Doctoral Program Computational Mathematics Numerical Analysis and Symbolic Computation

Der Wissenschaftsfonds.

A computable extension for D-finite functions

DD-finite functions

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Outline

1. D-finite functions
2. DD-finite functions
3. D^n-finite functions
4. Autonomous First Order Differential Equations (by Noordman, van der Put and Top)
5. Conclusions

Demo

You can try *yourself* from my personal webpage:

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Sage package: dd_functions

A Sage package for DD-finite functions: dd_functions

This software has been totally developed by myself and was funded by the *Austrian Science Fund (FWF): W1214-N15, project DK15*. The software provided on this web site is free; you can re
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This is a Sage package aimed to manipulate D-finite, DD-finite and, more generically, D^n-finite functions.

Link to the repository: git repository

Link to the code: dd_functions

Demo: launch binder

Installation and usage

To use this package you can either download the last stable version (see the *link to the code*) or cloning/downloading the last version from the git repository (see the *link to the repository*). Or

- Install the package in your Sage distribution using "*sage -pip install dd_functions*".
- Manually add the path to the folder *dd_functions* to your Sage init file.

Once installed, the code can be used after importing the package within a Sage session with the command "*from ajpastor.dd_functions import **".

You can also install the package directly using the command "*sage -pip install git+https://github.com/Antonio-JP/dd_functions*".

Further information on usage will be added soon.

D-finite functions: the holonomic framework

Notation

Throughout this talk we consider:

- K: a **computable** field
- $K[[x]]$: ring of formal power series over K.
- ∂ is the *standard* derivation over $K[[x]]$:

$$\partial\left(\sum_{n\geq 0} c_n x^n\right) = \sum_{n\geq 0} c_n \partial(x^n) = \sum_{n\geq 0} c_n n x^{n-1}$$

D-finite functions

Definition

Let $f \in K[[x]]$. We say that f is *D-finite* (or *holonomic*) if there exist $d \in \mathbb{N}$ and polynomials $p_0(x), \ldots, p_d(x)$ such that:

$$p_d(x)f^{(d)}(x) + \ldots + p_0(x)f(x) = 0.$$

When $p_d(x) \neq 0$, we say that d is the *order* of f.

Examples

A lot of special functions are D-finite:

- Exponential function: e^x.
- Trigonometric functions: $\sin(x)$, $\cos(x)$.
- Logarithm function: $\log(x+1)$.
- Bessel functions: $J_n(x)$.
- Hypergeometric functions: ${}_pF_q\left(\begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q\end{matrix}; x\right)$.
- Airy functions: $Ai(x), Bi(x)$.
- Combinatorial generating functions: $F(x), C(x)$,,...

Closure properties

$f(x), g(x)$ D-finite of order d_1, d_2.
$a(x)$ algebraic over $K(x)$ of degree p.

Property	**Is D-finite**	**Order bound**
Addition	$(f+g)$	d_1+d_2
Product	(fg)	$d_1 d_2$
Differentiation	f'	d_1
Integration	$\int f$	d_1+1
Be Algebraic	$a(x)$	p

Characterization Theorem

Theorem

The following are equivalent:

$$f(x) \text{ is D-finite}$$

$$\Updownarrow$$

$$\text{The } K(x)\text{-vector space } \langle f, f', f'', \dots \rangle \text{ has finite dimension.}$$

Working with D-finite functions

There are several implementations of D-finite functions:

- *gfun*: Maple package, by B. Salvy and P. Zimmermann
- *HolonomicFunctions*: Mathematica package, by C. Koutschan
- *ore_algebra*: Sage package, by M. Kauers et al.

DD-finite functions: extending the class

Non-D-finite examples

There are power series that are not D-finite:

- Double exponential: $f(x) = e^{e^x}$.
- Tangent: $\tan(x) = \frac{\sin(x)}{\cos(x)}$.

- Gamma function: $f(x) = \Gamma(x+1)$.
- Partition Generating Function: $f(x) = \sum_{n\geq 0} p(n)x^n$.

