Computing reduced forms of block-triangular (reducible) differential system with applications to integrals and transcendence questions.

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Introduction

We consider a *reducible* linear diff system over a diff. field $(\mathbf{k}, \partial)$

$$[A] : \partial Y = AY, \text{ with } A = \left(\begin{array}{c|c} A_2 & 0 \\ \hline S & A_1 \end{array}\right) \in \mathrm{Mat}(\mathbf{k}). \quad (1)$$

Let $A_{\text{diag}} := \left(\begin{array}{c|c} A_2 & 0 \\ \hline 0 & A_1 \end{array}\right)$.

- Assume that A_{diag} is in *reduced form*.
  GOAL : find gauge transformation turning $[A]$ into "reduced form".
- Given a general reducible system where the Galois group of the diagonal part is known, this gives the differential Galois group of the full system and puts it into a more suitable form (with S "smallest" in some sense)

TWO TOY EXAMPLES (1)

Let $f_1(x)$, $f_2(x)$ be solutions (Heun functions) of $L(y) = 0$

$$L(y) := \frac{\mathrm{d}^2}{\mathrm{d}x^2} y(x) + \frac{2}{3}\left(\frac{1}{x} + \frac{1}{x-1}\right)\frac{\mathrm{d}}{\mathrm{d}x} y(x) - \frac{(3x^2 - 6x + 7)}{144\, x\, (x-1)} y(x) = 0$$

$$f_1(x) = \exp\left(\frac{\sqrt{3}}{12} x\right) \mathrm{HeunC}\left(\frac{\sqrt{3}}{6}, -\frac{1}{3}, -\frac{1}{3}, \frac{1}{48}, \frac{11}{48}, x\right)$$

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Let $F_1(x) := \int^x f_1(t)dt$. Can one get integral F_1 from f_i, f_i' or is it new ?
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$$A = \begin{pmatrix} 0 & 1 & 0 \\ \frac{3x^2-6x+7}{144(x-1)^3x^2} & -\frac{2}{3}\left(\frac{1}{x-1} + \frac{1}{x}\right) & 0 \\ 1 & 0 & 0 \end{pmatrix}, U := \begin{pmatrix} f_1(x) & f_2(x) & 0 \\ f_1'(x) & f_2'(x) & 0 \\ \int^x f_1(t)\mathrm{dt} & \int^x f_2(t)\mathrm{dt} & 1 \end{pmatrix}$$

TWO TOY EXAMPLES (1)

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Let G be Diff. Galois gp of $Y' = A.Y$. (...) It has dimension 5.
Method : find gauge transformation $A \to R$ s.t $Lie(R)$ has minimal dimension.

$\dim(Lie(A)) = 6$ and $\dim(Lie(R)) = 5$ Let $\mathfrak{g} := Lie(G)$: then $\mathfrak{g} = Lie(R)$.

Two toy examples (2)

Second example ; take $f_1(x) = {}_2F_1([-\frac{1}{3}, \frac{1}{12}], [\frac{7}{12}])(x)$ and $f_2(x) = x^{5/12}{}_2F_1([\frac{1}{12}, \frac{1}{2}], [\frac{17}{12}])(x)$.

$$A_1 = \begin{pmatrix} 0 & 1 \\ \frac{1}{36}\frac{1}{x(x-1)} & -\frac{7}{12\,x} - \frac{1}{6(x-1)} \end{pmatrix} \text{ and } A = \begin{pmatrix} A_1 & 0 \\ 1, 0 & 0 \end{pmatrix}$$

Reduction technique show that $\dim(Gal([A])) = 3$ and we have relations

$$\int^x f_i\,(t)\ \mathrm{dt} = -\frac{9}{11}\,x\,(x-1)\,f_i'\,(x) + \frac{15}{44}\,(3\,x-1)\,f_i\,(x) + \frac{9}{11}\,c_i$$

MOTIVATION (1 : INTEGRALS VIA REDUCIBLE SYSTEMS)

$$[A] : \partial Y = AY, \text{ with } A = \left(\begin{array}{c|c} A_2 & 0 \\ \hline S & A_1 \end{array}\right) \in \mathrm{Mat}(\mathbf{k}). \tag{2}$$

Why not solve this directly by variations of constants ?
Fundamental solution matrix U is

$$U = \left(\begin{array}{c|c} U_2 & 0 \\ \hline V\, U_1 & U_1 \end{array}\right) \quad \text{with} \quad \left\{ \begin{array}{lcl} {U_i}' & = & A_i U_i \,, \\ V & = & \int \left(U_1^{-1} S U_2 \right) dx \end{array} \right.$$

Involves integrals of complicated functions..
Our approach: "reduce" S with only rational manipulations to prepare for easier solving. *gives (algebraic properties of) integrals*

MOTIVATION (2 : REDUCIBLE SYSTEMS)

Reducible system $[A]$: $Y' = AY$ with $A = \begin{pmatrix} A_k & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & A_2 & 0 \\ S_k & \cdots & S_2 & A_1 \end{pmatrix}$.

