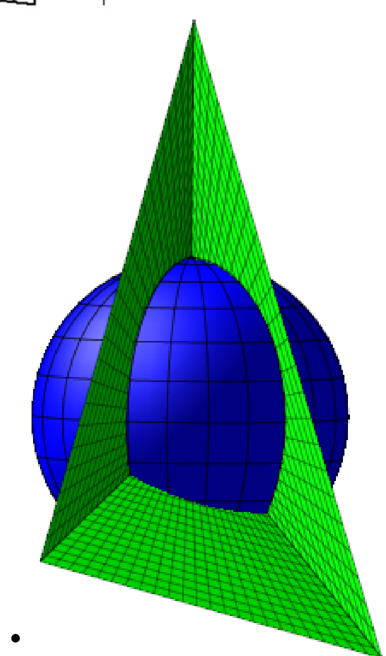
$$\boxed{\#\text{excursions} \sim C\,\rho^n n^{-\alpha_{\mathcal{S}}},}$$

$$\rho := |\mathcal{S}| \min_{(x_1,\ldots,x_d)\in\mathbb{R}_+^d} \chi(x_1,\ldots,x_d)$$

$$\alpha := \sqrt{\lambda_1 + (d/2-1)^2} + 1,$$

λ_1: fundamental eigenvalue of $\Delta_{\mathbb{S}^{d-1}}$ on $\mathbb{S}^{d-1} \cap MK$.



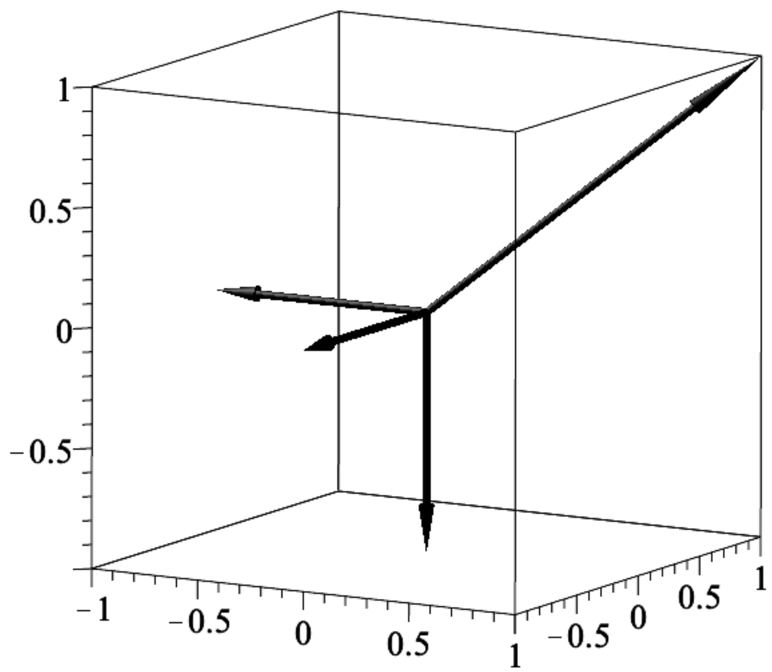


Partial combinatorial explanation for ρ

[DenisovWachtel15,JohnsonMishnaYeats18]

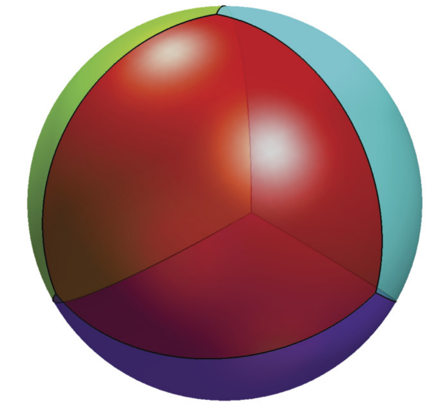
Example: Kreweras 3D

The group of the walk is finite



$$\mathcal{S} = \left\{ \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

Excursions: $e_n \sim C\,4^n n^{-\alpha_\kappa}$



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Previous estimates lead to:

$\alpha_\kappa \in [3.323, 3.326]$	(Costabel, 2008)
$\alpha_\kappa \simeq 3.32572$	(Ratzkin, Treibergs, 2009)
$\alpha_\kappa \simeq 3.3261$	(Balakrishna, 2013)
$\alpha_\kappa \simeq 3.325757004174456$	(Guttmann, 2015)
$\alpha_\kappa \simeq 3.3257569$	(Bacher et al., 2016)
$\alpha_\kappa \simeq 3.325757004175$	(Bogosel et al., 2020)

Initial goal of this work

New: If $\alpha_\kappa = p/q \in \mathbb{Q}$, then $q > 10^{51}$.

D-finiteness more and more doubtful

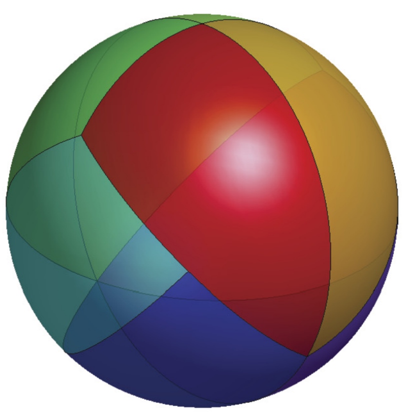
Walk-mining in $\mathbb{N}^3$

(Bogosel, Perrollaz, Raschel, Trotignon 2020)

17 spherical triangles associated to finite groups

+ computation of the corresponding angles (all in $\{\pi/4, \pi/3, \pi/2, 2\pi/3, 3\pi/4\}$)

+ exact or estimated value of the exponent



$(\pi/3, \pi/2, 2\pi/3)$

They all correspond to tilings of the sphere

Results

Angles	BPRT	new
$(3\pi/4,\pi/3,\pi/2)$	12.400051	$12.400051652843377905\ldots \pm 10^{-47}$
$(2\pi/3,\pi/3,\pi/2)$	13.744355	$13.744355213213231835\ldots \pm 10^{-84}$
$(2\pi/3,\pi/4,\pi/2)$	20.571973	$20.571973537984730557\ldots \pm 10^{-30}$
$(2\pi/3,\pi/3,\pi/3)$	21.309407	$21.309407630190445260\ldots \pm 10^{-206}$
$(3\pi/4,\pi/4,\pi/3)$	24.456913	$24.456913796299111694\ldots \pm 10^{-73}$
$(2\pi/3,\pi/4,\pi/4)$	49.109945	$49.109945263284609920\ldots \pm 10^{-153}$
$(2\pi/3,3\pi/4,3\pi/4)$	4.261734	$4.2617347552939870857\ldots \pm 10^{-22}$
$(2\pi/3,2\pi/3,2\pi/3)$	5.159145	$5.1591456424665417112\ldots \pm 10^{-104}$
$(\pi/2,2\pi/3,3\pi/4)$	6.241748	$6.2417483307263342368\ldots \pm 10^{-20}$
$(\pi/2,2\pi/3,2\pi/3)$	6.777108	$6.7771080545983009573\ldots \pm 10^{-35}$

finite elements &
convergence acceleration

[BogoselPerrollazRaschelTrotignon20]

Next: How do we do it?

and why are the precisions so different?