DD-finite Functions

Definition

Let $f \in K[[x]]$. We say that f is *D-finite* if there exist $d \in \mathbb{N}$ and polynomials $p_0(x), \dots, p_d(x)$ such that:

$$p_d(x)f^{(d)}(x) + \dots + p_0(x)f(x) = 0.$$

DD-finite Functions

Definition

Let $f \in K[[x]]$. We say that f is *DD-finite* if there exist $d \in \mathbb{N}$ and D-finite elements $r_0(x), \ldots, r_d(x)$ such that:

$$r_d(x)f^{(d)}(x) + \ldots + r_0(x)f(x) = 0.$$

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Non-D-finite examples

There are power series that are DD-finite:

- Double exponential: $f(x) = e^{e^x} \to f'(x) - e^x f(x) = 0$
- Tangent: $\tan(x) = \frac{\sin(x)}{\cos(x)} \to \cos^2(x)\tan''(x) - 2\tan(x) = 0.$

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$\Gamma(z)$ is not diff. algebraic (Hölder's Theorem), so it is not DD-finite

Non-D-finite examples

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- Gamma function: $f(x) = \Gamma(x+1)$.
- Partition Generating Function: $f(x) = \sum_{n \geq 0} p(n)x^n$.

It has a dense set of singularities in $|z| = 1$

Differentially Definable Functions

Definition

Let $f \in K[[x]]$. We say that f is *DD-finite* if there exist $d \in \mathbb{N}$ and D-finite elements $r_0(x), \ldots, r_d(x)$ such that:

$$r_d(x)f^{(d)}(x) + \ldots + r_0(x)f(x) = 0.$$

Differentially Definable Functions

Definition

Let $f \in K[[x]]$ and $R \subset K[[x]]$ a diff. subring. We say that f is differentially definable over R if there exist $d \in \mathbb{N}$ and elements in R $r_0(x), \ldots, r_d(x)$ such that:

$$r_d(x)f^{(d)}(x) + \ldots + r_0(x)f(x) = 0.$$

D(R): differentially definable functions over R.

Characterization Theorem

Theorem

The following are equivalent:

$$f(x) \text{ is D-finite}$$

$$\Updownarrow$$

$$\text{The } K(x)\text{-vector space } \langle f, f', f'', \dots \rangle \text{ has finite dimension.}$$

Characterization Theorem

Theorem

The following are equivalent:

$$f(x) \text{ is in } D(R)$$

$$\Updownarrow$$

The F-vector space $\langle f, f', f'', \dots \rangle$ has finite dimension.

- $R \subset K[[x]]$ is a differential subring
- F is its field of fractions.

Closure properties

$f(x), g(x) \in \mathrm{D}(R)$ of order d_1, d_2.
F the field of fractions of R.
$a(x)$ algebraic over F of degree p.

Property	**Is in** $\mathrm{D}(R)$	**Order bound**
Addition	$(f+g)$	$d_1 + d_2$
Product	(fg)	$d_1 d_2$
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Be Algebraic	$a(x)$	p

$\longrightarrow$ Proof by direct formula
$\longrightarrow$ Proof by linear algebra

D^n-finite functions: iterating the process

D^n-finite functions

Remark

$$R \subset K[[x]] \text{ diff. ring} \Rightarrow \mathrm{D}(R) \subset K[[x]] \text{ diff. ring}$$

$$\downarrow$$

Iterate the process

D^n-finite functions

Remark

$$R \subset K[[x]] \text{ diff. ring} \Rightarrow D(R) \subset K[[x]] \text{ diff. ring}$$

$$\downarrow$$

Iterate the process

D^n-finite functions

D^n-finite functions are the nth iteration over $K[x]$, i.e., $D^n(K[x])$.

$$K[x] \subset D(K[x]) \subset D^2(K[x]) \subset \ldots \subset D^n(K[x]) \subset \ldots$$

New Properties

$f(x) \in D^n(K[x])$ of order d_1.
$g(x) \in D^m(K[x])$ of order d_2.
$a(x)$ algebraic over $D^m(K[x])$ of degree p.

Property	**Function**	**Is in**	**Order bound**
Composition	$f \circ g$	$D^{n+m}(K[x])$	d_1
Alg. subs.	$f \circ a$	$D^{n+m}(K[x])$	pd_1

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- $a(x)$ algebraic over $D^m(K[x])$ implies $a(x) \in D^{m+1}(K[x])$.
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- Then $f(a(x))$ is in $D^{n+m+1}(K[x])$.

