Differential Algebraic Generating Series for Walks in the Quarter Plane

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Transcendence et Combinatoire

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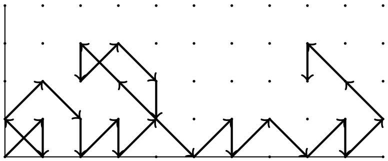
Walks

Consider the walks in the quarter plane starting from $(0,0)$ with steps in a fixed set

$$\mathcal{D} \subset \{\leftarrow, \nwarrow, \uparrow, \nearrow, \rightarrow, \searrow, \downarrow, \swarrow\} \leftrightarrow \{(i,j) \mid i,j \in \{-1,0,1\}\}$$

Example with possible directions

$$\mathcal{D} = \{\nwarrow, \nearrow, \searrow, \downarrow\}.$$



Assign probabilities to each $(i,j) \in \mathcal{D}$ and ask what is

$$\mathbb{P}\left((0,0) \to^k (l,s)\right)$$

the probability that a walk starting at $(0,0)$ ending at (l,s) after k steps?

Models and Generating Series

Weighted Model: Fix a set of probabilistic weights

$$\mathcal{W} = \{(d_{i,j})_{i,j=-1,0,1} \in (\mathbb{Q} \cap [0,1])^9 \text{ with } \sum d_{i,j} = 1\},$$

associated with a set of directions $\mathcal{D} := \{(i,j) | d_{i,j} \neq 0\}$

Unweighted Model: the $d_{i,j} = \frac{1}{|\mathcal{D}|}$ for all $(i,j) \in \mathcal{D}$ and $d_{0,0} = 0$. In this case

$$\mathbb{P}\left((0,0) \to^k (l,s)\right) = \frac{\#(\text{walks from } (0,0) \text{ to } (l,s) \text{ with } k \text{ steps})}{|\mathcal{D}|^k}$$

Generating Series: Fix $\mathcal{W}$ (and therefore $\mathcal{D}$)

$$Q_{\mathcal{W}}(x,y,t) = \sum_{l,s,k} \mathbb{P}\left((0,0) \to^k (l,s)\right) x^l y^s t^k$$

converges for $|x|, |y| \leq 1$ and $|t| < 1$.

Classification

$$\text{Algebraic/Analytic properties of } Q_{\mathcal{W}}(x, y, t)$$
$$\Updownarrow$$
$$\text{Asymptotic properties of } \mathbb{P}\left((0,0) \to^{k} (l, s)\right)$$

Classification problem: when is $Q_{\mathcal{D}}(x, y, t)$

- Algebraic over $\mathbb{C}(x, y, t)$?
- Holonomic over $\mathbb{C}(x, y, t)$? (x-, y-, and t-holonomic)
- Differentially Algebraic over $\mathbb{C}(x, y, t)$? (x-,y-, and t-diff. algebraic)

$f(x, y, t)$ is <u>x-holonomic</u> if for some n and $a_i \in \mathbb{C}(x, y, t)$,

$$a_n(x, y, t)\frac{\partial^n f}{\partial x^n} + \ldots + a_1(x, y, t)\frac{\partial f}{\partial x} + a_0(x, y, t) f = 0$$

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- Differentially Algebraic over $\mathbb{C}(x, y, t)$? (x-,y-, and t-diff. algebraic)

$f(x, y, t)$ is <u>x-differentially algebraic</u> if for some n and polynomial $P \neq 0$,

$$P(x, y, t, f, \frac{\partial f}{\partial x}, \ldots, \frac{\partial^n f}{\partial x^n}) = 0$$

Classification

Fayolle, Iasnogorodski, Malyshev (1999), Bousquet-Mélou, Mishna (2010) - associate to a model $\mathcal{W}$,

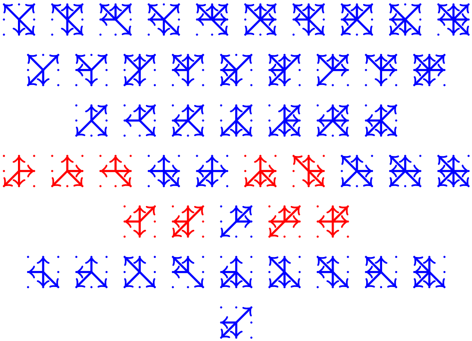
- an algebraic curve $E_{\mathcal{W}}$ of genus 0 or 1, and
- a group $G_{\mathcal{W}}$, finite or infinite.