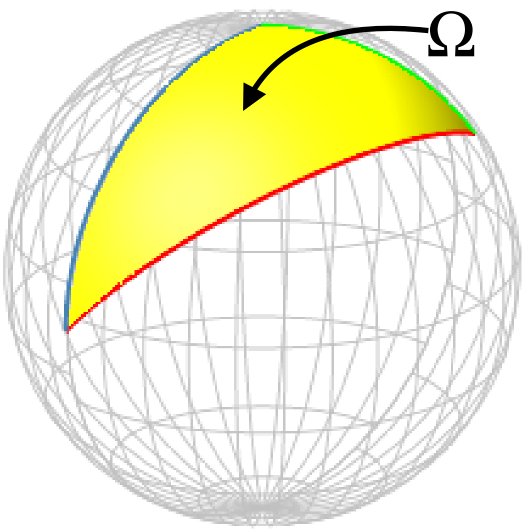
II. Laplacian on Spherical Triangles

Fundamental Eigenvalue of the Laplace-Beltrami Operator on the Unit Sphere

Laplace operator in spherical coordinates in $\mathbb{R}^d$

$$\Delta f = r^{1-d}\frac{\partial}{\partial r}\left(r^{d-1}\frac{\partial f}{\partial r}\right) + r^{-2}\Delta_{\mathbb{S}^{d-1}} f$$

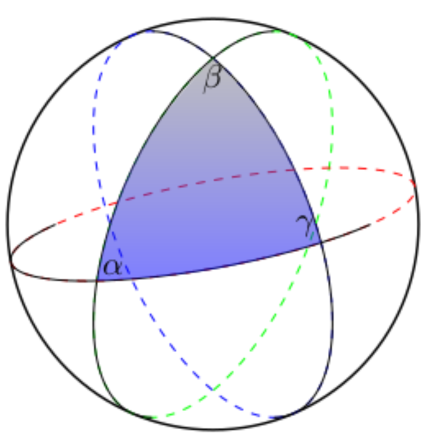
Laplace-Beltrami on the sphere



Eigenvalue problem for $\Omega \subset \mathbb{S}^{d-1}$:

$$\Delta_{\mathbb{S}^{d-1}} f + \lambda f = 0 \text{ in } \Omega, \quad f|_{\partial\Omega} = 0.$$

Dirichlet condition



Goal: $(\alpha, \beta, \gamma) \mapsto \lambda_1$ with high precision (dimension d=3)

Basic Properties of the Laplacian over a Bounded Domain $\Omega \subset \mathbb{R}^d$

Hold also for $\Delta_{\mathbb{S}^{d-1}}$

$$(u, v) := \frac{1}{\mathrm{Vol}\,\Omega} \int_\Omega uv\, d\sigma$$

. self-adjoint: $(\Delta u, v) = (u, \Delta v)$ on $\{f \in C^2(\Omega),\ f|_{\partial\Omega} = 0\}$

. discrete spectrum with no accumulation point

fundamental eigenvalue

$$0 < \lambda_1 < \lambda_2 \leq \cdots, \quad \lambda_n \to \infty$$

. corresponding eigenfunctions (u_n) Hilbert basis of $L_2(\Omega)$

. maximum principle: $\Delta u \geq 0$ in $\Omega \Rightarrow \sup\limits_{x\in\Omega} u(x) \leq \sup\limits_{x\in\partial\Omega} u(x)$

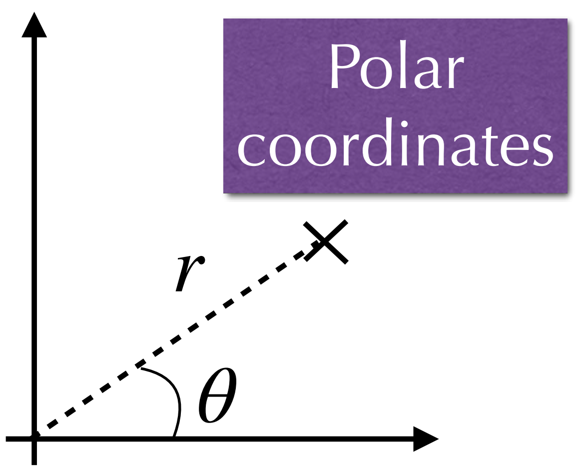
. monotonicity: $\Omega \subset \Omega' \Rightarrow \lambda_i(\Omega) \geq \lambda_i(\Omega'),\quad$ for all i

. Faber-Krahn inequality: $\lambda_1(\Omega) \geq \lambda_1(\Omega^\star)$,

$$\Omega^\star = \begin{cases} \text{ball with the same volume for } \Delta, \\ \text{spherical cap with the same area for } \Delta_{\mathbb{S}^{d-1}}. \end{cases}$$

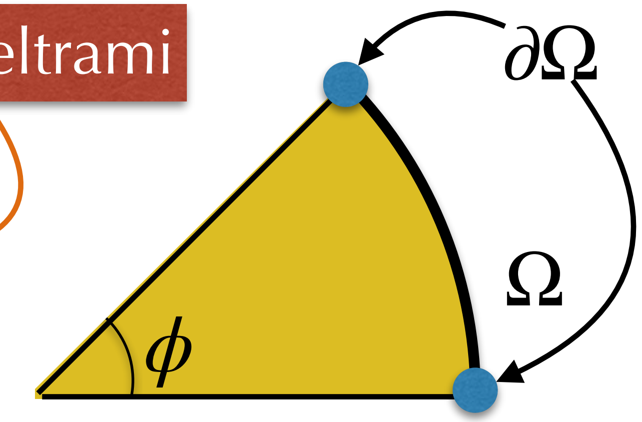
Matlab logo

Planar Case

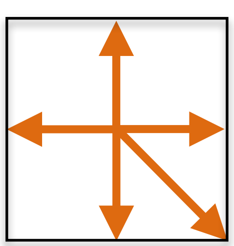


Laplace-Beltrami

$$\Delta f(r,\theta) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}$$



Eigenfunctions: $\sin(\sqrt{\lambda}\theta + c)$



Boundary: $\begin{cases} \theta = 0 \to c = 0 \\ \theta = \phi \to \lambda = \lambda_k := \left(\dfrac{k\pi}{\phi}\right)^2, \; k \in \mathbb{N}^\star . \end{cases}$

Ex. $\mathcal{S} = \{\uparrow, \downarrow, \rightarrow, \leftarrow, \searrow\}$, $\phi = \arccos(u)$, $\mu_u(u) = 0$, $\mu_u = 8t^3 - 8t^2 + 6t - 1$

exponent $\alpha = 1 - \dfrac{\pi}{\arccos(u)} \notin \mathbb{Q} \Rightarrow$ U not D-finite.

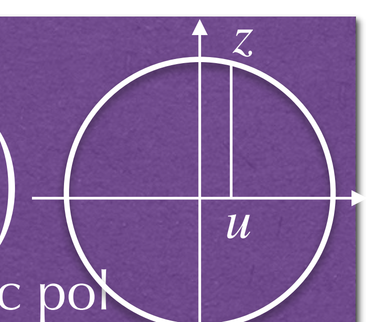
Automatic proof of 51 of the 56 non-Dfinite cases

Proof: $\alpha \in \mathbb{Q}$

$\Rightarrow$ numer $\mu_u\left(\dfrac{z + 1/z}{2}\right)$

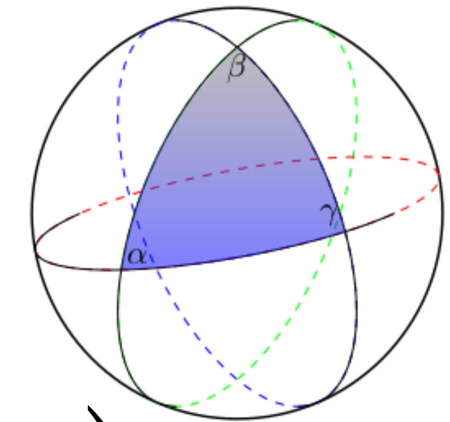
divisible by a cyclotomic pol

Contradiction.