$D^n \subsetneq D^{n+1}$: Iterated exponentials

Iterated exponentials

$$K[x] \subsetneq D(K[x]) \subset D^2(K[x]) \subset \ldots \subset D^n(K[x]) \subset \ldots$$

Iterated exponentials

$$K[x] \subsetneq \mathrm{D}(K[x]) \subsetneq \mathrm{D}^2(K[x]) \subset \ldots \subset \mathrm{D}^n(K[x]) \subset \ldots$$

$$e^x \in \mathrm{D}(K[x]), \qquad e^{e^x - 1} \in \mathrm{D}^2(K[x])$$

Iterated exponentials

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$$e^x \in \mathrm{D}(K[x]), \qquad e^{e^x-1} \in \mathrm{D}^2(K[x])$$

Iterated Exponentials

- $e_0(x) = 1$,
- $\hat{e}_n(x) = \int_0^x e_n(t)dt$,
- $e_{n+1}(x) = \exp(\hat{e}_n(x))$.

Increasing chain

Proposition

- $e_n(x) \in D^n(K[x])$.
- $e_n(x) \notin D^{n-1}(K[x])$.

Increasing chain

Proposition

- $e_n(x) \in \mathrm{D}^n(K[x])$.
- $e_n(x) \notin \mathrm{D}^{n-1}(K[x])$.

First is trivial: $e_n'(x) = e_{n-1}(x)e_n(x)$.

Increasing chain

Proposition

- $e_n(x) \in \mathrm{D}^n(K[x])$.
- $e_n(x) \notin \mathrm{D}^{n-1}(K[x])$.

Second: proof using Differential Galois Theory (M. F. Singer)

Picard-Vessiot

Picard-Vessiot closure

Let (K, ∂) be a differential field with constants C. The *Picard-Vessiot* closure is the *smallest* field with same constants such that **all** linear differential equation with coefficients in K have all the C-linearly independent solutions.

Picard-Vessiot

Picard-Vessiot closure

Let (K, ∂) be a differential field with constants C. The *Picard-Vessiot* closure is the *smallest* field with same constants such that **all** linear differential equation with coefficients in K have all the C-linearly independent solutions.

$$\begin{array}{ccccccccc}
C[x] & \subset & \mathrm{D}(C[x]) & \subset & \ldots & \subset & \mathrm{D}^{n-1}(C[x]) & \subset & \ldots \\
\cap & & \cap & & \ddots & & \cap & & \\
C(x) & \subset & F_1 & \subset & \ldots & \subset & F_{n-1} & \subset & \ldots \\
\cap & & \cap & & \ddots & & \cap & & \\
K_0 & \subset & K_1 & \subset & \ldots & \subset & K_{n-1} & \subset & \ldots
\end{array}$$

G.T. required result

Proposition

Let (K,∂) be a differential field with algebraically closed field of constants C. Let E be a PV-extension of K. Let $u, v \in E \setminus \{0\}$ such that:

$$\frac{u'}{u} = a \in K, \qquad \frac{v'}{v} = u,$$

then u is algebraic over K.

G.T. required result

Proposition

Let (K, ∂) be a differential field with algebraically closed field of constants C. Let E be a PV-extension of K. Let $u, v \in E \setminus \{0\}$ such that:

$$\frac{u'}{u} = a \in K, \qquad \frac{v'}{v} = u,$$

then u is algebraic over K.

Corollary

Let $c \in C^*$ and $n \in \mathbb{N} \setminus \{0\}$. Then $e_n^c = \exp(c\hat{e}_{n-1}) \notin K_{n-1}$.

End of the proof

$$\begin{array}{ccccccccc}
C[x] & \subset & \mathrm{D}(C[x]) & \subset & \dots & \subset & \mathrm{D}^{n-1}(C[x]) & \subset & \dots \\
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\end{array}$$

$e_n(x)$

End of the proof

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$e_n(x) \notin K_{n-1}$, and...

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\end{array}$$

$e_n(x) \notin F_{n-1}$, and...

End of the proof

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\end{array}$$

$e_n(x) \notin D^{n-1}(K[x])$, finishing the proof.

$D^n \subsetneq$ Diff. Algebraic

Non linear differential equations

- Diff. definable over $R \longrightarrow$ linear differential equation.
- Diff. algebraic over $R \longrightarrow$ non-linear differential equation.