256 choices for $\mathcal{D} \xrightarrow{\textit{triviallity}, \textit{symmetries}}$ 79 interesting ones

Results: For the **79** unweighted models

- $|G_{\mathcal{D}}| < \infty$ for **23** walks $\Rightarrow Q_{\mathcal{D}}(x, y, t)$ algebraic or holonomic.
  $\rightarrow$ A. Bostan, M. Bousquet-Mélou, M. van Hoeij, M. Kauers, M. Mishna, ...
- $|G_{\mathcal{D}}| = \infty$ for **56** walks $\Rightarrow Q_{\mathcal{D}}(x, y, t)$ **not** holonomic.
  - 5 walks with genus$(E_{\mathcal{W}}) = 0$ $\rightarrow$ S. Melzcer, M. Mishna, A. Rechnitzer, ...
  - 51 walks with genus$(E_{\mathcal{W}}) = 1$ $\rightarrow$A. Bostan, I. Kurkova, K. Raschel, B. Salvy, ...
- Differentially Algebraic???

The unweighted 51 models with $|G_{\mathcal{D}}| = \infty$, genus$(E_{\mathcal{W}}) = 1$



Theorem (D-H-R-S, 2018): For $t \in \mathbb{C}\backslash\overline{\mathbb{Q}}$

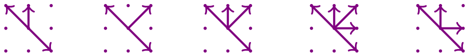
1. In 42 cases, $x \mapsto Q_{\mathcal{D}}(x, 0, t)$ is not x-DA, $y \mapsto Q_{\mathcal{D}}(0, y, t)$ is not y-DA.
2. In 9 cases, $x \mapsto Q_{\mathcal{D}}(x, 0, t)$ is x-DA, $y \mapsto Q_{\mathcal{D}}(0, y, t)$ is y-DA but neither is holon.

- 1. implies $Q_{\mathcal{D}}(x, y, t)$ is not x- or y-DA (and so not holon.) in these cases.
- 2. $Q_{\mathcal{D}}(x, y, t)$ not x-,y-, or t-DA first shown by O. Bernardi, M. Bousquet-Mélou, K. Raschel

What about weighted models?

Weighted Models

- For 5 models with $|G_{\mathcal{D}}| = \infty$, genus$(E_{\mathcal{W}}) = 0$:



<u>Theorem (D-H-R-S, 2020):</u> For $t \in \mathbb{R}\backslash\overline{\mathbb{Q}}$

In all weighted cases, $x \mapsto Q_{\mathcal{W}}(x, 0, t)$, is not x-DA and $y \mapsto Q_{\mathcal{W}}(0, y, t)$ is not y-DA.

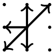
This implies $Q_{\mathcal{W}}(x, y, t)$ is not DA (and so not holon.) in these cases.

For 51 weighted models with $|G_{\mathcal{W}}| = \infty$, genus$(E_{\mathcal{W}}) = 1$

some are and some are not

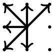
Examples

Ex.1 The weighted model



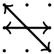
is differentially algebraic iff $d_{-1,-1}d_{1,1} - d_{0,-1}d_{0,1} = 0$

Ex.2 The weighted model



is differentially algebraic iff $d_{-1,1}d_{0,1}^2 - d_{0,1}d_{-1,-1}d_{0,-1} + d_{1,1}d_{-1,-1}^2 = 0$

Ex.3 The weighted model



is differentially algebraic iff $d_{1,0}d_{-1,0} - d_{-1,1}d_{1,-1} = 0$. In this case the group is D_4 or D_8 and the generating series is holonomic.

- Generalities about Walks: Functional Equation, Curve, Group
- Theorems for Differential Algebraicity: Certificates, Decoupling, Orbit Residues
- Algorithms for Differential Algebraicity: Mordell-Weil Lattices, Néron-Tate Height

Generalities: Functional Equation of the Walk

Generating series: Fix $\mathcal{W}$ (and therefore $\mathcal{D}$)

$$Q_{\mathcal{W}}(x,y,t) = \sum_{l,s,k} \mathbb{P}\left((0,0) \to^{k} (l,s)\right) x^{l} y^{s} t^{k}$$

