Linear differential equations with finite differential Galois group

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What is this talk about?

Question: Let $G \subseteq GL_n(\mathbb{C})$ be a finite (primitive) group.
Given a linear ordinary differential equation with coefficients in $\mathbb{C}(t)$ and differential Galois group G

$$\left(\frac{d}{dt}\right)^n y + a_{n-1}\left(\frac{d}{dt}\right)^{n-1} y + \ldots + a_1 \frac{d}{dt} y + a_0 y = 0,$$

how to write down a solution for this equation?
Irreducible LODE with only algebraic solutions will be called "algebraic LODEs".

Previous work - F. Klein (1877)

If $n = 2$, the solutions to an algebraic LODE are of the form

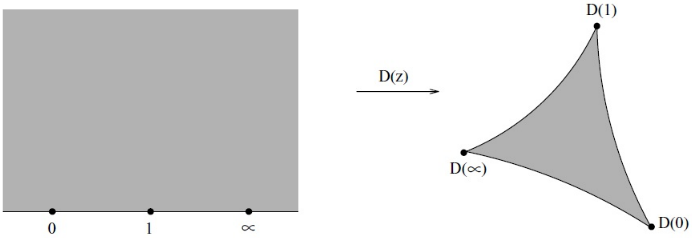
$$y(t) = f(t) \cdot {}_2F_1\left(\alpha, \beta; \gamma | s(t)\right)$$

where $f'/f \in \mathbb{C}(t)$, $s(t) \in \mathbb{C}(t)$ and (α, β, γ) are chosen according to the Schwarz spherical triangles.
Indeed, if y_1, y_2 are a two linearly independent solutions of the equation, the Schwarz map

$$[\mathbf{y}] : t \longmapsto (y_1(t) : y_2(t))$$

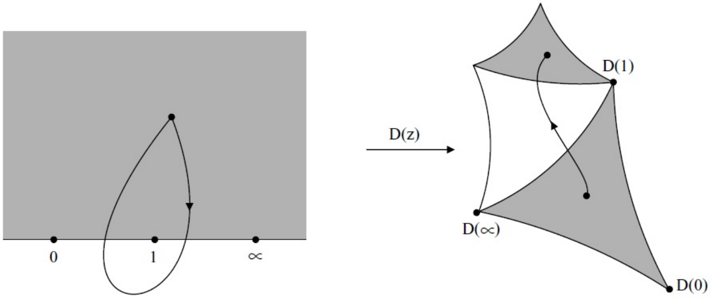
defines a tessellation of the sphere.

Schwarz maps for 2nd order LODEs



F. Beukers, *Gauss' hypergeometric function*

Schwarz maps for 2nd order LODEs



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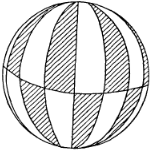
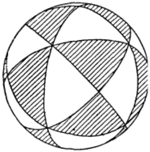
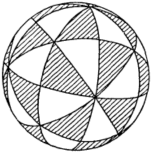
Schwarz maps for 2nd order LODEs



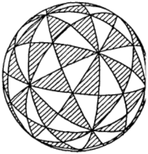
$(2, 2, n)$



$(2, 3, 3)$



$(2, 3, 4)$



$(2, 3, 5)$

M. Yoshida, *Hypergeometric functions, My Love*

Previous work - M. Berkenbosch, M. van Hoeij, J.-A. Weil (2005)

If $n = 2$, the solutions to an algebraic LODE are of the form

$$y(t) = f(t) \cdot {}_2F_1\left(\alpha, \beta; \gamma | s(t)\right)$$

where $f'/f \in \mathbb{C}(t)$, $s(t) \in \mathbb{C}(t)$ and $(\alpha, \beta; \gamma)$ is:

$$\begin{aligned}
(-\frac{1}{12}, \frac{1}{4}; \frac{2}{3}) &\text{ if } G/Z(G) \simeq A_4 \\
(-\frac{1}{24}, \frac{5}{24}; \frac{2}{3}) &\text{ if } G/Z(G) \simeq S_4 \\
(\frac{11}{60}, -\frac{1}{60}; \frac{2}{3}) &\text{ if } G/Z(G) \simeq A_5 \\
(-\frac{1}{2n}, \frac{1}{2n}; -\frac{1}{2}) &\text{ if } G/Z(G) \simeq D_{2n}
\end{aligned}$$

Previous work - J. Kovacic (1986)

If $n = 2$, the solutions to an algebraic LODE are of the form

$$y(t) = e^{\int \omega(t)dt}$$

where ω is algebraic over $\mathbb{C}(t)$ and one can compute its minimal polynomial which is of degree:

$$\begin{aligned} 4 &\text{ if } G \simeq A_4^{SL_2} \\ 6 &\text{ if } G \simeq S_4^{SL_2} \\ 12 &\text{ if } G \simeq A_5^{SL_2} \\ 2 &\text{ if } G \simeq D_{2n}^{SL_2} \end{aligned}$$

Previous work - M.F. Singer, F. Ulmer (1998)

If $n = 2$, one can compute the minimal polynomial of the solutions to a given algebraic LODE which is of degree:

$$24 \text{ if } G \simeq A_4^{SL_2}$$
$$48 \text{ if } G \simeq S_4^{SL_2}$$
$$120 \text{ if } G \simeq A_5^{SL_2}$$
$$4n \text{ if } G \simeq D_{2n}^{SL_2}$$

Previous work - M. Berkenbosch (2006)

If $n = 3$, the solutions to an algebraic LODE are of the form

$$y(t) = f(t) \cdot F_G\left(s(t)\right)$$

where $f'/f \in \mathbb{C}(t)$, $s(t) \in \mathbb{C}(t)$ and F_G is a solution to a "standard equation", i.e.:

$$\mathbb{C}\left(\frac{x_2}{x_1}, \frac{x_3}{x_1}\right)^G = \mathbb{C}(t)$$

where x_1, x_2, x_3 form a system of solutions of the algebraic LODE defining F_G.