Non linear differential equations

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Theorem

Let $f \in K[[x]]$. If there is $n \in \mathbb{N}$ with $f \in \mathrm{D}^n(R)$, then f is differentially algebraic over R.

Non linear differential equations

- Diff. definable over R $\longrightarrow$ linear differential equation.
- Diff. algebraic over R $\longrightarrow$ non-linear differential equation.

Theorem

Let $f \in K[[x]]$. If there is $n \in \mathbb{N}$ with $f \in D^n(R)$, then f is differentially algebraic over R.

The proof is constructive and it is implemented.

Non linear differential equations

- Double exponential ($\exp(\exp(x) - 1)$):

$$f'(x) - \exp(x)f(x) = 0$$
$$\downarrow$$
$$f''(x)f(x) - f'(x)^2 - f'(x)f(x) = 0$$

Non linear differential equations

- Tangent ($\tan(x)$):

$$\cos(x)^2 f''(x) - 2f(x) = 0$$

$$\downarrow$$

$$-2f^{(5)}(x)f''(x)^2 f(x) + 12f^{(4)}(x)f'''(x)f''(x)f(x) - 6f^{(4)}(x)f''(x)^2 f'(x) - 12f'''(x)^3 f(x) + 12f'''(x)^2 f''(x)f'(x) - 4f'''(x)f''(x)^3 - 8f'''(x)f''(x)^2 f(x) + 8f''(x)^3 f'(x) = 0$$

Non linear differential equations

- Mathieu functions:

$$f''(x) - (a - 2q\cos(2x))f(x) = 0$$

$$\downarrow$$

$$f^{(5)}(x)f(x)^3 - 3f^{(4)}(x)f'(x)f(x)^2 - 4f'''(x)f''(x)f(x)^2 + 6f'''(x)f'(x)^2f(x) + 4f'''(x)f(x)^3 + 6f''(x)^2f'(x)f(x) - 6f''(x)f'(x)^3 - 4f''(x)f'(x)f(x)^2 = 0$$

Noordman, van der Put and Top paper

The paper

- **Authors:** Noordman, van der Put and Top
- **Title:** Autonomous first order Differential Equations
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Main idea

Use algebraic geometry to solve differential problems.

Autonomous differential equations

Autonomous Diff. Equations

Let $P(y_0, y_1, \ldots, y_n) \in K[y_0, \ldots, y_n]$. The equation

$$P(y, y', \ldots, y^{(n)}) = 0$$

is an *autonomous differential equation*.

Autonomous differential equations

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Let $P(y_0, y_1, \dots, y_n) \in K[y_0, \dots, y_n]$. The equation

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A diff. algebraic equation is autonomous if the coefficients are all constants.

Autonomous differential equations

Autonomous Diff. Equations

Let $P(y_0, y_1, \ldots, y_n) \in K[y_0, \ldots, y_n]$. The equation

$$P(y, y', \ldots, y^{(n)}) = 0$$

is an *autonomous differential equation*.

P defines a hypersurface in $\mathbb{P}^{n+1}(F)$ for any field $F \supset K$

AFODE

Let $P(y_0, y_1)$ be a *Autonomous First Order Differential Equation* (AFODE) with P irreducible.

AFODE

Let $P(y_0, y_1)$ be a *Autonomous First Order Differential Equation* (AFODE) with P irreducible.

Solutions of AFODE

Let u be a solution (i.e., $P(u, u') = 0$). Then for any field $F \supset K$ containing u and u':

$$\begin{array}{rccc} \varphi: & K(y_0, y_1)/(P) & \longrightarrow & F \\ & \bar{x} & \mapsto & u \\ & \bar{y} & \mapsto & u' \end{array}$$

is injective.

AFODE

Let $P(y_0, y_1)$ be a *Autonomous First Order Differential Equation* (AFODE) with P irreducible.

Varieties of AFODE

Given $P(y_0, y_1)$, we can build a plane curve $X \subset \mathbb{P}^2(K)$.
Derivations on $K(X)$ are characterized with 1-forms ω such that: $D(f)\omega = df$.

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Varieties of AFODE

Given $P(y_0, y_1)$, we can build a plane curve $X \subset \mathbb{P}^2(K)$. Derivations on $K(X)$ are characterized with 1-forms ω such that: $D(f)\omega = df$.