Step Inventory: $\mathcal{S}_{\mathcal{W}}(x,y) = \sum_{(i,j)} d_{i,j} x^{i} y^{j}$
Kernel of the Walk: $K_{\mathcal{W}}(x,y,t) = xy(1 - t\mathcal{S}_{\mathcal{W}}(x,y))$ - biquadratic
Functional Equation:

$$\begin{aligned} K_{\mathcal{W}}(x,y,t)Q_{\mathcal{W}}(x,y,t) &= xy \\ &\quad - K_{\mathcal{W}}(x,0,t)Q_{\mathcal{W}}(x,0,t) - K_{\mathcal{W}}(0,y,t)Q_{\mathcal{W}}(0,y,t) \\ &\quad + K_{\mathcal{W}}(0,0,t)Q_{\mathcal{W}}(0,0,t). \end{aligned}$$

- $Q_{\mathcal{W}}(x,y,t)$ is x-DA $\Longleftrightarrow$ $Q_{\mathcal{W}}(x,0,t)$ is x-DA (similarly for y-DA)
- $Q_{\mathcal{W}}(x,0,t)$ is x-DA $\Longleftrightarrow$ $Q_{\mathcal{W}}(0,y,t)$ is y-DA.
- *(Dreyfus-Hardouin 2019)* $K_{\mathcal{W}}(x,y,t)$ is x-DA $\Longleftrightarrow$ $K_{\mathcal{W}}(x,y,t)$ is t-DA

Curve of the Walk

Step Inventory: $\mathcal{S}_{\mathcal{W}}(x,y) = \sum_{(i,j) q_{i,j} \in \mathcal{W}} x^i y^j$

Kernel of the Walk: $K_{\mathcal{W}}(x,y,t) = xy(1 - t\mathcal{S}_{\mathcal{W}}(x,y))$ - biquadratic

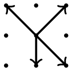
Functional Equation:

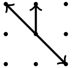
$$K_{\mathcal{W}}(x,y,t)Q_{\mathcal{W}}(x,y,t) = xy - K_{\mathcal{W}}(x,0,t)Q_{\mathcal{W}}(x,0,t) - K_{\mathcal{W}}(0,y,t)Q_{\mathcal{W}}(0,y,t) + K_{\mathcal{W}}(0,0,t)Q_{\mathcal{W}}(0,0,t).$$

Fix $t \in \mathbb{C} \backslash \overline{\mathbb{Q}}$. The Curve of the Walk is the curve

$$E_{\mathcal{W}} = \overline{\{(x,y) \mid K_{\mathcal{W}}(x,y,t) = 0\}}^{Zariski} \subset \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$$

Fact: $K_{\mathcal{W}}$ irreducible $\Rightarrow$ $E_{\mathcal{W}}$ has genus 0 or 1.

Ex: 1) $\mathcal{D} =$  $E_{\mathcal{W}} : xy - t(y^2 + x^2y^2 + x^2 + x) = 0 \quad \Rightarrow g(E_{\mathcal{W}}) = 1$

2) $\mathcal{D} =$  $E_{\mathcal{W}} : xy - t(y^2 + xy^2 + x^2) = 0 \quad \Rightarrow g(E_{\mathcal{W}}) = 0$

for $t \in \mathbb{C} \backslash \overline{\mathbb{Q}}$

Group of the Walk

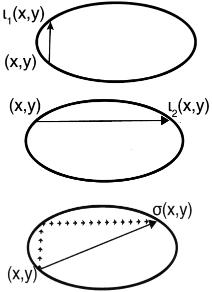
$$E_{\mathcal{W}} = \overline{\{(x,y) \mid K_{\mathcal{W}}(x,y,t) = 0\}}^{Zariski} \subset \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$$

We define two involutions of $E_{\mathcal{W}}$ and an automorphism:

$$\iota_1(x,y) = \left(x, \frac{1}{y}\frac{\sum_i q_{i,-1}x^i}{\sum_i q_{i,1}x^i}\right)$$

$$\iota_2(x,y) = \left(\frac{1}{x}\frac{\sum_j q_{-1,j}y^j}{\sum_j q_{1,j}y^j}, y\right)$$

$$\sigma_{\mathcal{W}} = \iota_2 \circ \iota_1$$



The Group of the Walk $G_{\mathcal{W}}$ is the group generated by ι_1, ι_2.

Facts: 1) $\sigma_{\mathcal{W}}$ is a QRT-map. (Duistermaat - *Discrete Integrable Systems*)
2) $G_{\mathcal{W}}$ is infinite iff $\sigma_{\mathcal{W}}$ is infinite.
3) $g(E_{\mathcal{W}}) = 0 \Rightarrow G_{\mathcal{W}}$ are fractional linear trans.
4) $g(E_{\mathcal{W}}) = 1 \Rightarrow \exists \mathbf{P} \in E_{\mathcal{W}}$, s.t. $\sigma_{\mathcal{W}}(\mathbf{Q}) = \mathbf{Q} \oplus \mathbf{P}$.