There are infinitely many standard equations for each group :(

Previous work - van Hoeij, Ragot, Ulmer, Weil (1998)

If $n = 3$, the solutions to an algebraic LODE are of the form

$$y(t) = e^{\int \omega(t)dt}$$

where ω is algebraic over $\mathbb{C}(t)$ and one can compute its minimal polynomial which is of degree:

$$\begin{aligned} 6 \text{ or } 15 \text{ if } G &\simeq A_5^{SL_3} \text{ or } A_5^{SL_3} \times C_3 \\ 45 \text{ if } G &\simeq A_6^{SL_3} \\ 21 \text{ if } G &\simeq G_{168} \text{ or } G_{168} \times C3 \\ 9 \text{ if } G &\simeq H_{216}^{SL_3}, H_{72}^{SL_3} \text{ or } F_{36}^{SL_3} \end{aligned}$$

Previous work - M.F. Singer, F. Ulmer (1998)

If $n = 3$, one can compute the minimal polynomial of the solutions to a given algebraic LODE which is of degree:

$$
\begin{aligned}
12 &\text{ if } G \simeq A_5^{SL_3} \text{ or } A_5^{SL_3} \times C_3 \\
216 &\text{ if } G \simeq A_6^{SL_3} \\
42 &\text{ if } G \simeq G_{168} \text{ or } G_{168} \times C3 \\
81 &\text{ if } G \simeq H_{216}^{SL_3} \\
27 &\text{ if } G \simeq H_{72}^{SL_3} \\
36 &\text{ if } G \simeq F_{36}^{SL_3}
\end{aligned}
$$

Previous work - C. Sanabria (2017)

The solutions to an algebraic LODE are of the form

$$y(t) = f(t) \cdot F_G\left(s(t)\right)$$

where $f'/f \in \mathbb{C}(t)$, $s(t) \in \mathbb{C}(t)$ and F_G is a solution to a "standard equation", i.e.:

$$\mathbb{C}\left(\frac{x_2}{x_1}, \ldots, \frac{x_n}{x_1}\right)^G = \mathbb{C}(t)$$

where $x_1, x_2, \ldots, x_n$ form a system of solutions to the algebraic LODE defining F_G.

There are infinitely many standard equations for each group...

Previous work - C. Sanabria (2017)

Whenever $x_1, x_2, \ldots, x_n$ form a system of solutions to a stantard LODE, the associated Schwarz map:

$$[\mathbf{x}]t \longmapsto (x_1(t) : x_2(t) : \ldots : x_n(t))$$

is a solution to a G-homogeneous dynamical system on $\mathbb{P}^{n-1}(\mathbb{C})$. (i.e. a system whose first integrals are given by G-invariant polynomials).

Proof - Compoint's Theorem

Let

$$\left(\frac{d}{dt}\right)^n y + a_{n-1}\left(\frac{d}{dt}\right)^{n-1} y + \ldots + a_1 \frac{d}{dt} y + a_0 y = 0,$$

$a_i \in \mathbb{C}(t)$, be an algebraic LODE, $\mathbf{y} = (y_1, \ldots, y_n)$ a system of solutions and $G \subseteq GL_n(\mathbb{C})$ its (finite and primitive) differential Galois group. Let $I \subset \mathbb{C}(t)[Y_1, \ldots, Y_n]$ be the kernel of the evaluation $\mathbb{C}(t)$-morphism:

$$\begin{aligned} \Phi : \mathbb{C}(t)[Y_1, \ldots, Y_n] &\longrightarrow \mathbb{C}(t)(y_1, \ldots, y_n) \\ Y_j &\longmapsto y_j \end{aligned}$$

Galois correspondence: if $P \in \mathbb{C}(t)[Y_1, \ldots, Y_n]$ is G-invariant, then $\Phi(P) \in \mathbb{C}(t)$.

Theorem (Compoint's theorem for the algebraic case)

The ideal I is generated by the G-invariants contained in it.

Proof - Compoint's Theorem

Let $I \subset \mathbb{C}(t)[Y_1, \ldots, Y_n]$ be the kernel of the evaluation $\mathbb{C}(t)$-morphism:

$$\begin{array}{rcl} \Phi : \mathbb{C}(t)[Y_1, \ldots, Y_n] & \longrightarrow & \mathbb{C}(t)(y_1, \ldots, y_n) \\ Y_j & \longmapsto & y_j \end{array}$$

Galois correspondence: if $P \in \mathbb{C}(t)[Y_1, \ldots, Y_n]$ is G-invariant, then $\Phi(P) \in \mathbb{C}(t)$.

Theorem (Compoint's theorem for the algebraic case)

If $P_1, \ldots, P_N \in \mathbb{C}[Y_1, \ldots, Y_n]$ is a set of generators of the G-invariant subring $\mathbb{C}[Y_1, \ldots, Y_n]^G$, then

$$I = \langle P_1 - f_1, \ldots, P_N - f_N \rangle,$$

where $f_i = \Phi(P_i)$, $i = 1, \ldots, N$.

Proof - Projection onto G-orbits

Let G act on $\mathbb{C}^n$. Since G is finite, the orbit space $\mathbb{C}^n/G$ has dimension n, therefore, the derivative of the projection onto the orbits

$$\begin{array}{rcl} \Pi_G : \mathbb{C}^n & \longrightarrow & \mathbb{C}^n/G \subseteq \mathbb{C}^N \\ \mathbf{Y} = (Y_1, \ldots, Y_n) & \longmapsto & (P_1(\mathbf{Y}), \ldots, P_N(\mathbf{Y})) \end{array}$$

is non-singular in a dense subset $U \subseteq \mathbb{C}^n$. In particular, after arranging the indexes of the P_i's if necessary, we may assume that

$$M(\mathbf{Y}) = \begin{bmatrix} \partial P_1/\partial Y_1 & \ldots & \partial P_1/\partial Y_n \\ \vdots & \ddots & \vdots \\ \partial P_n/\partial Y_1 & \ldots & \partial P_n/\partial Y_n \end{bmatrix} (\mathbf{Y})$$

is invertible over U.