Several interests of reducible linear differential systems

- Arise naturally as *variational equations* of non-linear differential systems along a particular solution

  $\longrightarrow$ Morales, Ramis, Simó, Casale, Maciejewski/Przybylska, Combot, Aparicio etc.

- Arise naturally in ϵ-expansions for systems $\frac{d}{dx}Y = B(x, \epsilon)Y$

  $\longrightarrow$ J. Blümlein, C. Raab, C. Schneider, J. Henn etc.

- Operators from statistical physics, combinatorics (minimal operators of generating series) are often reducible

  $\longrightarrow$ J.-M. Maillard, A. Bostan, C. Koutschan, M. Kauers, Y. Abdelaziz, etc.

- Allows to study reducible monodromies (Kalmykov)

1. Introduction
   - The problem
   - Motivation
2. Reduced forms of linear differential systems
   - Ingredient #1 : Differential Galois Group and its Lie Algebra
   - Ingredient #2 : Lie algebra $Lie(A(x))$ associated to $A(x)$
3. How we Compute a Reduced Form of a Reducible System?
   - Shape of the gauge transformation
   - Isotypical decomposition
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   - Reduction and rational solutions of systems
   - Algorithm for reducing $\frac{d}{dx}Y = AY$
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II. Reduced forms of linear differential systems

INGREDIENT #1 : DIFFERENTIAL GALOIS GROUP

$$\boldsymbol{Y}' = A(x)\boldsymbol{Y}. \tag{3}$$

k = differential field generated by coefficients of A, $C = Const(k)$
Picard-Vessiot Ext.: $K = k(U_1)$, U_1 fundamental solution matrix
Differential Galois Group : $G := \mathrm{Aut}_\partial(K/k)$

$$\mathrm{Aut}_\partial(K/k) := \{\sigma \in \mathrm{Aut}(K) \ : \ \sigma|_k \equiv \mathrm{id}_k \text{ and } \sigma \circ \partial \equiv \partial \circ \sigma\}$$

G is a group of matrices (linear algebraic group).

G STABILIZES THE *ideal of differential relations* BETWEEN SOLUTIONS

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$\mathfrak{g} = Lie(Y' = AY)$ Lie algebra of G : tangent space of G at Id.
$\mathfrak{g}$ is called the *Galois-Lie algebra of* $[A]$

$\mathfrak{g}$ MEASURES THE TRANSCENDENCE OF K OVER k.
$\dim_{\mathbb{C}} \mathfrak{g} = \mathrm{trdeg}(K/k)$.

Some linear algebraic groups & Lie algebras

G linear algebraic group , $\mathfrak{g}$ its Lie algebra.

$$N \in \mathfrak{g} \iff Id + \epsilon N \in G\,(C[\epsilon]),\ \epsilon^2 = 0.$$

- $M \in SL_n(C)\ : \det(M) = 1 \longrightarrow N \in \mathfrak{sl}_n(C)\ : Tr(N) = 0.$
- $M \in Sp_{2n}(C) : M^T \cdot J \cdot M = J\ \rightarrow\ N \in \mathfrak{sp}_n(C) \colon N^T J + JN = 0.$

$$J = \begin{pmatrix} 0 & Id \\ -Id & 0 \end{pmatrix}$$

SOME LINEAR ALGEBRAIC GROUPS & LIE ALGEBRAS

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- $\mathbb{G}_a = \{\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, a \in C\} \longrightarrow \mathfrak{g}_a = Span_C\{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\}$.
- $\mathbb{G}_m = \{\begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix}, a \in C^*\} \longrightarrow \mathfrak{g}_m = Span_C\{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\}$

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- $\mathbb{G}_a = \{\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, a \in C\} \longrightarrow \mathfrak{g}_a = Span_C\{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\}.$
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- $M \in SO(3) : M^T M = Id_{\ \&\ \det(M)=1} \longrightarrow N \in \mathfrak{so}(3) : N^T + N = 0$

INGREDIENT #2 : LIE ALGEBRA ASSOCIATED TO A

Decompose the matrix $A \in \mathcal{M}_n(k)$ as

→ Magnus, Feynman, Wei & Norman 63/64

$$A := \sum_{i=1}^{r} a_i(x) M_i, \qquad M_i \in \mathcal{M}_n(C)$$

The $a_i \in k$ are basis of C-vector space spanned by coeffs of A.
The C-vector space generated by the M_i is unique.

Definition

The (algebraic envelope of the) Lie algebra generated by $M_1, \ldots, M_r$ is called the *Lie algebra $Lie(A)$ associated to $A(x)$*

i.e. gen. by the M_i and the Lie brackets

This is also the (algebraic envelop of the) Lie algebra generated by all $A(x_0)$ for all $x_0 \in C$.