From now on, we will assume that $E_{\mathcal{W}}$ has genus 1 and $G_{\mathcal{W}}$ is infinite.

$E_{\mathcal{W}}$ has genus 1 $\Longleftrightarrow$ $E_{\mathcal{W}}$ is irreducible and smooth.

$G_{\mathcal{W}}$ is infinite $\Longleftrightarrow$ order of $\sigma > 6$.

Theorems for Differential Algebraicity

FIx $\mathcal{W} = \{d_{i,j}\}$

Curve: $E_{\mathcal{W}} = \overline{\{(x, y) \mid K_{\mathcal{W}}(x, y, t) = 0\}}^{Zariski} \subset \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$

Group: $G_{\mathcal{W}} = \langle \iota_1, \iota_2 \rangle, \sigma = \iota_2 \circ \iota_1,\ \sigma(\mathbf{Q}) = \mathbf{Q} \oplus \mathbf{P}$; $\mathbf{P}, \mathbf{Q} \in E_{\mathcal{W}}$, $\mathbf{P}$ infinite order.

There is a derivation δ on functions on $E_{\mathcal{W}}$ such that $\delta \circ \sigma = \sigma \circ \delta$.

Prop.(D-H-R-S, 2018) Let $b = x(\iota_1(y) - y) \in \mathbb{C}(E_{\mathcal{W}})$.TFAE:

1. $Q_{\mathcal{W}}(0, y, t)$ is y-DA.
2. There exist an integer $n \geq 0$, $c_i \in \mathbb{C}$, and $g \in \mathbb{C}(E_{\mathcal{W}})$ such that

$$\delta^n(b) + c_{n-1}\delta^{n-1}(b) + \ldots + c_1\delta(b) + c_0 b = \sigma(g) - g.$$

The proof relies heavily on

- the fact that $Q_{\mathcal{W}}(0, y, t)$ converges in a nbhd of 0 and can be continued to a meromorphic fnc on $\mathbb{C}$. *F-I-M (1999), Kurkova/Raschel (2012) (unweighted), Dreyfus/Raschel (2019) (weighted)*
- A Galois theory measuring differential dependencies of a function satisfying a difference equation. *H-S (2008)*

Theorems for Algebraicity: Certificates

Prop.(1. $\Leftrightarrow$ 2. *D-H-R-S (2018);* 2. $\Leftrightarrow$ 3. *H-S (2020)*) Let $b \in \mathbb{C}(E_{\mathcal{W}})$. TFAE

1. There exist an integer $n \geq 0$, $c_i \in \mathbb{C}$, and $g \in \mathbb{C}(E_{\mathcal{W}})$ such that

$$\delta^n(b) + c_{n-1}\delta^{n-1}(b) + \ldots + c_1\delta(b) + c_0 b = \sigma(g) - g.$$

2. There exist $\mathbf{P}_0 \in E_{\mathcal{W}}$ and $g, h \in \mathbb{C}(E_{\mathcal{W}})$ with h having only poles at $\mathbf{P}_0, \mathbf{P}_0 \ominus \mathbf{P}$ of order at most 1 such that

$$b = \sigma(g) - g + h.$$

3. Assume $\iota_1(b) = -b$. There exists $g \in \mathbb{C}(E_{\mathcal{W}})$ such that

$$b = \sigma(g) - g.$$

   This g is called a certificate

Note: For $b = x(\iota_1(y) - y)$, we have $\iota_1(b) = -b$.

$$Q_{\mathcal{W}}(x, y, t) \text{ is DA}$$
$$\Updownarrow$$
$$b = x(\iota_1(y) - y) = \sigma(g) - g \text{ for some } g \in \mathbb{C}(E_{\mathcal{W}})$$

Theorems for Algebraicity: Decoupling

Def. A model $\mathcal{W}$ is decoupled if there exist $f(x) \in \mathbb{C}(x)$ and $g(y) \in \mathbb{C}(y)$ s.t.

$$xy = f(x) + g(y)$$

in $\mathbb{C}(E_{\mathcal{W}})$. In this case $(f(x), g(y))$ are called a decoupling pair.