Proof - Projection & Compoint's theorem

For the system of solutions $\mathbf{y} = (y_1, \ldots, y_n)$, from Compoint's theorem, we have

$$\begin{aligned} P_1(\mathbf{y}) &= f_1(t) \\ &\vdots \\ P_N(\mathbf{y}) &= f_N(t). \end{aligned}$$

We differentiate

$$\begin{bmatrix} \partial P_1(\mathbf{y})/\partial Y_1 & \ldots & \partial P_1(\mathbf{y})/\partial Y_n \\ \vdots & \ddots & \vdots \\ \partial P_n(\mathbf{y})/\partial Y_1 & \ldots & \partial P_n(\mathbf{y})/\partial Y_n \end{bmatrix} \begin{bmatrix} y_1' \\ \vdots \\ y_n' \end{bmatrix} = \begin{bmatrix} \frac{d}{dt} f_1 \\ \vdots \\ \frac{d}{dt} f_n \end{bmatrix}.$$

Therefore $\mathbf{y}(t)$ is a solution to the (non-autonomous) dynamical system $\frac{d}{dt}\mathbf{Y} = F(t, \mathbf{Y})$ where

$$F(t, \mathbf{Y}) = \left[M(\mathbf{Y})\right]^{-1} \begin{bmatrix} \frac{d}{dt} f_1 \\ \vdots \\ \frac{d}{dt} f_n \end{bmatrix}.$$

Proof - Projection & Compoint's theorem

If we choose an adapted coordinate system of $\mathbb{C}^n/G$ for the curve parametrized by $t \longmapsto \mathbf{y}(t) \cdot G$:

$$\begin{array}{rcl} Q_1(\mathbf{y}) & = & s(t) \\ Q_2(\mathbf{y}) & = & c_2 \\ & \vdots & \\ Q_n(\mathbf{y}) & = & c_n. \end{array}$$

Then $\mathbf{y}(t)$ is a solution to the (non-autonomous) dynamical system $\frac{d}{dt}\mathbf{Y} = E(t, \mathbf{Y})$ where

$$E(t, \mathbf{Y}) = \begin{bmatrix} \partial Q_1/\partial Y_1 & \dots & \partial Q_1/\partial Y_n \\ \partial Q_2/\partial Y_1 & \dots & \partial Q_2/\partial Y_n \\ \vdots & \ddots & \vdots \\ \partial Q_n/\partial Y_1 & \dots & \partial Q_n/\partial Y_n \end{bmatrix}^{-1} \begin{bmatrix} \frac{d}{dt}s(t) \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Proof - Projection & Compoint's theorem

If we choose an adapted coordinate system of $\mathbb{C}^n/G$ for the curve parametrized by $t \longmapsto \mathbf{y}(t) \cdot G$:

$$\begin{array}{rcl} Q_1(\mathbf{y}) & = & s(t) \\ Q_2(\mathbf{y}) & = & c_2 \\ & \vdots & \\ Q_n(\mathbf{y}) & = & c_n. \end{array}$$

Then $\mathbf{y}(t)$ covers a solution to the autonomous dynamical system $\frac{d}{ds}\mathbf{Y} = E_0(\mathbf{Y})$ where

$$E_0(\mathbf{Y}) = \begin{bmatrix} \partial Q_1/\partial Y_1 & \dots & \partial Q_1/\partial Y_n \\ \partial Q_2/\partial Y_1 & \dots & \partial Q_2/\partial Y_n \\ \vdots & \ddots & \vdots \\ \partial Q_n/\partial Y_1 & \dots & \partial Q_n/\partial Y_n \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Proof - Projective Space

We can apply the same principle to $\mathbb{P}\left(\mathbb{C}^n/G\right)$ instead of $\mathbb{C}^n/G$ by choosing a homogeneous coordinates system

$$t \overset{\mathbf{y}}{\longmapsto} \mathbf{y}(t) \in \mathbb{C}^n$$

$$\downarrow$$

$$\left(P_1(\mathbf{y}), \ldots, P_N(\mathbf{y})\right)(t) \in \mathbb{C}^n/G$$

$$\downarrow$$

$$\left(P_1(\mathbf{y})^{d_1} : \ldots : P_N(\mathbf{y})^{d_N}\right)(t) \in \mathbb{P}(\mathbb{C}^n/G)$$

$$d_1 \deg P_1 = \ldots = d_N \deg P_N$$

Proof - Projective Space

If we choose an adapted homogeneous coordinate system $Q_1, \ldots, Q_n$ of $\mathbb{P}(\mathbb{C}^n/G)$ for the curve parametrized by

$$t \longmapsto [\mathbf{y}(t)] \cdot G = (y_1 : \ldots : y_n)(t) \cdot G$$

then the Schwarz map $[\mathbf{y}(t)] = (y_1 : \ldots : y_n)(t)$ covers a solution to the autonomous dynamical system $\frac{d}{ds}\mathbf{Y} = G_0(\mathbf{Y})$ where

$$G_0(\mathbf{Y}) = \begin{bmatrix} \partial Q_1/\partial Y_1 & \ldots & \partial Q_1/\partial Y_n \\ \partial Q_2/\partial Y_1 & \ldots & \partial Q_2/\partial Y_n \\ \vdots & \ddots & \vdots \\ \partial Q_n/\partial Y_1 & \ldots & \partial Q_n/\partial Y_n \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Proof - G-homogeneous dynamical systems and Standard Equations

Solutions to autonomous G-homogeneous dynamical system correspond to standard LODEs.
Indeed, every linear ODEs with algebraic solutions come from curves of the form

$$P_1(\mathbf{x}) = f_1(t), \quad \dots \quad , P_N(\mathbf{y}) = f_N(t) \tag{1}$$

where $P_1, \dots, P_N \in \mathbb{C}[X_1, \dots, X_n]^G$ and G is a finite group in $GL_n(\mathbb{C})$.
Differentiating (1) n-times one finds a G-invariant linear relation in-between

$$\mathbf{x}(t), \quad \frac{d}{dt}\mathbf{x}(t), \quad \dots \quad \left(\frac{d}{dt}\right)^n \mathbf{x}(t)$$

and thus obtains explicitly the LODE.

Proof - G-homogeneous dynamical systems and Standard Equations

Solutions to any other algebraic LODE covers solutions to autonomous G-homogeneous dynamical system (which correspond to standard LODEs).
Therefore: the solutions to an algebraic LODE are of the form

$$y(t) = f(t) \cdot F_G\left(s(t)\right)$$

where $f'/f \in \mathbb{C}(t)$, $s(t) \in \mathbb{C}(t)$ and F_G is a solution to a "standard equation".