INGREDIENT #2 : EXAMPLE OF $Lie(A)$

$$A := \begin{pmatrix} 1 & \frac{1}{x} & \frac{1}{x-1} & 0 \\ 0 & 1 & 0 & \frac{1}{x-1} \\ 0 & 0 & 1 & -\frac{1}{x} \\ 0 & 0 & 0 & 1 \end{pmatrix}. \qquad A = M_1 + \frac{1}{x} M_2 + \frac{1}{x-1} M_3, \text{ where}$$

$$M_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; M_2 := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; M_3 := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$M_4 := [M_2, M_3] = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \text{Lie}(A) = \langle M_1, M_2, M_3, M_4 \rangle.$$

LIE ALGEBRA ASSOCIATED TO $A \in \mathcal{M}_n(k)$ (2)

$$Y' = AY, \quad A := \sum_{i=1}^{r} a_i(x) M_i, \quad Lie(A) := \overline{Lie(M_1, \ldots, M_r)}.$$

Theorem (Kovacic, Kolchin)

$$Galois - Lie(Y' = AY) \subset Lie(A)$$

Definition

A system $Y' = A(x)Y$ is in *reduced form* if the Lie algebra $Lie(A)$ associated to $A(x)$ equals the Galois-Lie algebra $Lie(Y' = AY)$.

EXISTENCE : KOLCHIN-KOVACIC'S THEOREM

$$A := \sum_{i=1}^{r} a_i(x) M_i, \quad Lie(A) := \overline{Lie(M_1, \ldots, M_r)}$$
$$G = Gal(Y' = AY), \mathfrak{g} = GaloisLie(Y' = AY).$$

Theorem (Kolchin, Kovacic)

$$GaloisLie(Y' = AY) \subset Lie(A)$$

Theorem (Kolchin-Kovacic Reduction (non constructive))

Let $\mathfrak{h} = Lie(A)$, Lie algebra of a connected group H. Then $G \subset H$ and $\exists P \in H(\bar{k})$ such that system $F' = \tilde{A}F$, with $Y = PF$ and $\tilde{A} = P[A] = P^{-1}(AP - P')$, satisfies $\mathfrak{g} = Lie(\tilde{A})$

Reduced form EXISTS : we compute it.

Ingredient #2 : Lie algebra $Lie(A(x))$ associated to $A(x)$

$$A := \begin{pmatrix} 1 & \frac{1}{x} & \frac{1}{x-1} & 0 \\ 0 & 1 & 0 & \frac{1}{x-1} \\ 0 & 0 & 1 & -\frac{1}{x} \\ 0 & 0 & 0 & 1 \end{pmatrix}. \ \mathrm{Lie}(A) = \langle M_1, M_2, M_3, M_4 \rangle.$$

$$U = e^x \begin{pmatrix} 1 & \ln(x) & \ln(x-1) & 2\,\mathrm{dilog}(x) + \ln(x-1)\ln(x) \\ 0 & 1 & 0 & \ln(x-1) \\ 0 & 0 & 1 & -\ln(x) \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where dilog is defined by $\mathrm{dilog}'(x) = \frac{\ln(x)}{1-x}$.

$\mathrm{dilog}(x) = \mathrm{Li}_2(1-x)$.

CONSTRUCTIVE APPROACHES TO REDUCED FORMS

Used for inverse problems — Kovacic 71, Kolchin, Singer & Mitschi 96-02-05, Hartmann 01, Juan 04, etc.

Studied for Lie-Vessiot systems — Blazquez & Morales 10, etc.

Kolchin-Kovacic reduction , constructive/algorithmic approaches:

Philosophy : it is "often" easier to compute $\mathfrak{g}$ than the Galois group G .

- REDUCTIVE CASE (irreducible or block-diagonal)
  $\longrightarrow$ Aparicio-Compoint-Weil 2013, criterion+ decision procedure, not efficient
  $\longrightarrow$ Barkatou-Cluzeau-Di Vizio-Weil : algorithm in 2015 (irreducible) and 2017-21 (reductive), much more efficient
- ABELIAN DIAGONAL (effective Morales-Ramis-Simó) $\rightarrow$ Aparicio-Dreyfus-Weil 2011-16
- BLOCK-TRIANGULAR (REDUCIBLE) CASE — Dreyfus-Weil'20 + today

III : How we Compute a Reduced Form of a Reducible System?

Our goal in this part

We consider a *reducible* system over a differential field $(\mathbf{k}, \partial)$

$$[A] : \partial Y = AY, \text{ with } A = \left(\begin{array}{c|c} A_2 & 0 \\ \hline S & A_1 \end{array}\right) \in \mathrm{Mat}(\mathbf{k}). \quad (4)$$

Let $A_{\mathrm{diag}} := \left(\begin{array}{c|c} A_2 & 0 \\ \hline 0 & A_1 \end{array}\right)$. Assume that A_{diag} is in *reduced form* .

We show how to PUT $[A]$ INTO REDUCED FORM .

This will give a general reduction algorithm.