Bernardi, Bousquet-Mélou, Raschel (2017) show of 79 unweighted models,

- 13 are decoupled; 9 with $|G_{\mathcal{W}}| = \infty$, 4 with $|G_{\mathcal{W}}| < \infty$.
- decoupled models have DA (in all variables) generating series and give analytic expressions for generating series when $|G_{\mathcal{W}}| = \infty$.
- decoupled models with $|G_{\mathcal{W}}| < \infty$ have algebraic generating series which can be produced.

Theorems for Algebraicity: Decoupling

Def. A model $\mathcal{W}$ is decoupled if there exist $f(x) \in \mathbb{C}(x)$ and $g(y) \in \mathbb{C}(y)$ s.t.

$$xy = f(x) + g(y)$$

in $\mathbb{C}(E_{\mathcal{W}})$. In this case $(f(x), g(y))$ are called a decoupling pair.

Bernardi, Bousquet-Mélou, Raschel (2017) show for unweighted models

decoupling pair $\Longrightarrow$ DA generating series

(Dreyfus (in preparation) for weighted models)

Prop. Assume the model $\mathcal{W}$ has an infinite group. TFAE

- $\mathcal{W}$ is decoupled.
- The element $b = x(\iota_1(y) - y)$ has a certificate in $\mathbb{C}(E_{\mathcal{W}})$, i.e. $b = \sigma(g) - g$ for some $g \in \mathbb{C}(E_{\mathcal{W}})$.

In this case, if $(f(x), g(y))$ is a decoupling pair, then $g(y)$ is a certificate for b.
If g is a certificate for b, then $(f = xy - g, g)$ is a decoupling pair.

Theorems for Algebraicity: Decoupling

Prop. Assume the model $\mathcal{W}$ has an infinite group. TFAE

- $\mathcal{W}$ is decoupled.
- The element $b = x(\iota_1(y) - y)$ has a certificate in $\mathbb{C}(E_{\mathcal{W}})$, i.e. $b = \sigma(g) - g$.

In this case, if $(f(x), g(y)$ is a decoupling pair, then $g(y)$ is a certificate for b. If g is a certificate for b, then $(f = xy - g, g)$ is a decoupling pair.

Half of a Proof: Decoupled $\Rightarrow$ Certificate.

$\mathbb{C}(E_{\mathcal{W}})$ = quotient field of $\mathbb{C}[x, y]/(K(x, y))$.

$\mathbb{C}(x) \hookrightarrow \mathbb{C}(E_{\mathcal{W}})$ is fixed field of ι_1. $\mathbb{C}(y) \hookrightarrow \mathbb{C}(E_{\mathcal{W}})$ is fixed field of ι_2.

Assume $xy = f(x) + g(y)$, then

$$\iota_1(xy) = x\iota_1(y) = \iota_1(f(x) + g(y) = f(x) + \iota_1(g(y))$$

Subtract $\Rightarrow x(\iota_1(y) - y) = \iota_1(g(y)) - g(y)$

Since $\sigma(g(y)) = \iota_1(\iota_2(g(y))) = \iota_1(g(y))$, we have

$$x(\iota_1(y) - y) = \iota_1(g(y)) - g(y) = \sigma(g(y)) - g(y)$$

Theorems for Algebraicity: Orbit Residues

Def. $E_{\mathcal{W}}$ elliptic curve, σ the addition by a non-torsion point $\mathbf{P}$, $K = \mathbb{C}(E_{\mathcal{W}})$

- $\{u_{\mathbf{Q}} \mid \mathbf{Q} \in E_{\mathcal{W}}\}$ local param. are **coherent** if $u_{\mathbf{Q} \ominus \mathbf{P}} = \sigma(u_{\mathbf{Q}})$.
- For $g \in \mathbb{C}(E_{\mathcal{W}}),\ \mathbf{Q} \in E_{\mathcal{W}}$, write

$$g = \frac{c_{\mathbf{Q},N}}{{u_{\mathbf{Q}}}^N} + \cdots + \frac{c_{\mathbf{Q},i}}{u_{\mathbf{Q}}^i} + \cdots + \frac{c_{\mathbf{Q},1}}{u_{\mathbf{Q}}} + f$$

with f regular at $\mathbf{Q}$. Then, the $\mathbf{i^{th}}$ **orbit residue** of g at $\mathbf{Q}$ is

$$\mathrm{ores}^i_{\mathbf{Q}}(g) = \sum_{n \in \mathbb{Z}} c^i_{\sigma^n(\mathbf{Q})}.$$

Prop. *(D-H-R-S (2018))* The following are equivalent for $b \in \mathbb{C}(E_{\mathcal{W}}),\ \iota_1(b) = -b$:

- b has a certificate.
- For all $i \in \mathbb{N}_{>0}, Q \in E_{\mathcal{W}}$, $\mathrm{ores}^i_Q(b) = 0$.