Standard equations

Let

$$\left(\frac{d}{dt}\right)^n y + a_{n-1}\left(\frac{d}{dt}\right)^{n-1} y + \ldots + a_1 \frac{d}{dt} y + a_0 y = 0,$$

$a_i \in \mathbb{C}(t)$, be an algebraic LODE, $\mathbf{y} = (y_1, \ldots, y_n)$ a system of solutions and $G \subseteq GL_n(\mathbb{C})$ its (finite and primitive) differential Galois group.

Theorem: The equation is standard if and only if the Schwarz map $[\mathbf{y}(t)] = (y_1 : \ldots : y_n)(t)$ is a solution to a G-homogeneous dynamical system on $\mathbb{P}^{n-1}(\mathbb{C})$. (i.e. a system whose first integrals are given by G-invariant polynomials).

Example

Klein's simple group of order 168, G_{168}, is isomorphic to the subgroup of $SL_3(\mathbb{C})$ generated by the matrices

$$\begin{bmatrix} \beta & 0 & 0 \\ 0 & \beta^2 & 0 \\ 0 & 0 & \beta^4 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \text{ and } \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix},$$

where β is a primitive 7-th root of unity (i.e. $\beta^6 + \beta^5 + \beta^4 + \beta^3 + \beta^2 + \beta^1 = 0$), $a = \beta^4 - \beta^3$, $b = \beta^2 - \beta^5$, and $c = \beta - \beta^6$.

Example

The invariant subring $\mathbb{C}[X_1, X_2, X_3]^{G_{168}}$ is generated by

$$
\begin{aligned}
F_4 &= X_1^3 X_2 + X_2^3 X_3 + X_3^3 X_1, \\
F_6 &= \frac{1}{54} \det \begin{bmatrix} \partial^2 F_4/\partial X_1 \partial X_1 & \partial^2 F_4/\partial X_1 \partial X_2 & \partial^2 F_4/\partial X_1 \partial X_3 \\ \partial^2 F_4/\partial X_2 \partial X_1 & \partial^2 F_4/\partial X_2 \partial X_2 & \partial^2 F_4/\partial X_2 \partial X_3 \\ \partial^2 F_4/\partial X_3 \partial X_1 & \partial^2 F_4/\partial X_3 \partial X_2 & \partial^2 F_4/\partial X_3 \partial X_3 \end{bmatrix}, \\
F_{14} &= \frac{1}{9} \det \begin{bmatrix} \partial^2 F_4/\partial X_1 \partial X_1 & \partial^2 F_4/\partial X_1 \partial X_2 & \partial^2 F_4/\partial X_1 \partial X_3 & \partial F_6/\partial X_1 \\ \partial^2 F_4/\partial X_2 \partial X_1 & \partial^2 F_4/\partial X_2 \partial X_2 & \partial^2 F_4/\partial X_2 \partial X_3 & \partial F_6/\partial X_2 \\ \partial^2 F_4/\partial X_3 \partial X_1 & \partial^2 F_4/\partial X_3 \partial X_2 & \partial^2 F_4/\partial X_3 \partial X_3 & \partial F_6/\partial X_3 \\ \partial F_6/\partial X_1 & \partial F_6/\partial X_2 & \partial F_6/\partial X_3 & 0 \end{bmatrix},
\end{aligned}
$$

and

$$
F_{21} = \frac{1}{14} \det \begin{bmatrix} \partial F_4/\partial X_1 & \partial F_4/\partial X_2 & \partial F_4/\partial X_3 \\ \partial F_6/\partial X_1 & \partial F_6/\partial X_2 & \partial F_6/\partial X_3 \\ \partial F_{14}/\partial X_1 & \partial F_{14}/\partial X_2 & \partial F_{14}/\partial X_3 \end{bmatrix}.
$$

Example

As a ring, $\mathbb{C}[X_1, X_2, X_3]^{G_{168}}$ is isomorphic to $\mathbb{C}[\Phi_4, \Phi_6, \Phi_{14}, \Phi_{21}]/(T)$ where

$$\begin{aligned} T &= 2048\Phi_4^9\Phi_6 - 22016\Phi_4^6\Phi_6^3 - 256\Phi_{14}\Phi_4^7 + 60032\Phi_4^3\Phi_6^5 \\ &\quad +1088\Phi_{14}\Phi_4^4\Phi_6^2 - 1728\Phi_6^7 + 1008\Phi_{14}\Phi_4\Phi_6^4 + 88\Phi_{14}^2\Phi_4^2\Phi_6 \\ &\quad +\Phi_{14}^3 - \Phi_{21}^2. \end{aligned}$$

Klein Quartic- Hurwitz equation (1886)

$$\begin{aligned}0 \;=\; & \left(\frac{d}{dt}\right)^3 x + \frac{7t-4}{t(t-1)}\left(\frac{d}{dt}\right)^2 x \\ & + \frac{1}{252}\frac{2592t^2 - 2963t + 560}{t^2(t-1)^2}\left(\frac{d}{dt}\right) x \\ & + \frac{1}{24696}\frac{57024t - 40805}{t^2(t-1)^2} x\end{aligned}$$

$$F_4(x_1(t), x_2(t), x_3(t)) = 0$$

$$F_6(x_1(t), x_2(t), x_3(t)) = \frac{-3^3}{t^4(t-1)^3}$$

$$F_{14}(x_1(t), x_2(t), x_3(t)) = -\frac{2^2 3^8}{t^9(t-1)^7}$$

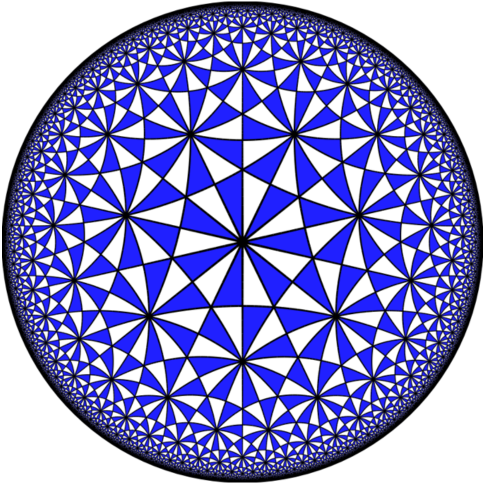
Klein Quartic- Hurwitz equation (1886)