[BostanRaschelSalvy14]

Spherical Triangles



Eigenvalues known when $(\alpha, \beta, \gamma) = \left(\frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{r}\right)$

Only possible (p, q, r) that give triangles:

. $(2,3,3) \longrightarrow \lambda = k(k+1),\ k \in 6 + 3\mathbb{N} + 4\mathbb{N}$

. $(2,3,4) \longrightarrow \lambda = k(k+1),\ k \in 9 + 6\mathbb{N}$

. $(2,3,5) \longrightarrow \lambda = k(k+1),\ k \in 15 + 6\mathbb{N} + 10\mathbb{N}$

. $(2,2,r) \longrightarrow \lambda = k(k+1),\ k \in r + 1 + 2\mathbb{N} + r\mathbb{N}$

This solves 7 of the 17 finite groups for 3D walks

$$\alpha := \sqrt{\lambda_1 + (d/2 - 1)^2} + 1 \in \mathbb{Q}$$

No other value known —> turn to numerical computation

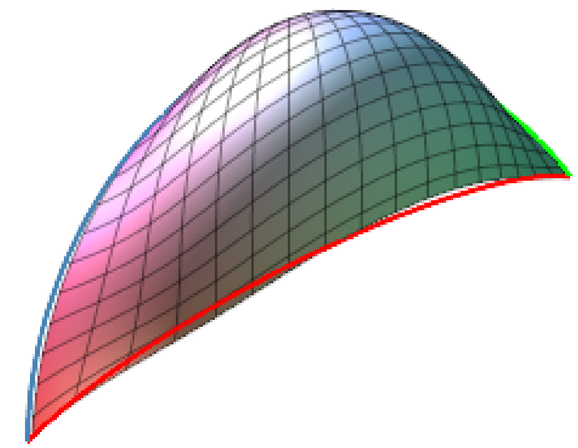
[Berard1983,BogoselPerrollazRaschelTrotignon18]

Bounds from an Approximate Eigenvalue

Eigenvalue problem for Ω :

$$\Delta f + \lambda f = 0 \text{ in } \Omega, \quad f|_{\partial\Omega} = 0.$$

replace by f small



Thm. If $\Delta f^\star + \lambda^\star f^\star = 0$ in Ω, then there exists λ s.t.

$$\frac{|\lambda - \lambda^\star|}{\lambda} \leq \frac{\sup_{x\in\partial\Omega} |f^\star(x)|}{\|f^\star\|_2}.$$

Method:

1. Find a good approximate pair $(f^\star, \lambda^\star)$
2. Compute the bound in a certified way
3. Certify the index

Cost of certification: not large

[FoxHenriciMoler67,MolerPayne68]

$$\boxed{\frac{|\lambda - \lambda^\star|}{\lambda} \leq \frac{\sup_{x\in\partial\Omega} |f^\star(x)|}{\|f^\star\|_2}.}$$

Proof

Use

(u_n) orthonormal basis, with $\Delta u_n = \lambda_n u_n$, $u_n|_{\partial_\Omega} = 0$

w solution of $\Delta w = 0$, $w|_{\partial\Omega} = f^\star|_{\partial\Omega}$

If small, $\|w\|$ small

Coefficients of w: $(w, u_n) = (w - f^\star, u_n) + (f^\star, u_n) = \frac{1}{\lambda_n}(w - f^\star, \Delta u_n) + (f^\star, u_n),$

Δ self-adjoint

$$= \frac{1}{\lambda_n}(\Delta w - \Delta f^\star, u_n) + (f^\star, u_n) = \left(1 - \frac{\lambda^\star}{\lambda_n}\right)(f^\star, u_n).$$

Take λ where min is reached

$$|(w, u_n)| \geq \left|1 - \frac{\lambda^\star}{\lambda}\right| |(f^\star, u_n)|,$$

discrete spectrum with no accumulation point

Square and sum

$$\left(\sup_{x\in\partial\Omega} f^\star\right)^2 \geq \|w\|^2 \geq \left|1 - \frac{\lambda^\star}{\lambda}\right|^2 \|f^\star\|^2.$$

max. ppl

Orthonormality and completeness

[MolerPayne68]

1. Find a good approximate pair $(f^\star, \lambda^\star)$
2. Compute the bound in a certified way
3. Certify the index

Step 1. Find a good approximate pair $(f^\star, \lambda^\star)$

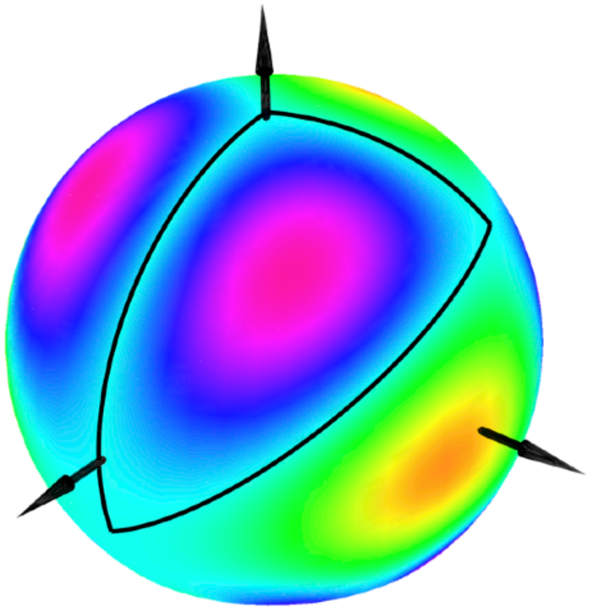
$$\frac{|\lambda - \lambda^\star|}{\lambda} \leq \frac{\sup_{x\in\partial\Omega} |f^\star(x)|}{\|f^\star\|_2}.$$

High precision needed, and no guarantee

Method of Particular Solutions

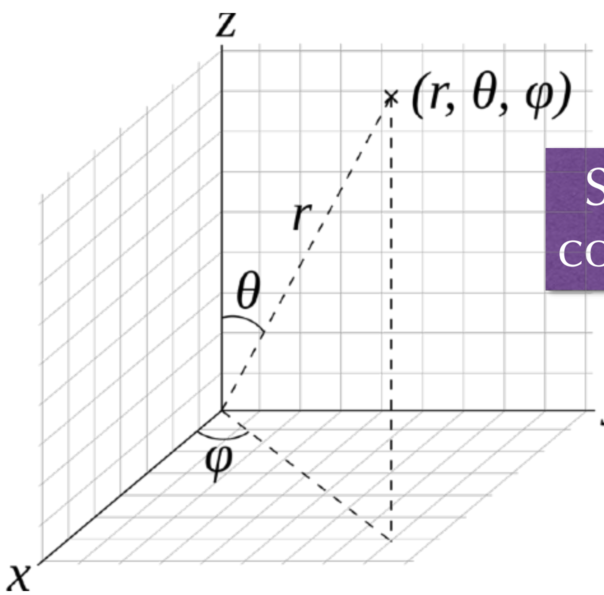
$$\boxed{\text{Target: } \Delta f^\star + \lambda^\star f^\star = 0 \text{ in } \Omega, \ \sup_{x\in\partial\Omega} |f^\star| \text{ small.}}$$