To make $K(X)$ differentially isomorphic to $\varphi(K(X))$ in any field F we need to fixed a particular derivation (i.e., a particular 1-form ω):

$$\omega = \frac{dx}{y}$$

AFODE

Let $P(y_0, y_1)$ be a *Autonomous First Order Differential Equation* (AFODE) with P irreducible.

Correspondance with curves

There is a correspondance between any AFODE $P(u, u') = 0$ and plane, irreducible, smooth, and projective curves X paired with a 1-form ω

$$P(u, u') = 0 \longleftrightarrow (X, \omega).$$

Equivalence and relation between AFODE

Using rational maps between curves, we can relate two different AFODE:

Equivalence of AFODE

We say that (X, ω) and (Y, ν) are *equivalent* if there is a birational map $\psi : X \to Y$ such that $\psi^*(\nu) = \omega$.

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Pull-backs of AFODE

We say that (X, ω) is a *pull-back* of (Y, ν) if there is a rational map $m : X \rightarrow Y$ with $m^*\nu = \omega$.

Equivalence and relation between AFODE

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Pull-backs of AFODE

We say that (X, ω) is a *pull-back* of (Y, ν) if there is a rational map $m : X \to Y$ with $m^*\nu = \omega$.

- If (X, ω) is equivalent to (Y, ν), then both are pull-backs of each other.
- If both are pull-backs of each other, then they are equivalent.

Solutions to AFODE as geometric elements

Let (X, ω) be the variety associated to a AFODE $P(u, u') = 0$, and $F \supset K$ a differential extension of K where $C(F) = K$.

Solutions to AFODE as geometric elements

Let (X, ω) be the variety associated to a AFODE $P(u, u') = 0$, and $F \supset K$ a differential extension of K where $C(F) = K$.

For any solution $u \in F$ we can build a K-linear map φ_u from $K(X)$ to F mapping:

- $\varphi_u(\bar{x}) = u$,
- $\varphi_u(\bar{y}) = u'$.

Solutions to AFODE as geometric elements

Let (X, ω) be the variety associated to a AFODE $P(u, u') = 0$, and $F \supset K$ a differential extension of K where $C(F) = K$.

For any solution $u \in F$ we can build a K-linear map φ_u from $K(X)$ to F mapping:

- $\varphi_u(\bar{x}) = u$,
- $\varphi_u(\bar{y}) = u'$.

These conditions make φ_u an injective differential map from $(K(X), D_\omega)$ to (F, ∂).

- $S_{(X,\omega)}(F)$: set of solutions (as K-linear maps) of (X, ω) in F.

Solutions between pull-backs

Let (X,ω) and (Y,ν) two AFODE and F a differential field.

Pull-back case

If X is a pull-back of Y with a map m with $\deg(m) = n$, then m induces a map

$$m^* : \begin{array}{ccc} S_{(X,\omega)}(F) & \rightarrow & S_{(Y,\nu)} \\ \varphi & \mapsto & \varphi \circ m^* \end{array}$$

This map **collapses** n different solutions of (X,ω) into one solution of (Y,ν).

Solutions between pull-backs

Let (X, ω) and (Y, ν) two AFODE and F a differential field.

Equivalent AFODE

In the case (X, ω) and (Y, ν) are equivalent, there is a **bijective correspondance** between solutions on the same field.

Clasification of AFODE

Depending of basic geometric properties, we can classify a AFODE (X, ω):

Due to properties of ω

- It is *exact* if there is $f \in K(X)$ with $\omega = df$.
- It is *exponential* if there is $finK(X)$ with there is $c \in K$ with

$$cf\omega = df.$$

- It is *Weierstrass* if (X, ω) is a pullback of an elliptic curve with a translation-invariant 1-form.
- Otherwise, it is called of *general type*.

Clasification of AFODE

Depending of basic geometric properties, we can classify a AFODE (X, ω):

Due to the pull-backs available

- It is *new* if all pull-backs (Y, ν) are equivalent to (X, ω).
- It is *old* otherwise.

Proving results for *new* AFODE is usually enough!

Main theorems of the paper

Algebraic independence between solutions

Theorem 1

Let (X, ω) be of **general** and **new** type. Then any finite set of solutions $\{\varphi_1, \ldots, \varphi_n\}$ is K-algebraically independent.