$$\begin{aligned} 0 &= \left(\frac{d}{dt}\right)^3 x + \frac{7t-4}{t(t-1)}\left(\frac{d}{dt}\right)^2 x \\ &\quad + \frac{1}{252}\frac{2592t^2 - 2963t + 560}{t^2(t-1)^2}\left(\frac{d}{dt}\right) x \\ &\quad + \frac{1}{24696}\frac{57024t - 40805}{t^2(t-1)^2} x \end{aligned}$$

$$F_4(x_1(t), x_2(t), x_3(t)) = 0$$

$$\frac{-F_{14}^3}{1728F_6^7}(x_1(t), x_2(t), x_3(t)) = t$$

Klein Quartic- Hurwitz equation (1886)



taken from Wikipedia.

Hessian of Klein Quartic - Beukers & Heckman (1989)

$$
\begin{aligned}
0 \;=\; & \left(\frac{d}{dt}\right)^3 x + \frac{3}{2}\frac{3t-2}{t(t-1)}\left(\frac{d}{dt}\right)^2 x + \frac{3}{112}\frac{116t-35}{t^2(t-1)}\left(\frac{d}{dt}\right) x \\
& + \frac{195}{2744}\frac{1}{t^2(t-1)} x
\end{aligned}
$$

$$
\begin{aligned}
F_4(x_1(t), x_2(t), x_3(t)) &= \frac{1}{t} \\
F_6(x_1(t), x_2(t), x_3(t)) &= 0 \\
F_{14}(x_1(t), x_2(t), x_3(t)) &= \frac{16}{t^3} \\
\frac{F_{14}^2}{256 F_4^7}(x_1(t), x_2(t), x_3(t)) &= t
\end{aligned}
$$

Klein Quartic - ${}_3F_2(-1/14, 3/14, 5/14; 2/3, 1/3|t)$

$$\begin{aligned} 0 \;=\; & \left(\frac{d}{dt}\right)^3 x + \frac{1}{2}\frac{7t-4}{t(t-1)}\left(\frac{d}{dt}\right)^2 x \\ & + \frac{1}{252}\frac{387t-56}{t^2(t-1)}\left(\frac{d}{dt}\right) x \\ & - \frac{1}{2744}\frac{15}{t^2(t-1)} x \end{aligned}$$

$$\begin{aligned} F_4(x_1(t), x_2(t), x_3(t)) &= 0 \\ F_6(x_1(t), x_2(t), x_3(t)) &= 1 \\ \frac{-F_{14}^3}{1728F_6^7}(x_1(t), x_2(t), x_3(t)) &= t \end{aligned}$$

Hessian of Klein Quartic - ${}_3F_2(19/28, 3/28, -1/28; 1, 1/2|t)$

$$
\begin{aligned}
0 \;=\; & \left(\frac{d}{dt}\right)^3 x + \frac{3}{4}\frac{5t-3}{t(t-1)}\left(\frac{d}{dt}\right)^2 x + \frac{3}{112}\frac{67t-14}{t^2(t-1)}\left(\frac{d}{dt}\right) x \\
& - \frac{1}{21952}\frac{57}{t^2(t-1)} x
\end{aligned}
$$

$$
\begin{aligned}
F_4(x_1(t), x_2(t), x_3(t)) &= 1 \\
F_6(x_1(t), x_2(t), x_3(t)) &= 0 \\
\frac{F_{14}^2}{256 F_4^7}(x_1(t), x_2(t), x_3(t)) &= t
\end{aligned}
$$

Example - M. van der Put, F. Ulmer (2000)

$$\begin{aligned} 0 &= \left(\frac{d}{dt}\right)^3 y + \frac{9t-4}{t(t-1)}\left(\frac{d}{dt}\right)^2 y \\ &\quad + \frac{10}{63}\,\frac{117t^2-104t+14}{t^2(t-1)^2}\left(\frac{d}{dt}\right) y \\ &\quad - \frac{10}{1029}\,\frac{741t^2-988t+274}{t^2(t-1)^3} y \end{aligned}$$

Example - J. Top, M. van der Put, C. Sanabria (2018)

$$
\begin{aligned}
0 \;=\; & \left(\frac{d}{dt}\right)^3 y + \frac{9t-4}{t(t-1)}\left(\frac{d}{dt}\right)^2 y \\
& + \frac{10}{63}\,\frac{117t^2-104t+14}{t^2(t-1)^2}\left(\frac{d}{dt}\right) y \\
& - \frac{10}{1029}\,\frac{741t^2-988t+274}{t^2(t-1)^3} y
\end{aligned}
$$

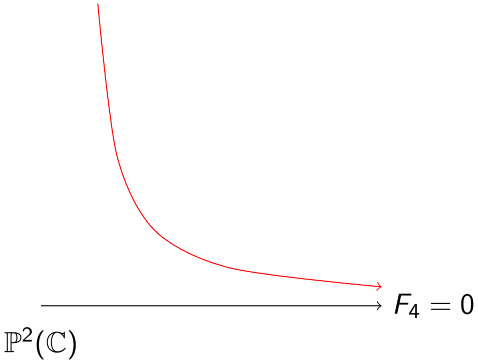
$$
\begin{aligned}
F_4(y_1(t), y_2(t), y_3(t)) &= 0 \\
F_6(y_1(t), y_2(t), y_3(t)) &= \frac{27}{t^4(t-1)^5} \\
\frac{-F_{14}^3}{1728F_6^7}(y_1(t), y_2(t), y_3(t)) &= -\frac{1}{2^6}\,\frac{t(9t-8)^3}{t-1}
\end{aligned}
$$