1. Fix λ
2. Find a set $(u_\lambda^{(k)})_{k=1}^N$ of solutions of $\Delta f + \lambda f = 0$ in Ω
3. Find a linear combination $\sum_{k=1}^{N} c_k u_\lambda^{(k)}$ that is
   - . small on $\partial\Omega$
   - . *not too small on* Ω
4. Repeat to minimize $\sup_{x\in\partial\Omega}$ over λ

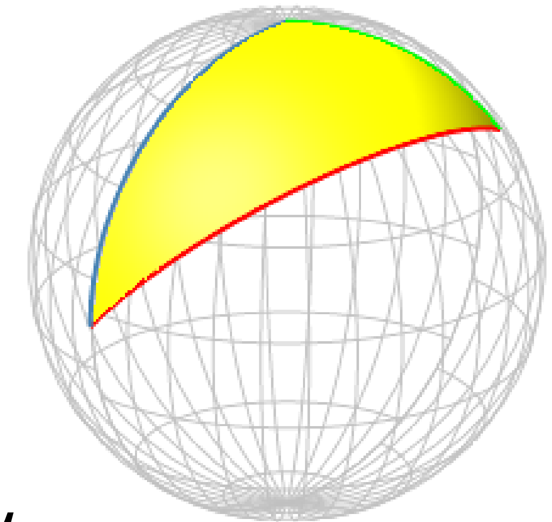


[Bergman47,Vekua48,FoxHenriciMoler67,BetckeTrefethen05,Betcke08]

2. Set of Eigenfunctions



Spherical coordinates



$$\Delta_{\mathbb{S}^2} u(\theta, \phi) = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}$$

Separation of variables: $u = f(\phi)g(\theta)$ gives

$$\frac{g''(\phi)}{g(\phi)} + \sin^2\theta\frac{f''(\theta)}{f(\theta)} + \sin\theta\cos\theta\frac{f'(\theta)}{f(\theta)} + \lambda\sin^2\theta = 0$$

$$g(\phi) = \sin(\mu\phi + c), \quad f(\theta) = \mathsf{P}_\nu^\mu(\cos\theta) \quad (\mu \leq 0)$$

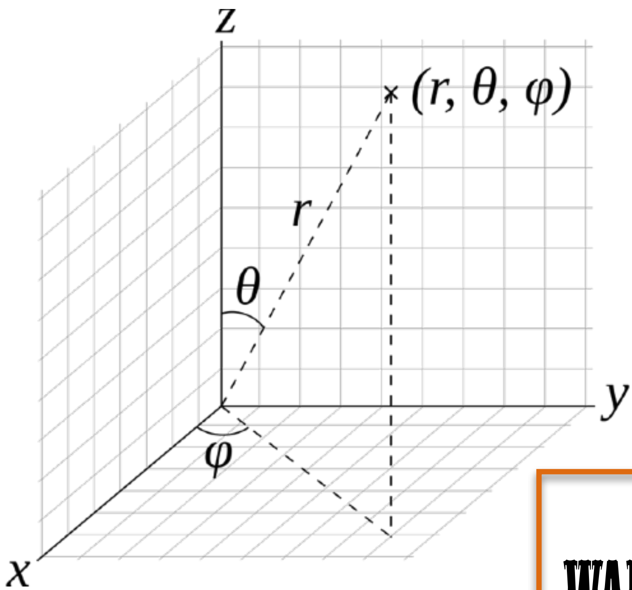
Ferrers function of the 1st kind (D-finite; generalize the Legendre functions)

with $\lambda = \nu(\nu+1)$.

First 2 boundaries:
$$\begin{cases} \phi = 0 & \to c = 0 \\ \phi = \phi_{\max} & \to \mu = \mu_k := -\dfrac{k\pi}{\phi_{\max}}, \ k \in \mathbb{N}. \end{cases}$$

$$\boxed{u_\lambda^{(k)}(\theta, \phi) = \sin(\mu_k\phi)\mathsf{P}_\nu^{\mu_k}(\cos\theta)}$$

3. Small on $\partial\Omega$, Not too Small on Ω

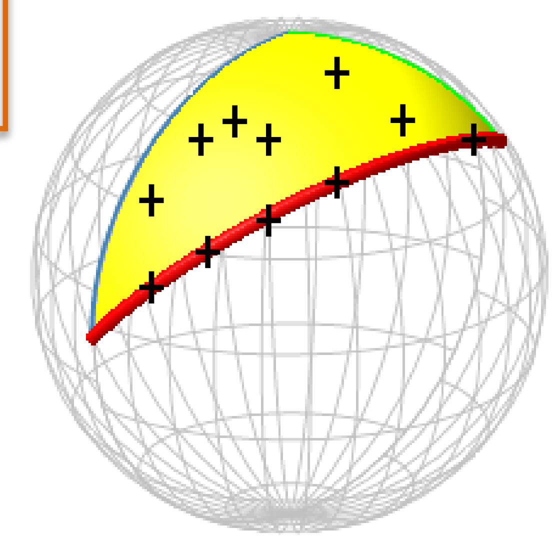


$$u_\lambda(\phi) := \sum_{k=1}^{N} c_k \sin(\mu_k \phi) \mathsf{P}_\nu^{\mu_k}(\cos\theta(\phi))$$

with $\lambda = \nu(\nu+1)$

satisfies $\Delta u_\lambda + \lambda u_\lambda = 0$

WANTED: c_k s.t. $u_\lambda|_{[0,\phi_{\max}]} \approx 0$, $\|u_\lambda\|_\Omega \approx 1$.



Choose $x_1, \ldots, x_{m_b}$ on $\partial\Omega$, $y_1, \ldots, y_{m_i}$ inside Ω;

Form a matrix $A := \begin{pmatrix} u_\lambda^{(k)}(x_i) \\ u_\lambda^{(k)}(y_i) \end{pmatrix}$

Orthonormal basis of Im A

Compute its QR factorization $A = \begin{pmatrix} Q_{\partial\Omega} \\ Q_\Omega \end{pmatrix} R$

$\sigma := \min_{\|v\|=1} \|Q_{\partial\Omega} v\|$ found together with v by SVD (least squares)