To determine if $Q_{\mathcal{W}}(x, y, t)$ is DA

find the orbits of the poles of $b = x(\iota_1(y) - y)$ and their orbit residues.

Theorems for Algebraicity: Orbit Residues

To prove DA, show: for all $i \in \mathbb{N}_{>0}$, $Q \in E_{\mathcal{W}}$, $\mathrm{ores}^i_Q(b) = \sum_{n\in\mathbb{Z}} c^i_{\sigma^n(\mathbf{Q})} = 0$

Ex. The weighted model

The polar divisor of b is $(b)_\infty = \mathbf{M} + \mathbf{N} + \iota_1(\mathbf{N})$ where $\mathbf{M}, \mathbf{N} \in \mathbb{P}^1 \times \mathbb{P}^1$ and

- $\mathbf{M} = ([1:0],[0:1])$ and $\iota_1(\mathbf{M}) = \mathbf{M}$; residue $= \alpha \neq 0$.
- $\mathbf{N} = ([-d_{0,1} : d_{1,1}],[1:0])$; residue $= \beta \neq 0$
- $\iota_1(\mathbf{N}) = ([-d_{0,1} : d_{1,1}], *)$; residue $= \beta \neq 0$

Classical Residue Theorem $\Rightarrow \alpha + 2\beta = 0$ so if all poles in same orbit (in particular, $\mathbf{M} = \sigma^n(\mathbf{N})$), then the orbit residues = 0.

If $\mathbf{M} = \sigma^n(\mathbf{N})$, then $\mathbf{M} = \iota_1(\mathbf{M}) = \iota_1(\sigma^n(\mathbf{N})) = \sigma^{-n}(\iota_1(\mathbf{N}))$ so all poles are in the same orbit and orbit residues are 0.

Differential Algebraicity $\iff$ $\mathbf{M} = \sigma^n(\mathbf{N})$ for some n.

Theorems for Algebraicity: Orbit Residues

In general, we have

the generating series of a weighted model $\mathcal{W}$ is differentially algebraic

$\Updownarrow$

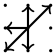
two specific poles of b lie in the same orbit.

The two poles giving this criterion depend on the relative positions of the (at most 6) poles of b and their behavior under ι_1, ι_2 and not on the weights.

The condition that these poles lie in the same orbit does depend on the weights and gives the NASC, in terms of weights, for the generating series to be DA.

How does one decide if $\exists n \in \mathbb{Z}$ s.t. $\mathbf{M} = \sigma^n(\mathbf{N})$?

Ex. The weighted model



has differential algebraic generating series if and only

$$([1:0],[0:1]) = \sigma^{1}([-d_{0,1} : d_{1,1}],[1:0]).$$

$$\Updownarrow$$

$$d_{1,0}d_{-1,0} - d_{-1,1}d_{1,-1} = 0$$

How does one find 1?

Algorithms for Algebraicity: Mordell-Weil Lattices, Néron-Tate Height

Mordell-Weil-Néron Theorem. If E be an elliptic curve defined over k, a finitely generated extension of $\mathbb{Q}$ then the group $E(k)$ of k-rational points, is a finitely generated abelian group,

$$E(k) = \mathbb{Z} \oplus \ldots \oplus \mathbb{Z} \oplus E(k)_{torsion}.$$

Denote $E(k)/E(k)_{torsion}$ by $\mathrm{MWL}(E)$.

Now assume E is defined over $k = \mathbb{Q}(t)$ and that E does not descend to $\mathbb{Q}$.

There is a $\mathbb{Q}$-valued symm. bilinear form

$$\langle *, * \rangle : E(k) \times E(k) \to \mathbb{Q}$$

called the Néron-Tate Pairing and the quadratic form

$$\hat{h}(\mathbf{Q}) = \langle \mathbf{Q}, \mathbf{Q} \rangle$$

is called the Néron-Tate Height. Furthermore, the lattice

$$(\mathrm{MWL}(E), \langle *, * \rangle)$$

is called the Mordell-Weil Lattice of E.