Algebraic independence between solutions

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Let (X, ω) be of **general** and **new** type. Then any finite set of solutions $\{\varphi_1, \ldots, \varphi_n\}$ is K-algebraically independent.

Remark 1.1

Assume (X, ω) is **old** and (Y, ν) is a new pull-back of X with a map $m : X \to Y$. Then any set of solutions $\{\varphi_1, \ldots, \varphi_n\}$ where $m^*(\varphi_i) \neq m^*(\varphi_j)$ is K-algebraically independent.

Algebraic independence between solutions

Theorem 1

Let (X, ω) be of **general** and **new** type. Then any finite set of solutions $\{\varphi_1, \ldots, \varphi_n\}$ is K-algebraically independent.

Remark 1.2

Let (X, ω) be of general type and F a differential field with $\text{trdeg}_K(F) < \infty$. Then

$$\left| S_{(X,\omega)}(F) \right| < \infty.$$

A more general theorem

Theorem 2

Let $(X_1, \omega_1), \ldots, (X_n, \omega_n)$ be general and new AFODE and let φ_i be a solution for (X_i, ω_i) in a fixed field F. If the solutions are algebraically dependent then there are $i \neq j$ and $\psi : X_i \to X_j$ such that:

- ψ is an isomorphism between the curves.
- $\psi^*\omega_j = \omega_i$ (i.e., X_i and X_j are equivalent)
- $\psi^*(\varphi_i) = \varphi_j$.

Theorem 2 $\Rightarrow$ Theorem 1

Theorem 1

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Assume that the set of solutions for (X, ω) is algebraically dependent.

Theorem 2 $\Rightarrow$ Theorem 1

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Assume that the set of solutions for (X, ω) is algebraically dependent.
By Theorem 2 we have a non-trivial automorphism $\psi : X \to X$ (i.e., $\psi \neq id_X$) with

- $\psi^*(\omega) = \omega$.
- $\psi^*(\varphi_i) = \psi^*(\varphi_j)$.

Theorem 2 ⇒ Theorem 1

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Since (X, ω) is general and new, this can not happen $\square$

Proof of Theorem 2 for $n = 2$

Let φ_1 and φ_2 be algebraically dependent and F_0 a field containing φ_1 and φ_2 *minimally*.

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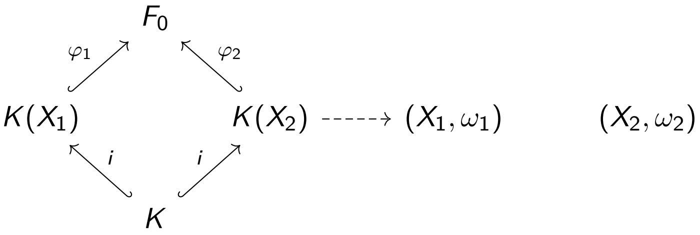

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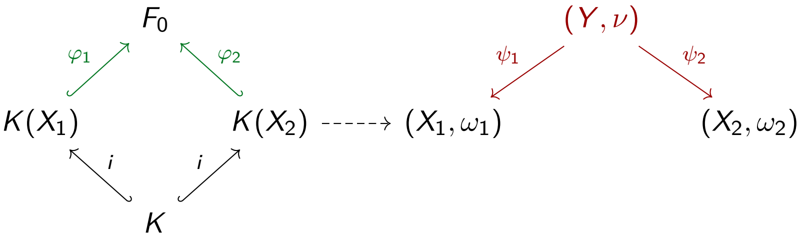

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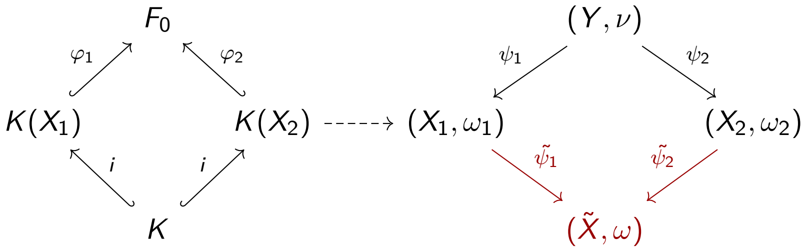

$\tilde{\psi}_1 \circ \tilde{\psi}_2^{-1}$ is the desired equivalence between X_1 and X_2.