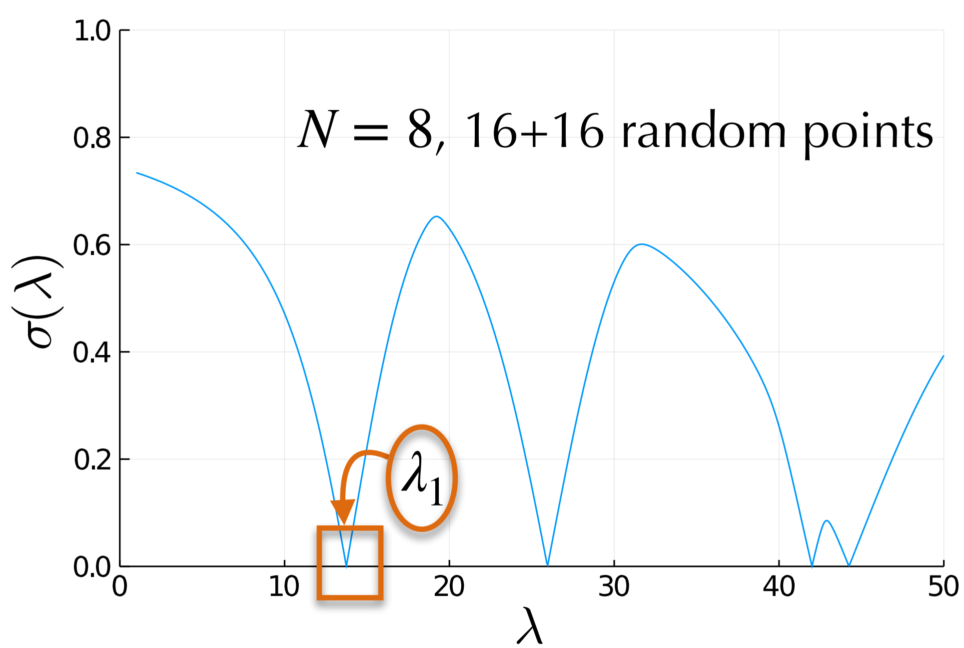
Then $\|Q_\Omega v\|^2 = 1 - \sigma^2$ not too small

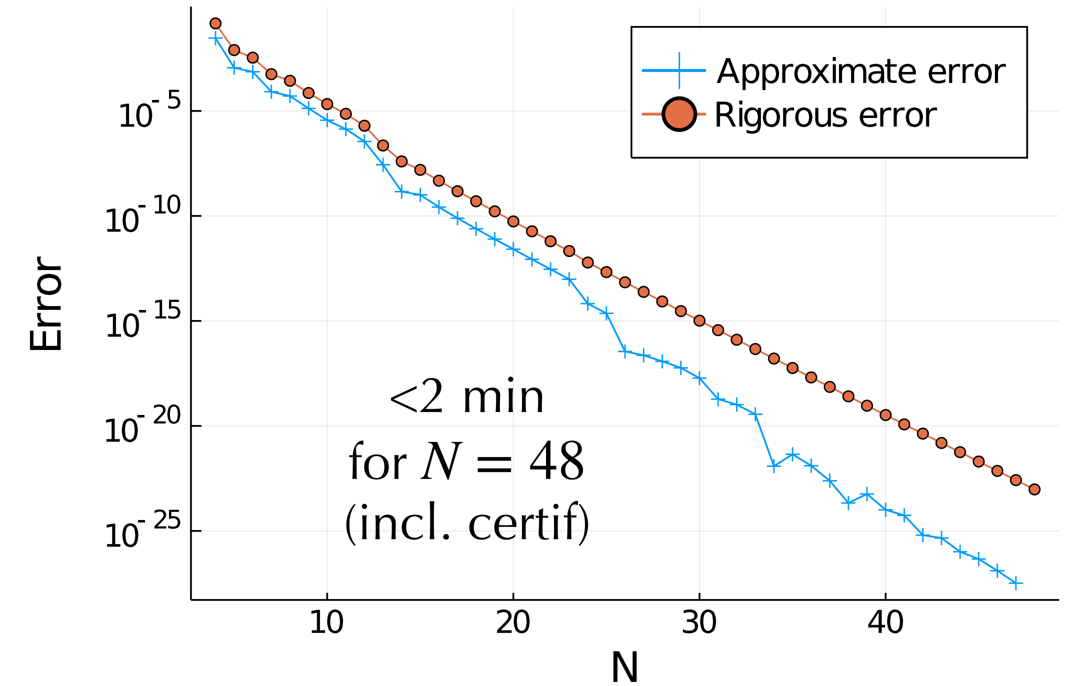
Recover c by solving $Rc = v$.

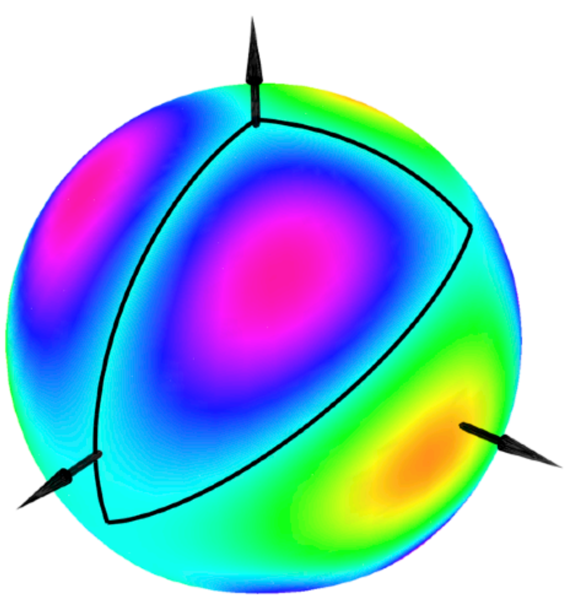
[BetckeTrefethen05]