THE MATRYOSHKA EXAMPLE

$$A(x) := \left(\begin{array}{cccc|cccc} 1 & 0 & \frac{1}{x} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{x-1} & 1 & 0 & -\frac{1}{x} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{x-1} & 1 & 0 & 0 & 0 & 0 \\ \hline \star & \star & \star & \star & 1 & 0 & \frac{1}{x} & 0 \\ \star & \star & \star & \star & \frac{1}{x-1} & 1 & 0 & -\frac{1}{x} \\ \star & \star & \star & \star & 0 & 0 & 1 & 0 \\ \star & \star & \star & \star & 0 & 0 & \frac{1}{x-1} & 1 \end{array}\right).$$

A_{diag} is in reduced form. $Lie(A_{\text{diag}})$ has dim. 4:
PV extension $C(x)\,(\exp(x), \ln(x), \ln(x-1), \text{dilog}(x))$

$dilog = \int \frac{\ln(x)}{x-1}$

SHAPE OF THE GAUGE TRANSFORMATION

Let $\mathfrak{h}_{sub}$ be the set of constant matrices of the form $\left(\begin{array}{c|c} 0 & 0 \\ \hline S & 0 \end{array}\right)$.

Lemma (Aparicio,Dreyfus,Weil 2015)

There exists a gauge transformation $P \in \left\{\mathrm{Id} + B, B \in \mathfrak{h}_{sub}(\mathbf{k})\right\}$, *such that* $\partial Y = P[A]Y$ *is in reduced form.*

Let $P = \mathrm{Id} + B$, $B \in \mathfrak{h}_{sub}(\mathbf{k})$.

If $\mathrm{Lie}_{\mathrm{alg}}(Q[P[A]]) \supseteq \mathrm{Lie}_{\mathrm{alg}}(P[A])$, $\forall Q \in \left\{\mathrm{Id} + B, B \in \mathfrak{h}_{sub}(\mathbf{k})\right\}$,

then $\partial Y = P[A]Y$ IS IN REDUCED FORM .

ITERATION LEMMA

$\left(\begin{array}{c|c|c} A_1 & 0 & 0 \\ \hline S_{2,1} & A_2 & 0 \\ \hline S_{3,1} & S_{3,2} & A_3 \end{array}\right)$ where the block diagonal part is in reduced form

Reduce the south-east into $\left(\begin{array}{c|c} A_2 & 0 \\ \hline S & A_3 \end{array}\right)$. Let P_1 be the reduction matrix.

By [DW'20, Lemma 5.1], the following is in reduced form

$$A_d := \left(\begin{array}{c|c|c} A_1 & 0 & 0 \\ \hline 0 & A_2 & 0 \\ \hline 0 & S & A_3 \end{array}\right).$$

so we may iterate.

$$A = \left(\begin{array}{cc|c|c} 1 & \frac{1}{x} & \frac{1}{x-1} & 0 \\ 0 & 1 & 0 & \frac{1}{x-1} \\ \hline 0 & 0 & 1 & -\frac{1}{x} \\ \hline 0 & 0 & 0 & 1 \end{array}\right).$$

THE ADJOINT ACTION OF THE DIAGONAL

$$A = \left(\begin{array}{c|c} A_2 & 0 \\ \hline S & A_1 \end{array}\right) \quad A_{\text{diag}} := \left(\begin{array}{c|c} A_2 & 0 \\ \hline 0 & A_1 \end{array}\right) \quad \mathfrak{h}_{sub} = \left\{ \left(\begin{array}{c|c} 0 & 0 \\ \hline S & 0 \end{array}\right) \right\}$$

Two simple calculations:

1. Let $P := \text{Id} + \sum f_i\, B_i$, with $f_i \in \mathbf{k}$, $B_i \in \mathfrak{h}_{sub}$. Then

$$P[A] = A + \sum f_i\, [B_i, A_{\text{diag}}] - \sum \partial(f_i) B_i.$$

2. $[B_i, A_{\text{diag}}] \in \mathfrak{h}_{sub}(\mathbf{k})$

Reduction will be governed by the *Adjoint Action* $\Psi : X \mapsto [X, A_{\text{diag}}]$ of A_{diag} on $\mathfrak{h}_{sub}(\mathbf{k})$. Ψ *is a linear map*

My example (2)

$$A(x) := \left(\begin{array}{cccc|cccc} 1 & 0 & \frac{1}{x} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{x-1} & 1 & 0 & -\frac{1}{x} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{x-1} & 1 & 0 & 0 & 0 & 0 \\ \hline \star & \star & \star & \star & 1 & 0 & \frac{1}{x} & 0 \\ \star & \star & \star & \star & \frac{1}{x-1} & 1 & 0 & -\frac{1}{x} \\ \star & \star & \star & \star & 0 & 0 & 1 & 0 \\ \star & \star & \star & \star & 0 & 0 & \frac{1}{x-1} & 1 \end{array}\right).$$

Matrix of the adjoint action of the diagonal on $\mathfrak{h}_{sub}$:

$$\Psi = \begin{pmatrix} 0 & -a & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ -a & 0 & 0 & -b & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 \\ b & 0 & 0 & 0 & 0 & -b & 0 & 0 & 0 & 0 & 0 & 0 & -a & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a & 0 & 0 \\ 0 & 0 & b & 0 & -a & 0 & 0 & -b & 0 & 0 & 0 & 0 & 0 & 0 & -a & 0 \\ 0 & 0 & 0 & b & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -b & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a & 0 & 0 & -b & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 & 0 & 0 & -b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b & 0 & -a & 0 & 0 & -b \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b & 0 & a & 0 & 0 \end{pmatrix}$$

with $a := \frac{1}{x}$ and $b := \frac{1}{x-1}$.