Example

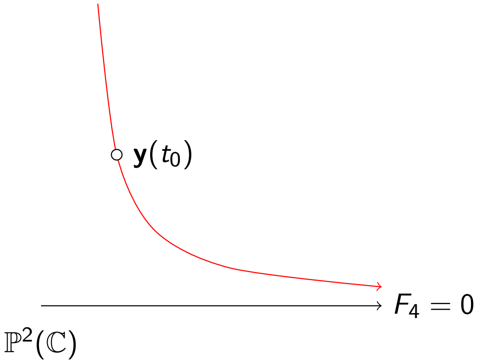
$$\begin{aligned} 0 &= \left(\frac{d}{dt}\right)^3 y + \frac{9t-4}{t(t-1)}\left(\frac{d}{dt}\right)^2 y \\ &\quad + \frac{10}{63}\frac{117t^2-104t+14}{t^2(t-1)^2}\left(\frac{d}{dt}\right) y \\ &\quad - \frac{10}{1029}\frac{741t^2-988t+274}{t^2(t-1)^3} y \end{aligned}$$

$$y(t) = \sqrt[6]{\frac{27}{t^4(t-1)^5}}\,{}_3F_2\left(-1/14, 3/14, 5/14; 2/3, 1/3 \middle| -\frac{1}{2^6}\frac{t(9t-8)^3}{t-1}\right)$$

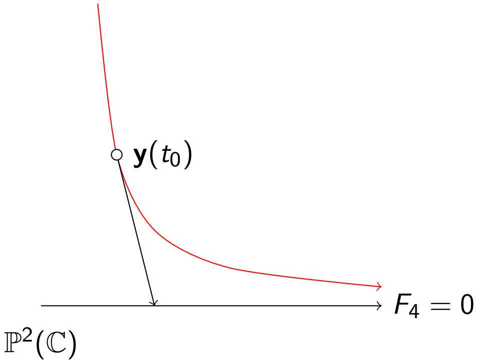
What about other parametric curves?



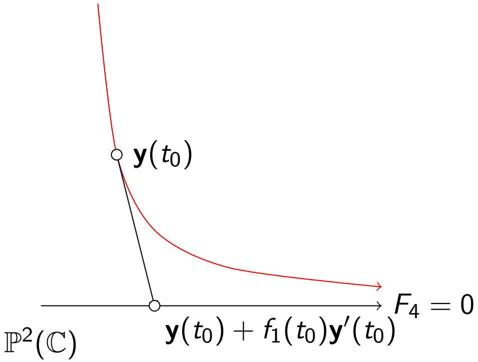
What about other parametric curves?



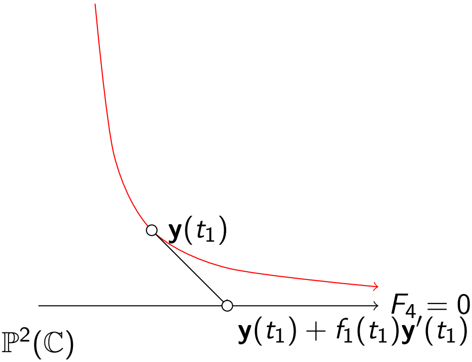
What about other parametric curves?



What about other parametric curves?



What about other parametric curves?



Example - M. van der Put, F. Ulmer (2000)

$$
\begin{aligned}
0 \quad = \quad & \left(\frac{d}{dt}\right)^3 y + \frac{7t-2}{t(t-1)}\left(\frac{d}{dt}\right)^2 y \\
& + \frac{1}{28}\frac{288t^2-161t-7}{t^2(t-1)^2}\left(\frac{d}{dt}\right) y \\
& + \frac{1}{2744}\frac{6336t^3-5273t^2+343t-686}{t^3(t-1)^3} y
\end{aligned}
$$

Example - J. Top, M. van der Put, C. Sanabria (2018)

$$
\begin{aligned}
0 \quad = \quad & \left(\frac{d}{dt}\right)^3 y + \frac{7t-2}{t(t-1)}\left(\frac{d}{dt}\right)^2 y \\
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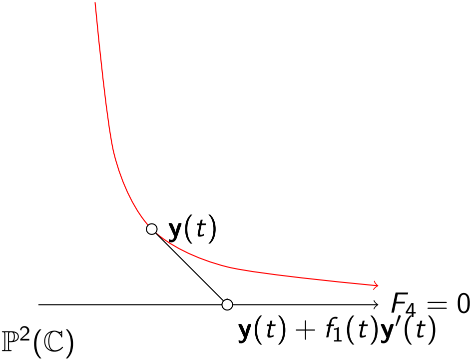
& + \frac{1}{2744}\frac{6336t^3-5273t^2+343t-686}{t^3(t-1)^3} y
\end{aligned}
$$

$$
\begin{aligned}
F_4(y_1(t), y_2(t), y_3(t)) &= \frac{14}{t^2(t-1)^3} \\
F_6(y_1(t), y_2(t), y_3(t)) &= \frac{3}{t^2(t-1)^5} \\
F_{14}(y_1(t), y_2(t), y_3(t)) &= \frac{4(-294+294t+t^2)}{t^6(t-1)^{12}}
\end{aligned}
$$

$$
-\frac{7}{53}F_6^3 - \frac{1}{8}F_4F_{14} + F_4^3F_6 = 0
$$



$\mathbf{y}(t)$
$F_4 = 0$
$\mathbf{y}(t) + f_1(t)\mathbf{y}'(t)$
$\mathbb{P}^2(\mathbb{C})$

Example

$$
\begin{aligned}
0 \;=\; & \left(\frac{d}{dt}\right)^3 y + \frac{7t-2}{t(t-1)}\left(\frac{d}{dt}\right)^2 y \\
& + \frac{1}{28}\frac{288t^2-161t-7}{t^2(t-1)^2}\left(\frac{d}{dt}\right) y \\
& + \frac{1}{2744}\frac{6336t^3-5273t^2+343t-686}{t^3(t-1)^3} y
\end{aligned}
$$

$$
y(t) + f_1(t)y'(t) = f(t)\,{}_3F_2\left(-1/14, 3/14, 5/14; 2/3, 1/3 \middle| s(t)\right)
$$