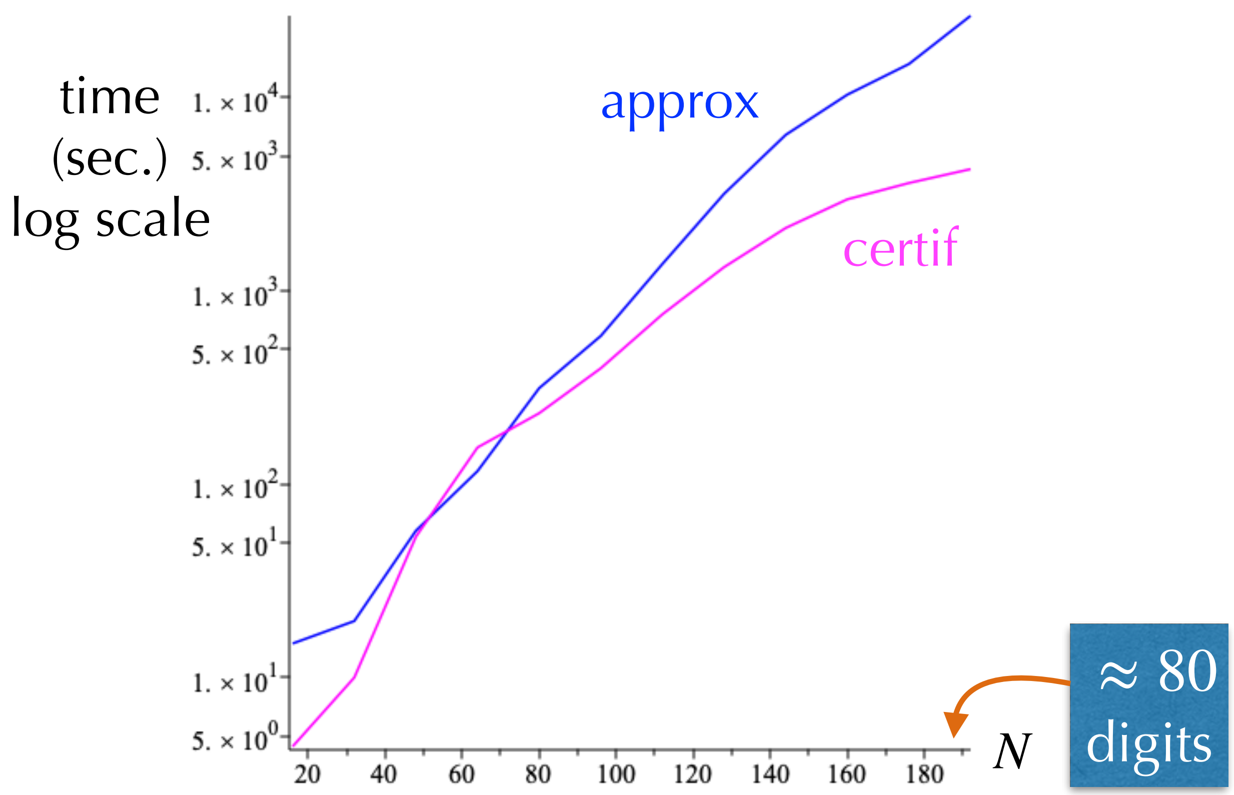
4. Optimize over λ

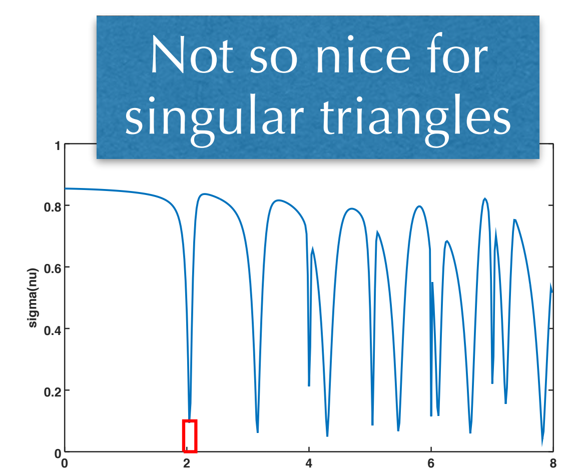
Ex. Regular Triangle: $(2\pi/3, \pi/3, \pi/2)$





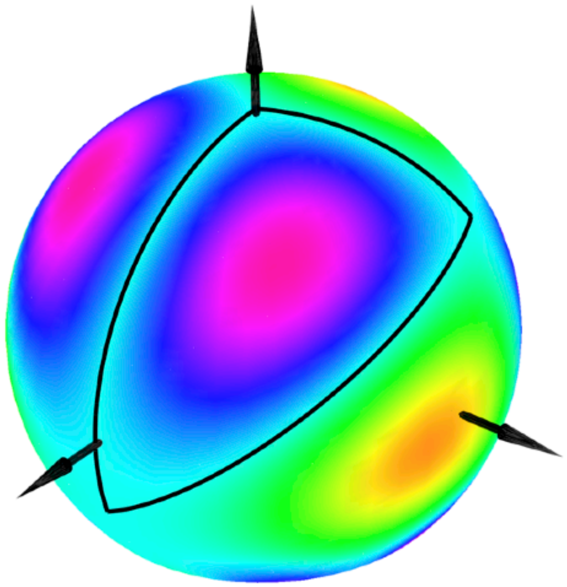






$(\pi/2, 2\pi/3, 3\pi/4)$

Regular vs Singular Triangles



Def. a corner is regular if its angle is π/k, $k \in \mathbb{N}^\star$, singular otherwise.

At a regular corner, eigenfunctions can be continued analytically (by reflection).

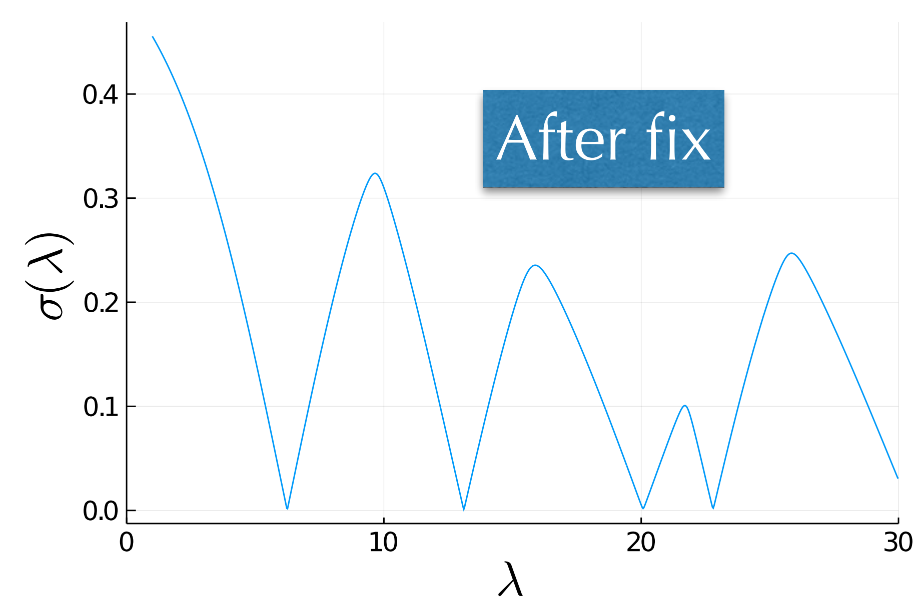
Expansions from a singular corner converge well when the other corners are regular, poorly otherwise.

3d-Kreweras is singular

Fix: use a sum of 4 expansions

$$f^\star(\theta,\phi) = u_1(\theta_1,\phi_1) + u_2(\theta_2,\phi_2) + u_3(\theta_3,\phi_3) + u_{\text{int}}(\theta_{\text{int}},\phi_{\text{int}})$$

One from each corner, one from an interior point



1. Find a good approximate pair $(f^\star, \lambda^\star)$
2. Compute the bound in a certified way
3. Certify the index

Step 2. Rigorous Bounds

$$\frac{|\lambda - \lambda^\star|}{\lambda} \leq \frac{\sup_{x \in \partial\Omega} |f^\star(x)|}{\|f^\star\|_2}.$$

Basic Tool: Interval Arithmetic

Replace all floating-point operations by set operations

$$[1.2,1.3] + [2.0,2.1] = [3.2,3.4]$$

$$[1.2,1.3] \times [2.0,2.1] = [2.40,2.73]$$

provides certified enclosures

Implementation requires care with rounding modes

We use https://arblib.org/

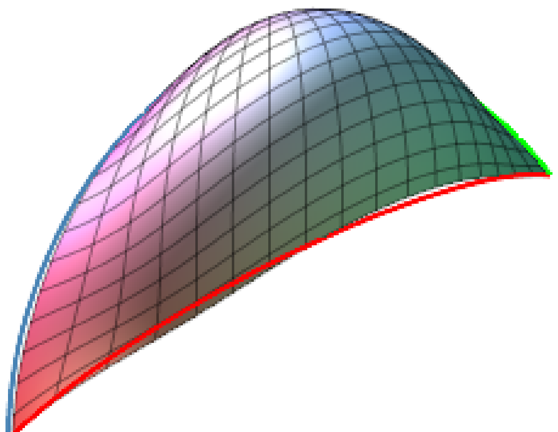
Weakness: wrapping effect

$$f := e^{-t} - (1 - t + t^2/2! + \cdots - t^9/9!)$$

$$f([1.0,1.1]) = [-0.161,0.161] \text{ while } f : [1.0,1.1] \mapsto [2.5\,10^{-7},6.5\,10^{-7}]$$

Situation very similar to our

$$f^\star = \sum c_k u_\lambda^{(k)} \text{ on } \partial\Omega$$



Upper Bound on the Boundary

$$\frac{|\lambda - \lambda^\star|}{\lambda} \le \frac{\sup_{x\in\partial\Omega}|f^\star(x)|}{\|f^\star\|_2}$$