INTERMEZZO : REDUCTION AND RATIONAL SOLUTIONS

First trivial case : condition on $P := \mathrm{Id} + \left(\begin{array}{c|c} 0 & 0 \\ \hline \beta & 0 \end{array}\right)$ to have

$$A = \left(\begin{array}{c|c} A_2 & 0 \\ \hline S & A_1 \end{array}\right) \longrightarrow P[A] = \left(\begin{array}{c|c} A_2 & 0 \\ \hline 0 & A_1 \end{array}\right)?$$

$\beta' = A_2\beta - \beta A_1 - S$, i.e $vec(\beta)' = \psi \cdot vec(\beta) - vec(S)$: rational solutions of a linear differential systems. $\longrightarrow$ algorithm of Barkatou'99

General tool : given matrix $\Psi(x)$ and vectors $\vec{b_1}, \dots, \vec{b_s}$, we will look for tuples $\left(\vec{F}(x), c_1, \dots, c_s\right)$, with $\vec{F}(x) \in \mathbf{k}^N$ and c_i constant, such that $\vec{F}'(x) = \Psi(x) \cdot \vec{F}(x) + \sum_i c_i \vec{b_i}$. This forms a *computable* vector space.

My example (2)

$$A(x) := \left(\begin{array}{cccc|cccc} 1 & 0 & \frac{1}{x} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{x-1} & 1 & 0 & -\frac{1}{x} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{x-1} & 1 & 0 & 0 & 0 & 0 \\ \hline \star & \star & \star & \star & 1 & 0 & \frac{1}{x} & 0 \\ \star & \star & \star & \star & \frac{1}{x-1} & 1 & 0 & -\frac{1}{x} \\ \star & \star & \star & \star & 0 & 0 & 1 & 0 \\ \star & \star & \star & \star & 0 & 0 & \frac{1}{x-1} & 1 \end{array}\right).$$

Matrix of the adjoint action of the diagonal on $\mathfrak{h}_{sub}$:

$$\Psi = \left(\begin{array}{cccccccccccccccc}
0 & -a & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 \\
-a & 0 & 0 & -b & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 \\
0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 \\
b & 0 & 0 & 0 & 0 & -b & 0 & 0 & 0 & 0 & 0 & 0 & -a & 0 & 0 & 0 \\
0 & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a & 0 & 0 \\
0 & 0 & b & 0 & -a & 0 & 0 & -b & 0 & 0 & 0 & 0 & 0 & 0 & -a & 0 \\
0 & 0 & 0 & b & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -b & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a & 0 & 0 & -b & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 & 0 & 0 & -b & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b & 0 & -a & 0 & 0 & -b \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b & 0 & a & 0 & 0
\end{array}\right)$$

with $a := \frac{1}{x}$ and $b := \frac{1}{x-1}$.

$\mathfrak{h}_{sub}$ decomposes as $\mathfrak{h}_{sub} = \mathfrak{h}_1 \oplus \mathfrak{h}_5 \oplus \mathfrak{h}_{10}$ (direct sum of Ψ-spaces).

Ψ-SPACES

$$A = \left(\begin{array}{c|c} A_2 & 0 \\ \hline S & A_1 \end{array}\right) \quad A_{\text{diag}} := \left(\begin{array}{c|c} A_2 & 0 \\ \hline 0 & A_1 \end{array}\right) \quad \mathfrak{h}_{sub} = \left\{ \left(\begin{array}{c|c} 0 & 0 \\ \hline S & 0 \end{array}\right) \right\}$$

$$\textit{Adjoint Action } \Psi : X \mapsto [X, A_{\text{diag}}]$$

Definition

A vector space $W \subset \mathfrak{h}_{sub}$ will be called a Ψ-*space* if $\Psi(W) \subset W \otimes_C \mathbf{k}$.

Lemma

Consider $P[A] = \left(\begin{array}{c|c} A_2 & 0 \\ \hline S & A_1 \end{array}\right)$ *and assume that* $\partial Y = P[A]Y$ *is in reduced form. Then,* $\text{Lie}_{\text{alg}}(P[A]) \cap \mathfrak{h}_{sub}$ *is a* Ψ*-space.*

There are algorithm to decompose and factor into Ψ*-spaces*

MY EXAMPLE (3.1)

$$A(x) := \left(\begin{array}{cccc|cccc}
1 & 0 & \frac{1}{x} & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{x-1} & 1 & 0 & -\frac{1}{x} & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{x-1} & 1 & 0 & 0 & 0 & 0 \\
\hline
\star & \star & \star & \star & 1 & 0 & \frac{1}{x} & 0 \\
\star & \star & \star & \star & \frac{1}{x-1} & 1 & 0 & -\frac{1}{x} \\
\star & \star & \star & \star & 0 & 0 & 1 & 0 \\
\star & \star & \star & \star & 0 & 0 & \frac{1}{x-1} & 1
\end{array}\right).$$

$\mathfrak{h}_{sub}$ decomposes as $\mathfrak{h}_{sub} = \mathfrak{h}_1 \oplus \mathfrak{h}_5 \oplus \mathfrak{h}_{10}$ (direct sum of Ψ-spaces).