Algorithms for Algebraicity: Mordell-Weil Lattices, Néron-Tate Height

(Oguiso-Shioda): As groups, there are 26 possibilites for for $E(k)$. The order of any element is at most 6 or infinite.

Properties of $\hat{h}$:

- If $\mathbf{N}$ is a torison point, then $\hat{h}(\mathbf{N}) = 0$.
- If $\mathbf{M} = n\mathbf{N}$, then $\hat{h}(\mathbf{M}) = n^2\hat{h}(\mathbf{N})$.
- $\hat{h}(\mathbf{N})$ is computable. *For the points we consider, this depends on the configuration of base points of the family $K(x, y, t) = 0$ and certain lines in a blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$, not on the weights.*

Using $\hat{h}$ to determine if $\mathbf{M} = \sigma^n(\mathbf{N})$ where $\sigma(\mathbf{Q}) = \mathbf{Q} \oplus \mathbf{P}$:

On an elliptic curve we can select the point we call $\mathbf{O}$. Let $\mathbf{O} = \mathbf{N}$. We have

$$\sigma(\mathbf{N}) = \mathbf{N} \oplus \mathbf{P} = \mathbf{O} \oplus \mathbf{P} = \mathbf{P}$$

so $\sigma^n(\mathbf{N}) = n\mathbf{P}$. Therefore

$$\mathbf{M} = \sigma^n(\mathbf{N}) \Leftrightarrow \mathbf{M} = n\mathbf{P} = n\sigma(\mathbf{N}).$$

So $\mathbf{M} = \sigma^n(\mathbf{N}) \Rightarrow \hat{h}(\mathbf{M}) = n^2\hat{h}(\sigma(\mathbf{N}))$.

Algorithms for Algebraicity

Fix a rational step set $\mathcal{W}$ such that the curve is an elliptic curve $E_{\mathcal{W}}$ and the $G_{\mathcal{W}} = \langle \iota_1, \iota_2 \rangle$ is infinite. Let $\sigma(\mathbf{Q}) = \mathbf{Q} \oplus \mathbf{P}$.

The generating series is DA

$$\Updownarrow$$

$b = x(\iota_1(y) - y)$ has a certificate.
In this case xy is decoupled and the results of BBMR give expression.

$$\Updownarrow$$

There are two poles of b, $\mathbf{M}, \mathbf{N} \in E(\mathbb{Q}(t))$, such that $\sigma^n(\mathbf{N}) = \mathbf{M}$, $n \in \mathbb{Z}$
$\mathbf{M}, \mathbf{N}$ depend only on $\mathcal{D}$, not on the weights.

$$\Updownarrow$$

Determine if $\exists n \in \mathbb{Z}$ s.t. $\hat{h}(\mathbf{M}) = n^2 \hat{h}(\sigma(\mathbf{N}))$.
If no, the generating series is not DA.
If yes, the condition $\sigma^n(\mathbf{N}) = \mathbf{M}$ yields polynomial conditions on the weights giving DA.

Ex.

DA gen. series $\Leftrightarrow$ $([1:0],[0:1]) = \sigma^{1}([-d_{0,1} : d_{1,1}],[1:0])$ $\Leftrightarrow d_{1,0}d_{-1,0} - d_{-1,1}d_{1,-1} = 0$

More About the Néron-Tate Height

$$K(x,y,t) = xy - txyS(x,y)$$
$$S(x,y) = \sum_{(i,j)} q_{i,j} x^i y^j$$

Fix $t \Rightarrow$ curve $E_{\mathcal{W}} \subset \mathbb{P}^1 \times \mathbb{P}^1$ coordinates $([x_0, x_1] : [y_0 : y_1])$

$$E_{\mathcal{W}} : x_1^2 y_1^2 K(\frac{x_0}{x_1}, \frac{y_0}{y_1}) = x_0 x_1 y_0 y_1 - t\tilde{S}(x_0, x_1, y_0, y_1)$$

Vary t $\Rightarrow$ family of curves. Points satisfying $x_0 x_1 y_0 y_1 = 0$ lie on all curves.

Called Base Points. There are 8 counting multiplicity.

Associate to $E_{\mathcal{W}}$ a surface $\mathcal{S}$

- $\mathcal{S}$ is a smooth projective rational surface with projection $\pi : \mathcal{S} \to \mathbb{P}^1$
- Almost all fibers are isomorphic to $E_{\mathcal{W}}$; $S_{[0:1]}$ is singular.
- Bijection between $\mathbb{Q}(t)$-points $\mathbf{P}$ of $E_{\mathcal{W}}$ and sections $\boldsymbol{\mathcal{P}} : \mathbb{P}^1 \to S$ over $\mathbb{Q}$.