Proof of Theorem 2 for $n > 2$

Assume the result true for any $m < n$. Then we can assume that any proper subset of solutions is algebraically independent.

- Let E be the algebraic closure of the field

$$K(\varphi_3(x_3), \varphi_3(y_3), \ldots, \varphi_n(x_n), \varphi_n(y_n)).$$

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- Apply Theorem 1, case $n = 2$.

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- Prove that φ_1 and φ_2 are K-algebraically dependent.

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- Apply Theorem 1, case $n = 2$.
- Prove that φ_1 and φ_2 are K-algebraically dependent.
- Use case $n = 2$ for finishing the proof.

Our desired result

Finally, we can prove the result about D^n-finiteness:

Proposition 3

Let (X, ω) be a AFODE of general type and φ a transcendental solution on $K((x))$. Then φ is not D^n-finite for any $n \in \mathbb{N}$.

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Sketch of the proof

- φ is D^n-finite $\Rightarrow$ there is a chain of PV-extensions

$$K[x] = K_0 \subset K_1 \subset \ldots \subset K_n \ni \varphi$$

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- φ is D^n-finite $\Rightarrow$ there is a chain of PV-extensions

$$K[x] = K_0 \subset K_1 \subset \ldots \subset K_n \ni \varphi$$

- Let $G_n = Gal(K_n/K_{n-1})$ and $\sigma \in G_n$. Then $\sigma(\varphi)$ is also a solution to (X, ω).

Our desired result

Proposition 3

Let (X, ω) be a AFODE of general type and φ a transcendental solution on $K((x))$. Then φ is not D^n-finite for any $n \in \mathbb{N}$.

Sketch of the proof

- Using Remark 1.2, $\operatorname{trdeg}_K(K_n) < \infty$ implies

$$|\sigma(\varphi) \;:\; \sigma \in G_n| < \infty,$$

so φ is algebraic over K_{n-1}.

Our desired result

Proposition 3

Let (X, ω) be a AFODE of general type and φ a transcendental solution on $K((x))$. Then φ is not D^n-finite for any $n \in \mathbb{N}$.

Sketch of the proof

- Let $Q(T) = T^m + a_{m-1}T^{m-1} + ... + a_0$ be the minimal polynomial of φ over K_{n-1}. Using Remark 1.2 we conclude that

$$|\sigma(Q) \ : \ \sigma \in G_{n-1}| < \infty,$$

and the same property hold for each of the a_i, so they are algebraic over K_{n-2}.

Our desired result

Proposition 3

Let (X, ω) be a AFODE of general type and φ a transcendental solution on $K((x))$. Then φ is not D^n-finite for any $n \in \mathbb{N}$.

Sketch of the proof

- Then φ is algebraic over K_{n-2}. By induction, φ is algebraic over K_0, contradicting the transcendency hypothesis.

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Sketch of the proof

- Then φ is algebraic over K_{n-2}. By induction, φ is algebraic over K_0, contradicting the transcendency hypothesis.
- So φ is not D^n-finite for any $n \in \mathbb{N}$

A non D^n-fnite example

Consider the differential equation:

$$y' = y^3 - y^2 = y^2(y-1).$$

The corresponding curve is defined with the polynomial $y = x^3 - x^2$ and it is proven that it is of *general type*.

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$$y' = y^3 - y^2 = y^2(y-1).$$

The corresponding curve is defined with the polynomial $y = x^3 - x^2$ and it is proven that it is of *general type*.

Then any solution in $K[[x]]$ is not D^n-finite for any n.

Conclusions

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Achievements

- Extended the framework of D-finite to wider class of computable functions
- Implemented closure properties for DD-finite
- Implemented composition of D^n-finite functions
- Explored the boundaries of the class of differentially definable
- Code available for Sage

Conclusions

Future work

- Improve performance of the current code
- Study analytic properties of DD-finite functions
- Study combinatorial properties of DD-finite functions
- Study the analogue of DD-finite functions in sequences

Thank you!

Contact webpage:

- `https://www.dk-compmath.jku.at/people/antonio`
- `https://www.risc.jku.at/home/ajpastor`

Sage code:

- `https://github.com/Antonio-JP/dd_functions`