Structure on $\mathfrak{h}_5$:

Action of $[\bullet, A_{\text{diag}}]$ on $\mathfrak{h}_5$ in a good basis $\tilde{N}_2, \ldots, \tilde{N}_6$:

$$\Psi_5 := \left(\begin{array}{cc|c|cc}
0 & 0 & \frac{1}{x-1} & 0 & 0 \\
0 & 0 & \frac{1}{x} & 0 & 0 \\
\hline
0 & 0 & 0 & \frac{1}{x} & \frac{1}{x-1} \\
\hline
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{array}\right).$$

Flag structure on $\mathfrak{h}_5$: try to remove $\tilde{N}_6, \tilde{N}_5$ if possible, then $\tilde{N}_4$, then $\tilde{N}_2, \tilde{N}_1$.

MY EXAMPLE (3.2)

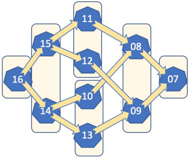
$$A(x) := \left(\begin{array}{cccc|cccc} 1 & 0 & \frac{1}{x} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{x-1} & 1 & 0 & -\frac{1}{x} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{x-1} & 1 & 0 & 0 & 0 & 0 \\ \hline \star & \star & \star & \star & 1 & 0 & \frac{1}{x} & 0 \\ \star & \star & \star & \star & \frac{1}{x-1} & 1 & 0 & -\frac{1}{x} \\ \star & \star & \star & \star & 0 & 0 & 1 & 0 \\ \star & \star & \star & \star & 0 & 0 & \frac{1}{x-1} & 1 \end{array}\right).$$

$\mathfrak{h}_{sub}$ decomposes as $\mathfrak{h}_{sub} = \mathfrak{h}_1 \oplus \mathfrak{h}_5 \oplus \mathfrak{h}_{10}$ (direct sum of Ψ-spaces).

Structure on $\mathfrak{h}_{10}$:

Action of $[\bullet, A_{\text{diag}}]$ on $\mathfrak{h}_{10}$ in a good basis $\tilde{N}_7, \ldots, \tilde{N}_{16}$:

$$\Psi_{10} := \left(\begin{array}{c|cc|cccc|cc|c} 0 & \frac{1}{x} & \frac{1}{x-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \frac{1}{x} & -\frac{1}{x-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{x} & \frac{1}{x-1} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{x-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{x-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{x} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{x} & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{x-1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{x} \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$



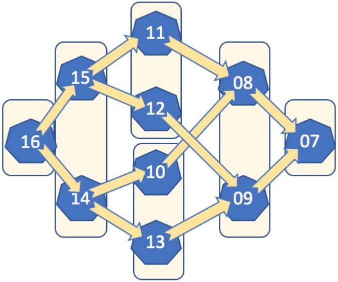
MY EXAMPLE (3.2)

$$A(x) := \left(\begin{array}{cccc|cccc} 1 & 0 & \frac{1}{x} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{x-1} & 1 & 0 & -\frac{1}{x} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{x-1} & 1 & 0 & 0 & 0 & 0 \\ \hline \star & \star & \star & \star & 1 & 0 & \frac{1}{x} & 0 \\ \star & \star & \star & \star & \frac{1}{x-1} & 1 & 0 & -\frac{1}{x} \\ \star & \star & \star & \star & 0 & 0 & 1 & 0 \\ \star & \star & \star & \star & 0 & 0 & \frac{1}{x-1} & 1 \end{array}\right).$$

$\mathfrak{h}_{sub}$ decomposes as $\mathfrak{h}_{sub} = \mathfrak{h}_1 \oplus \mathfrak{h}_5 \oplus \mathfrak{h}_{10}$ (direct sum of Ψ-spaces).

Structure on $\mathfrak{h}_{10}$:

Action of $[\bullet, A_{\text{diag}}]$ on $\mathfrak{h}_{10}$ in a good basis $\tilde{N}_7, \ldots, \tilde{N}_{16}$:



Isotypical decomposition

$$A = \left(\begin{array}{c|c} A_2 & 0 \\ \hline S & A_1 \end{array}\right), A_{\mathrm{diag}} := \left(\begin{array}{c|c} A_2 & 0 \\ \hline 0 & A_1 \end{array}\right), \mathfrak{h}_{sub} = \left\{ \left(\begin{array}{c|c} 0 & 0 \\ \hline S & 0 \end{array}\right) \right\}, \Psi : X \mapsto [X, A_{\mathrm{diag}}]$$

Isotypical decomposition

$$A = \left(\begin{array}{c|c} A_2 & 0 \\ \hline S & A_1 \end{array}\right),\ A_{\text{diag}} := \left(\begin{array}{c|c} A_2 & 0 \\ \hline 0 & A_1 \end{array}\right),\ \mathfrak{h}_{sub} = \left\{ \left(\begin{array}{c|c} 0 & 0 \\ \hline S & 0 \end{array}\right) \right\},\ \Psi : X \mapsto [X, A_{\text{diag}}]$$