Example

$$
\begin{aligned}
0 \;=\; & \left(\frac{d}{dt}\right)^3 y + \frac{7t-2}{t(t-1)}\left(\frac{d}{dt}\right)^2 y \\
& + \frac{1}{28}\,\frac{288t^2-161t-7}{t^2(t-1)^2}\left(\frac{d}{dt}\right) y \\
& + \frac{1}{2744}\,\frac{6336t^3-5273t^2+343t-686}{t^3(t-1)^3} y
\end{aligned}
$$

$$
y(t) + \frac{14t(t-1)}{19t-7} y'(t) =
$$

$$
3\sqrt[6]{\frac{t^3}{(19t-7)^6(t-1)^4}}\,{}_3F_2\left(-1/14, 3/14, 5/14; 2/3, 1/3 \,\middle|\, -\frac{1}{27}\frac{(t+3)^3}{(t-1)^2}\right)
$$

Solving in terms of ${}_3F_2(-1/14, 3/14, 5/14; 2/3, 1/3|t)$

We have

$$y + f_1 y' = f \cdot F_{KQ} \circ s$$

From $y''' + a_2 y'' + a_1 y' + a_0 y = 0$ we obtain

$$\begin{bmatrix} 1 & f_1 & 0 \\ 0 & 1 + f_1' & f_1 \\ -f_1 a_0 & f_1'' - a_1 f_1 & 1 + 2f_1'' - a_2 f_1 \end{bmatrix} \begin{bmatrix} y \\ y' \\ y'' \end{bmatrix} = \begin{bmatrix} f \cdot F_{KQ} \circ s \\ \left(f \cdot F_{KQ} \circ s\right)' \\ \left(f \cdot F_{KQ} \circ s\right)'' \end{bmatrix}$$

Solving in terms of ${}_3F_2(-1/14, 3/14, 5/14; 2/3, 1/3|t)$

We have

$$y + f_1 y' = f \cdot F_{KQ} \circ s$$

From $y''' + a_2 y'' + a_1 y' + a_0 y = 0$ we obtain

$$\begin{bmatrix} y \\ y' \\ y'' \end{bmatrix} = \begin{bmatrix} 1 & f_1 & 0 \\ 0 & 1 + f_1' & f_1 \\ -f_1 a_0 & f_1'' - a_1 f_1 & 1 + 2f_1'' - a_2 f_1 \end{bmatrix}^{-1} \begin{bmatrix} f \cdot F_{KQ} \circ s \\ \left(f \cdot F_{KQ} \circ s\right)' \\ \left(f \cdot F_{KQ} \circ s\right)'' \end{bmatrix}$$

Solving in terms of ${}_3F_2(-1/14, 3/14, 5/14; 2/3, 1/3|t)$

$$H(x) := 3\left(\frac{x^3}{(x-1)^4\,(19x-7)^6}\right)^{1/6} \mathrm{hypergeom}\left(\left[-\frac{1}{42}, \frac{5}{42}, \frac{17}{42}\right], \left[\frac{1}{3}, \frac{2}{3}\right], \frac{1}{27}\,\frac{(x+3)^3}{(x-1)^2}\right)$$

$$ODE := \frac{d^3}{dx^3}y(x) + \frac{(7x-2)\left(\frac{d^2}{dx^2}y(x)\right)}{x(x-1)} + \frac{1}{28}\,\frac{(288x^2-161x-7)\left(\frac{d}{dx}y(x)\right)}{x^2(x-1)^2} + \frac{1}{2744}\,\frac{(6336x^3-5273x^2+343x-686)\,y(x)}{x^3(x-1)^3} = 0$$

$$sol := y(x) = \frac{1932781}{6049137024}\,\frac{1}{\left(\frac{x^3}{(x-1)^4(19x-7)^6}\right)^{5/6}(x-1)^9(19x-7)^5}\left(\left(x(x-9)^2(x+3)^4\,\mathrm{hypergeom}\left(\left[\frac{83}{42}, \frac{89}{42}, \frac{101}{42}\right], \left[\frac{7}{3}, \frac{8}{3}\right], \frac{1}{27}\,\frac{(x+3)^3}{(x-1)^2}\right) + \frac{5821200}{113693}\left(\left(x^3 - \frac{351}{55}x^2 + \frac{243}{5}x + \frac{567}{55}\right)(x+3)\,\mathrm{hypergeom}\left(\left[\frac{41}{42}, \frac{47}{42}, \frac{59}{42}\right], \left[\frac{4}{3}, \frac{5}{3}\right], \frac{1}{27}\,\frac{(x+3)^3}{(x-1)^2}\right) + \frac{433944}{4675}\,\mathrm{hypergeom}\left(\left[-\frac{1}{42}, \frac{5}{42}, \frac{17}{42}\right], \left[\frac{1}{3}, \frac{2}{3}\right], \frac{1}{27}\,\frac{(x+3)^3}{(x-1)^2}\right)\left(x - \frac{21}{41}\right)(x-1)^2\right)(x-1)^2\right)x^2\right) \quad (3)$$

```
> simplify(odetest(sol, ODE));
```

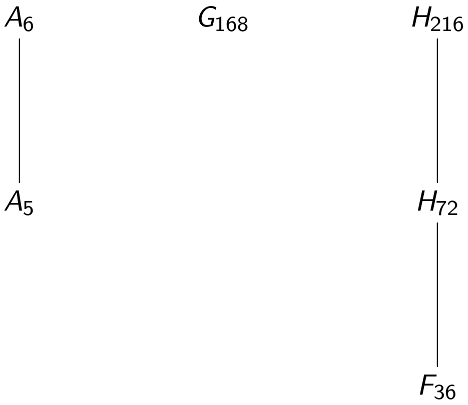
$$0 \quad (4)$$

```
> M;
```

$$\begin{bmatrix} 1 & \frac{14x(x-1)}{19x-7} & 0 \\ 0 & \frac{627x^2-462x+147}{(19x-7)^2} & \frac{14x(x-1)}{19x-7} \\ \frac{1}{196}\,\frac{-6336x^3+5273x^2-343x+686}{x^2(x-1)^2(19x-7)} & \frac{1}{2}\,\frac{-103968x^4+134729x^3-59115x^2+10731x+343}{(19x-7)^3\,x(x-1)} & \frac{-969x^2+560x+49}{(19x-7)^2} \end{bmatrix} \quad (5)$$

```
>
```

Solving in terms of ${}_3F_2$ for other groups



Solving in terms of $_3F_2$ for other groups

TABLE 1. Main results of the paper (see the text).