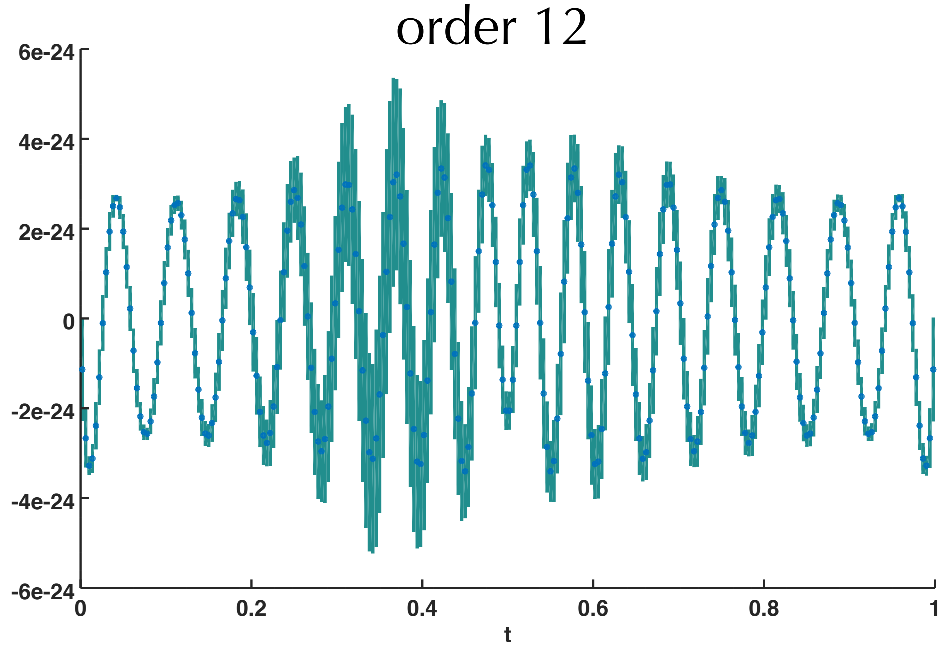
Taylor model:

$$\max_{t\in I} f^\star(\gamma(t)) \le \max_{t\in I} P_{\ell-1}(t) + \boxed{\frac{(|I|/2)^\ell}{\ell\,!}} \max_{t\in I} \left| \frac{d^\ell}{dt^\ell} f^\star(\gamma(t)) \right|$$

Taylor expansion at the midpoint
Coefficients via linear rec.

small

interval evaluation



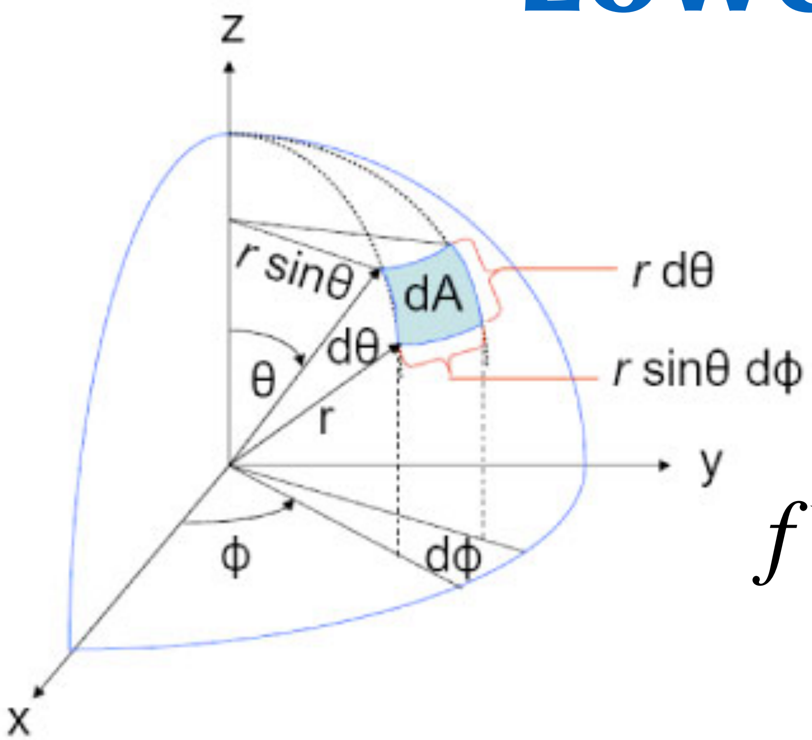
A lower bound on $\min f^\star$ also follows

Interval-evaluate
at enclosures of roots of the
derivative

The expensive part of the certification

[MakinoBerz03]

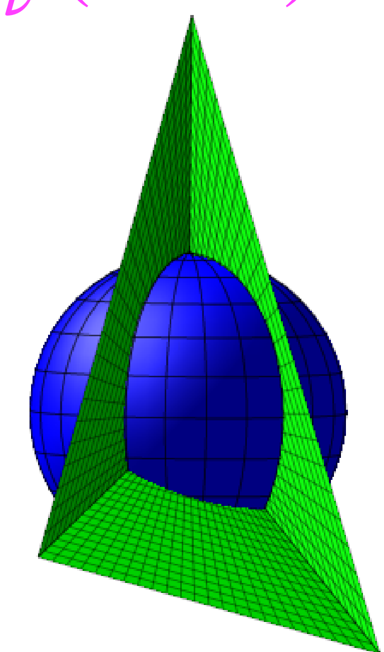
Lower Bound on the Norm



Regular Triangles

$$\frac{|\lambda - \lambda^\star|}{\lambda} \leq \frac{\sup_{x \in \partial\Omega} |f^\star(x)|}{\|f^\star\|_2}$$

$$f^\star = \sum c_k u_k^{(\lambda)} \qquad u_\lambda^{(k)}(\theta, \phi) = \sin(\mu_k \phi) \mathsf{P}_\nu^{\mu_k}(\cos\theta)$$



$$\int_0^{\phi_{\max}} \int_0^{\theta(\phi)} f^\star(\theta, \phi)^2 \, \sin\theta \, d\theta \, d\phi \geq \int_0^{\phi_{\max}} \int_0^{\theta_{\min}} f^\star(\theta, \phi)^2 \, \sin\theta \, d\theta \, d\phi$$

Orthogonality of $\sin(\mu_k \phi)$

$$\geq \frac{\phi_{\max}}{2} \sum c_k^2 \underbrace{\int_0^{\theta_{\min}} (\mathsf{P}_\nu^{\mu_k}(\cos\theta))^2 \sin\theta \, d\theta}$$

Rigorous interval evaluation (Arb)

Lower Bound on the Norm

Singular Triangles

$$\frac{|\lambda - \lambda^\star|}{\lambda} \le \frac{\sup_{x\in\partial\Omega}|f^\star(x)|}{\|f^\star\|_2}$$

1. When $f^\star \ge 0$ on $\Omega' \subset \Omega$, $\min_{\Omega'} f^\star \ge \min_{\partial\Omega'} f^\star$

Maximum principle

2. $f^\star|_{\partial\Omega'} > 0$ and Ω' sufficiently small $\Rightarrow f^\star > 0$ on Ω'