Lemma (Krull-Schmidt)

$\mathfrak{h}_{sub}$ *admits a unique* isotypical decomposition $\mathfrak{h}_{sub} = \bigoplus_{i=1}^{\kappa} W_i$:

- *Each* W_i *is a* Ψ*-space.*
- $W_i \simeq \nu_i V_i$, *where* V_i INDECOMPOSABLE Ψ*-space which admits a flag decomposition:* $V_i = V_i^{[\mu]} \supsetneq V_i^{[\mu-1]} \supsetneq \cdots \supsetneq V_i^{[1]} \supsetneq V_i^{[0]} = \{0\}$.
  $1 \leq j \leq \mu$:, $V_i^{[j]}/V_i^{[j-1]}$ *sum of* ISOMORPHIC IRREDUCIBLE Ψ*-spaces.*
- *For* $i \neq j$, *any* Ψ*-spaces* $V_i \subset W_i$ *and* $V_j \subset W_j$ *are non-isomorphic*

there are algorithms for that isotypical decomposition (eigenring, etc).

Lemma (Krull-Schmidt)

$\mathfrak{h}_{sub}$ *admits a unique* isotypical decomposition $\mathfrak{h}_{sub} = \bigoplus_{i=1}^{\kappa} W_i$:

- *Each* W_i *is a* Ψ*-space.*
- $W_i \simeq \nu_i V_i$, *where* V_i INDECOMPOSABLE Ψ*-space which admits a flag decomposition:*
  $V_i = V_i^{[\mu]} \supsetneq V_i^{[\mu-1]} \supsetneq \cdots \supsetneq V_i^{[1]} \supsetneq V_i^{[0]} = \{0\}$.
  For all $1 \leq j \leq \mu$, $V_i^{[j]}/V_i^{[j-1]}$ *direct sum of pairwise* ISOMORPHIC IRREDUCIBLE Ψ*-spaces.*
- *For* $i \neq j$, *any* Ψ*-spaces* $V_i \subset W_i$ *and* $V_j \subset W_j$ *are non-isomorphic*

At each stage: projection on a minimal Ψ-subspace in $V_{i,j}$ reduces to computing a matrix $E_{i,j}(x)$ (linear algebra) and tuples $\left(\vec{F}(x), c_1, \ldots, c_s\right)$, with $\vec{F}(x) \in \mathbf{k}^N$ and c_i constant, such that

$$\vec{F}'(x) = E_{i,j}(x) \cdot \vec{F}(x) + \sum_i c_i \vec{b_i}.$$

The resulting system $P[A]$ is "minimal": it is in REDUCED FORM .

REDUCTION ON $\mathfrak{h}_5$ (MATRYOSHKA EXAMPLE)

$$\Psi_5 := \left(\begin{array}{cc|c|cc} 0 & 0 & \frac{1}{x-1} & 0 & 0 \\ 0 & 0 & \frac{1}{x} & 0 & 0 \\ \hline 0 & 0 & 0 & \frac{1}{x} & \frac{1}{x-1} \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) . Y' = \Psi_5 . Y + b \quad \text{with} \quad b = \begin{pmatrix} \frac{3}{x^2} - \frac{1}{x} \\ \frac{1}{x^2} - \frac{1}{x} \\ \frac{1}{x^2} - \frac{1}{2\,x} \\ 0 \\ \frac{1}{x^2} \end{pmatrix}$$

Reduction equations

$$(W^{[3]}) : \begin{cases} f'_{3,1}(x) = \frac{1}{x^2} \\ f'_{3,2}(x) = 0 \end{cases}$$

$$(W^{[2]}) : \left\{ f'_{2,1}(x) = \frac{1}{x-1} f_{3,1}(x) + \frac{1}{x} f_{3,2}(x) + \frac{1}{x^2} - \frac{1}{2\,x} \right.$$

$$(W^{[1]}) : \begin{cases} f'_{1,1}(x) = \frac{1}{x} f_{2,1}(x) + \frac{1}{x^2} - \frac{1}{x} \\ f'_{1,2}(x) = \frac{1}{x-1} f_{2,1}(x) + \frac{3}{x^2} - \frac{1}{x} \end{cases}$$

Reduction on $\mathfrak{h}_{10}$

$$(W^{[5]}): \quad \left\{ f'_{5,1}(x) = 0 \right.$$

$$(W^{[4]}): \quad \left\{ \begin{array}{lcl} f'_{4,1}(x) & = & \frac{1}{x} f_{5,1}(x) - \frac{1}{2x} \\ f'_{4,2}(x) & = & \frac{1}{x-1} f_{5,1}(x) - \frac{1}{2(x-1)} \end{array} \right.$$