Min.	a_0	a_1	a_2	b_1	b_2	r_0, r_1, r_∞	n	p	g'	d
2	$\frac{9}{14}$	$\frac{15}{14}$	$\frac{11}{14}$	$\frac{4}{3}$	$\frac{5}{3}$	3, 2, 7	4	3	168	4, 6, 14
3	$\frac{3}{14}$	$\frac{5}{14}$	$\frac{13}{14}$	$\frac{5}{4}$	$\frac{3}{4}$	4, 2, 7	6	10	168	4, 6, 14
4.1	$\frac{-3}{14}$	$\frac{-1}{14}$	$\frac{3}{14}$	$\frac{4}{7}$	$\frac{-1}{7}$	7, 2, 7	36	19	168	4, 6, 14
4.2	$\frac{-3}{14}$	$\frac{-1}{14}$	$\frac{3}{14}$	$\frac{-3}{7}$	$\frac{6}{7}$	7, 2, 7	36	19	168	4, 6, 14
5	$\frac{3}{10}$	$\frac{5}{10}$	$\frac{7}{10}$	$\frac{4}{3}$	$\frac{2}{3}$	3, 2, 5	2	0	60	2, 6, 10
6	$\frac{5}{10}$	$\frac{7}{10}$	$\frac{13}{10}$	$\frac{8}{5}$	$\frac{7}{5}$	5, 2, 5	6	4	60	2, 6, 10
7	$\frac{-1}{30}$	$\frac{5}{30}$	$\frac{11}{30}$	$\frac{4}{5}$	$\frac{1}{5}$	5, 2, 5	12	19	360	6, 12, 30
8	$\frac{7}{30}$	$\frac{13}{30}$	$\frac{25}{30}$	$\frac{3}{4}$	$\frac{5}{4}$	4, 2, 5	6	10	360	6, 12, 30
9.1	$\frac{1}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	$\frac{5}{4}$	$\frac{3}{4}$	4, 3, 6	18	28	216	6, 9, 12
9.2	$\frac{7}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	$\frac{5}{4}$	$\frac{7}{4}$	4, 3, 6	18	28	216	6, 9, 12
9.3	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{5}{6}$	$\frac{5}{4}$	$\frac{3}{4}$	4, 3, 6	18	28	216	6, 9, 12
10.1	$\frac{1}{9}$	$\frac{4}{9}$	$\frac{7}{9}$	$\frac{7}{6}$	$\frac{3}{6}$	6, 3, 3	12	19	216	6, 9, 12
10.2	$\frac{10}{9}$	$\frac{4}{9}$	$\frac{7}{9}$	$\frac{7}{6}$	$\frac{9}{6}$	6, 3, 3	12	19	216	6, 9, 124
11	$\frac{2}{9}$	$\frac{5}{9}$	$\frac{8}{9}$	$\frac{5}{4}$	$\frac{3}{4}$	4, 3, 3	6	10	216	6, 9, 12
12	$\frac{-1}{12}$	$\frac{2}{12}$	$\frac{5}{12}$	$\frac{3}{4}$	$\frac{1}{4}$	4, 2, 4	3	1	36	3, 3, 6

Beukers & Heckman (1989)

Solving in terms of standard functions in higher order

If

$$\left(\frac{d}{dt}\right)^n y + a_{n-1}\left(\frac{d}{dt}\right)^{n-1} y + \ldots + a_1 \frac{d}{dt} y + a_0 y = 0,$$

is an algebraic LODE, for any system of solutions $\mathbf{y} = (y_1, \ldots, y_n)$, we have that $\mathbf{y}, \ldots, \mathbf{y}^{(n-2)}$ are linearly independent there for we have:

$$\begin{bmatrix} 1 & f_1 & \ldots & f_{n-2} & 0 \\ 0 & 1 + f_1' & \ldots & f_{n-3} + f_{n-2}' & f_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & \ldots & * & * \end{bmatrix} \begin{bmatrix} y \\ y' \\ \vdots \\ y^{(n-1)} \end{bmatrix} = \begin{bmatrix} f \cdot F_{Std} \circ s \\ \left(f \cdot F_{Std} \circ s\right)' \\ \vdots \\ \left(f \cdot F_{KQ} \circ s\right)^{(n-1)} \end{bmatrix}$$

Solving in terms of a standard functions in higher order

If

$$\left(\frac{d}{dt}\right)^n y + a_{n-1}\left(\frac{d}{dt}\right)^{n-1} y + \ldots + a_1 \frac{d}{dt} y + a_0 y = 0,$$

has finite and primitive differential Galois group, for any system of solutions $\mathbf{y} = (y_1, \ldots, y_n)$, we have that $\mathbf{y}, \ldots, \mathbf{y}^(n-2)$ are linearly independent there for we have:

$$\begin{bmatrix} y \\ y' \\ \vdots \\ y^{(n-1)} \end{bmatrix} = \begin{bmatrix} 1 & f_1 & \ldots & f_{n-2} & 0 \\ 0 & 1 + f_1' & \ldots & f_{n-3} + f_{n-2}' & f_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & \ldots & * & * \end{bmatrix}^{-1} \begin{bmatrix} f \cdot F_{Std} \circ s \\ \left(f \cdot F_{Std} \circ s\right)' \\ \vdots \\ \left(f \cdot F_{KQ} \circ s\right)^{(n-1)} \end{bmatrix}$$

Take-home message

Given a finite primitive group $G \subseteq GL_n(\mathbb{C})$, one can choose a standard equations such that the solutions to every LODE with differential Galois group G can be expressed in the form

$$y(t) = g_0(t)F_{Std}\big(s(t)\big) + \ldots + g_{n-1}(t)F_{Std}^{(n-1)}\big(s(t)\big)$$

where the terms $g_i(t)$ and $s(t)$ involve elements obtained from towers of field extensions of degree the degree of the $n-1$ smallest algebraically independent homogeneous invariants in $\mathbb{C}[X_1, \ldots, X_n]^G$.

Take-home message

Cheers!

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