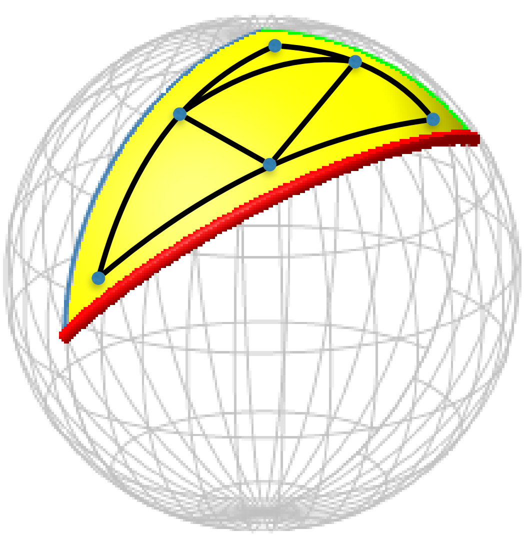
Consequence of Faber-Krahn

$$\Delta f^\star|_{\partial\Omega''} = 0 \text{ for } \Omega'' \subset \Omega' \Rightarrow \mathrm{Vol}\,\Omega'' \ge \mathrm{Vol}\,\Omega^\star_{\lambda^\star}$$

can be computed from zeros of Legendre functions

$\Omega^\star_{\lambda^\star}$: spherical cap with fundamental eigenvalue $\lambda^\star$

Conclusion: subdivide and minimize with Taylor models



4 triangles sufficient in our computations

[Gómez-SerranoOrriols21]

Results

Angles	eigenvalue
$(3\pi/4,\pi/3,\pi/2)$	$12.40005165284337790 5\ldots \pm 10^{-47}$
$(2\pi/3,\pi/3,\pi/2)$	$13.7443552132132318 35\ldots \pm 10^{-84}$
$(2\pi/3,\pi/4,\pi/2)$	$20.5719735379847305 57\ldots \pm 10^{-30}$
$(2\pi/3,\pi/3,\pi/3)$	$21.3094076301904452 60\ldots \pm 10^{-206}$
$(3\pi/4,\pi/4,\pi/3)$	$24.4569137962991116 94\ldots \pm 10^{-73}$
$(2\pi/3,\pi/4,\pi/4)$	$49.1099452632846099 20\ldots \pm 10^{-153}$
$(2\pi/3,3\pi/4,3\pi/4)$	$4.26173475529398708 57\ldots \pm 10^{-22}$
$(2\pi/3,2\pi/3,2\pi/3)$	$5.15914564246654171 12\ldots \pm 10^{-104}$
$(\pi/2,2\pi/3,3\pi/4)$	$6.24174833072633423 68\ldots \pm 10^{-20}$
$(\pi/2,2\pi/3,2\pi/3)$	$6.77710805459830095 73\ldots \pm 10^{-35}$

regular triangles

more work for this one

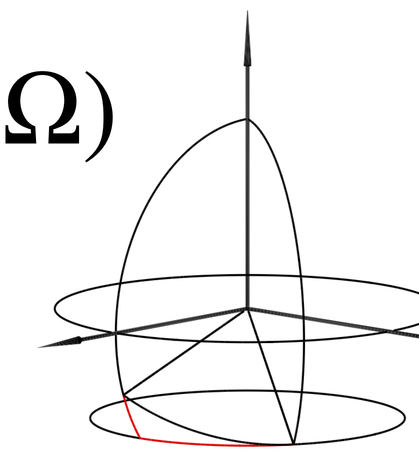
singular triangles

1. Find a good approximate pair $(f^\star, \lambda^\star)$
2. Compute the bound in a certified way
3. Certify the index

Step 3. Certify the index

Certification of the index

monotonicity $\lambda < \lambda_2(\Omega') \leq \lambda_2(\Omega)$ with $\Omega' \supset \Omega \Rightarrow \lambda = \lambda_1(\Omega)$



For the domain $0 \leq \phi \leq \phi_{\max}, 0 \leq \theta \leq \theta_{\max}$, the eigenvalues are $\nu(\nu+1)$ with ν root of

P_ν^μ satisfies $((1-x^2)w')' + q_{\mu,\nu}(x)w = 0$, $q_{\mu,\nu}(x) = \nu(\nu+1) - \frac{\mu^2}{1-x^2}$

$$\mathsf{P}_\nu^{\mu_k}(\cos\theta_{\max}) = 0$$

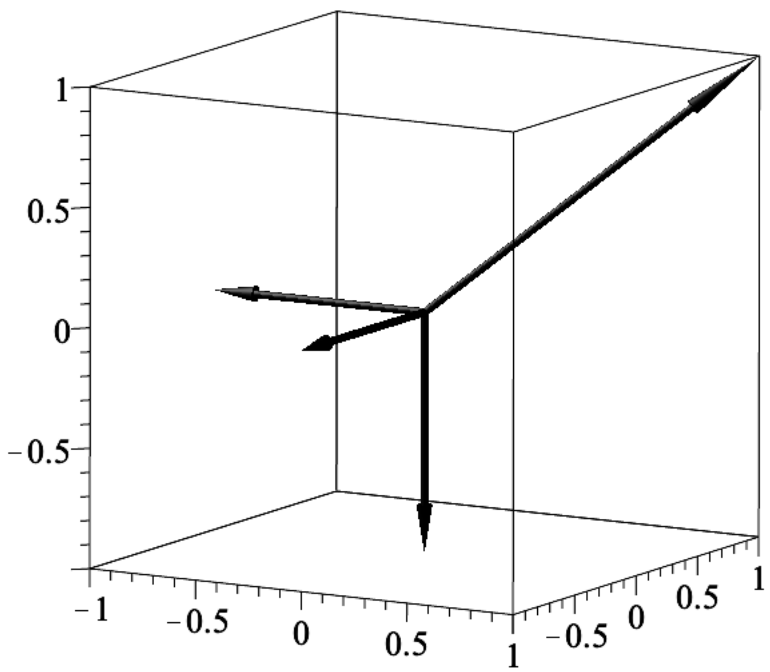
For each $k \in \mathbb{N}^\star$, an infinity of roots $\nu_{k,j}$, $j \in \mathbb{N}^\star$

It is sufficient to check λ against $\nu_{1,2}$ and $\nu_{2,1}$

Proof via Sturm's comparison theorem

Sometimes this fails, but another orientation of the triangle works

Bound on Denominator



$$\boxed{\#\text{excursions} \sim C\,4^n n^{-\alpha_\kappa}}$$

$$\lambda = 5.159145642466541\ldots \pm 10^{-104}$$

certified enclosure

$$\alpha_\kappa = 1 + \sqrt{\lambda + 1/4} = 3.32575700417\ldots \pm 10^{-104}$$

Interval arithmetic

Continued fraction: $\alpha_\kappa = 3 + \cfrac{1}{3+\cfrac{1}{14+\cfrac{1}{3+\cfrac{1}{100+\cfrac{1}{12+\cfrac{1}{\ddots}}}}}}$

Stop when a partial quotient cannot be certified

Last convergent: P/Q with $Q = 95716\ldots26933 > 10^{51}$

$$\boxed{\text{If } \alpha_\kappa = p/q \in \mathbb{Q}, \text{ then } q > 10^{51}.}$$

Summary & Conclusion

Linear recurrences with constant coefficients remain mysterious;

lattice walks provide a simple source of examples;

more and more tools are available;

numerical computation can yield rigorous results, useful in experimental mathematics.

Thank you.