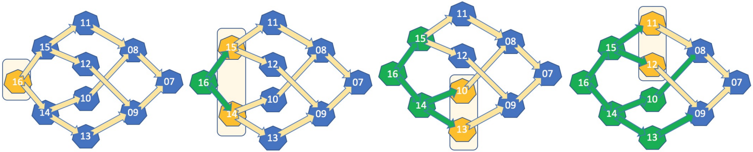
$$(W^{[3]}): \quad \left\{ \begin{array}{lcl} f'_{3,1}(x) & = & \frac{1}{x} f_{4,1}(x) + \frac{1}{x} \\ f'_{3,2}(x) & = & \frac{1}{x} f_{4,2}(x) - \frac{1}{2\,x} \\ f'_{3,3}(x) & = & \frac{1}{x-1} f_{4,1}(x) \\ f'_{3,4}(x) & = & \frac{1}{x-1} f_{4,2}(x) - \frac{1}{2\,(x-1)} \end{array} \right.$$

$$(W^{[2]}): \quad \left\{ \begin{array}{lcl} f'_{2,1}(x) & = & \frac{1}{x-1} f_{3,1}(x) - \frac{1}{x} f_{3,2}(x) - \frac{1}{2\,(x-1)} \\ f'_{2,2}(x) & = & -\frac{1}{x-1} f_{3,3}(x) + \frac{1}{x} f_{3,4}(x) + \frac{1}{x^2} - \frac{1}{2\,(x-1)} \end{array} \right.$$

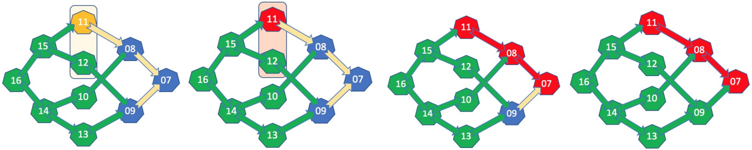
$$(W^{[1]}): \quad \left\{ f'_{1,1}(x) = \frac{1}{x-1} f_{2,1}(x) + \frac{1}{x} f_{2,2}(x) + \frac{2}{x^2} + \frac{1}{(x-1)}. \right.$$

Reduction on $\mathfrak{h}_{10}$

successive reductions



We reach an obstruction when trying to remove N_{11}



The reduction matrix is

$$P_{10} := \mathrm{Id} + \left(c_{1,1} - \frac{1}{x}\right) N_7 - \frac{1}{x} N_8 - N_9 - \frac{1}{2} N_{11} + \frac{1}{2} N_{13} + \frac{1}{2} N_{14} - N_{15} + \frac{1}{2} N_{16}$$

REDUCTION ON $\mathfrak{h}_{10}$

Reduced form $A_{\mathrm{red}} := P_{10}[P_5[A]]$:

$$A_{\mathrm{red}} := \left(\begin{array}{cccc|cccc} 1 & 0 & \frac{1}{x} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{x-1} & 1 & 0 & -\frac{1}{x} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{x-1} & 1 & 0 & 0 & 0 & 0 \\ \hline -\frac{1}{2\,(x-1)} & 0 & 0 & 0 & 1 & 0 & \frac{1}{x} & 0 \\ 0 & \frac{1}{2\,(x-1)} & 0 & 0 & \frac{1}{x-1} & 1 & 0 & -\frac{1}{x} \\ 0 & 0 & -\frac{1}{2\,(x-1)} & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2\,(x-1)} & 0 & 0 & \frac{1}{x-1} & 1 \end{array}\right).$$

REDUCTION ON $\mathfrak{h}_{10}$

$Lie(A)$ spanned by

$$\left(\begin{array}{cccc|cccc} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ \hline 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{array}\right), \left(\begin{array}{cccc|cccc} 0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ \hline -\frac{1}{2}&0&0&0&0&0&0&0\\ 0&\frac{1}{2}&0&0&1&0&0&0\\ 0&0&-\frac{1}{2}&0&0&0&0&0\\ 0&0&0&\frac{1}{2}&0&0&1&0 \end{array}\right), \left(\begin{array}{ccc} 0&0&1\\ 0&0&0\\ 0&0&0\\ 0&0&0\\ \hline 0&0&0\\ 0&0&0\\ 0&0&0\\ 0&0&0 \end{array}\right.$$

$$\left(\begin{array}{cccc|cccc} 0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ \hline 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{array}\right), \left(\begin{array}{cccc|cccc} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ \hline 0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{array}\right).$$

$\dim(Lie(A)) = 5$: *one more integration*

CONCLUSION : ALGORITHM FOR REDUCING $\partial Y = AY$

1. Apply a factorization algorithm. This puts the system into the form

$$A = \begin{pmatrix} A_k & & & 0 \\ & \ddots & & \\ & & A_2 & \\ S_k & & S_2 & A_1 \end{pmatrix}.$$

→ "more or less standard" algorithms

2. Compute the reduced form of *diagonal* $\frac{d}{dx}Y = \mathrm{Diag}(A_k, \ldots, A_1)Y$.

→ Barkatou, Cluzeau, Di Vizio, Weil.

3. ℓ from 2 to k: compute reduced form of $\frac{d}{dx}Y = \widetilde{A_\ell}Y$,

where $\widetilde{A_\ell}$ block-triangular with blocks $A_1, \ldots, A_k, S_2, \ldots, S_\ell$ as in A and with zeros elsewhere. → see above.

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Result: A REDUCED FORM OF $\frac{d}{dx}Y = AY$ and the *Galois-Lie algebra* $\mathfrak{g}$.