Eight base points distinct $\Rightarrow \mathcal{S} : t_1 x_0 x_1 y_0 y_1 - t_0 \tilde{S}(x_0, x_1, y_0, y_1) = 0$
If not, $\mathcal{S}$ is a blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at the eight points.

More About the Néron-Tate Height

$$E_{\mathcal{W}} \Rightarrow \pi : \mathcal{S} \to \mathbb{P}^1$$

$$\mathbf{P} \in E_{\mathcal{W}}(\mathbb{Q}(t)) \Leftrightarrow \boldsymbol{\mathcal{P}} : \mathbb{P}^1 \to \mathcal{S}$$

Def. $\hat{h}(\mathbf{P}) := \hat{h}(\boldsymbol{\mathcal{P}}) = 2 - \sum_{\nu \in R} contr_\nu(\boldsymbol{\mathcal{P}})$

$$\sum_{\nu \in R} contr_\nu(\boldsymbol{\mathcal{P}}) = ????$$

What we know about $\pi : \mathcal{S} \to \mathbb{P}^1$:

- R = set of reducible fibers
  - Finite number of types *(Kodaira, Néron)*
  - Algorithm/List *(Tate)* - determine the type of any fiber at t in terms of the orders of vanishing at t of the coefficients of the Weierstrass normal form.
- Type of a reducible fiber at $t = \nu \Rightarrow$ Root lattice $T_\nu \subset E_8$
  $T_\nu = \{\boldsymbol{\mathcal{P}} \,|\boldsymbol{\mathcal{P}}$ and $\boldsymbol{\mathcal{O}}$ intersect same component in$S_\nu\} \subset \mathrm{MWL}(E)$,
  - Fiber type $\Rightarrow$ Root lattice T_ν *(Oguiso-Shioda)*
  - Possible $\oplus_{\nu \in R} T_\nu$ *(Oguiso-Shioda)*
- Root Lattice $T_\nu \Rightarrow$ finite set of values for $contr_\nu(\boldsymbol{\mathcal{P}})$ *(Shiota)*
  $contr_\nu(\boldsymbol{\mathcal{P}})$ defined in terms of intersections in reducible fiber at ν.

More About the Néron-Tate Height

Ex. The weighted model

For which weights does there exist an $n \in \mathbb{Z}$ s.t.

$$\mathbf{M} = \sigma^{n}(\mathbf{N}), \quad \text{i.e. } \mathbf{M} = n\sigma(\mathbf{N}).$$

where $\mathbf{M} = ([1:0],[0:1])$ and $\mathbf{N} = ([-d_{0,1} : d_{1,1}], [1:0])$ Let $\mathcal{P} = \mathbf{M}$ or $\sigma(\mathbf{N})$.

- Tate algorithm applied to Weierstrass eqn for $E_{\mathcal{W}} \Rightarrow$ type of $S_0 = I_7$.
- Oguiso-Shioda list: $I_7 \Rightarrow A_6$
  Oguiso-Shioda list: $\Rightarrow$ at most 2 reducible fibers $\mathcal{S}_0, \mathcal{S}_\alpha$ and
  $\oplus_{\nu \in R} T_\nu = A_6$ or $A_6 \oplus A_1$.
- Shiota list: $T_0 = A_6 \Rightarrow contr_\nu(\mathcal{P}) \in \{0, 6/7, 10/7, 12/7\}$
  $T_\alpha = A_1 \Rightarrow contr_\nu(\mathcal{P}) \in \{0, 1/2\}$

Determine all possibilites for $\hat{h}(\mathcal{P}) = 2 - contr_0(\mathcal{P}) - \epsilon contr_\alpha(\mathcal{P}), \epsilon = 0, 1$.

$\hat{h}(\mathbf{M}), \hat{h}(\sigma(\mathbf{N}))$ will lie in this set. If $\hat{h}(\mathbf{M})/\hat{h}(\sigma(\mathbf{N})) = n^2$ for $n \in \mathbb{Z}$ then

$$n = -4, -3, -2, -1, 0, 1, 2, 3, 4$$

Equate $\mathbf{M} = \sigma^n(\mathbf{N})$ for these $\Rightarrow$ only n= 1 and $d_{1,0}d_{-1,0} - d_{-1,1}d_{1,-1} = 0$