Intrinsic approach to Galois theory of q-difference equations

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with the preface to Part 5

The Galois D-groupoid of a q-difference system

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Contents

Introduction

The Galois theory of difference equations has witnessed a major evolution in the last two decades. In the particular case of q-difference equations, authors have introduced several different Galois theories. In this memoir we consider an arithmetic approach to the Galois theory of q-difference equations and we use it to establish the relations among the different theories in the literature.

Let q be a non-zero element of the field $\mathbb{C}$ of complex numbers. A (linear) q-difference system is a functional equation of the form

$$Y(qx) = A(x)Y(x), \text{ with } A(x) \in \mathrm{GL}_\nu(\mathbb{C}(x)). \tag{0.1}$$

The *leitmotif* of the paper, which is sometimes hidden, sometimes openly used, is the Galoisian properties of the so-called dynamics of the system (0.1), namely the set of maps obtained by iteration of the maps $(x, X) \longmapsto (qx, A(x)X)$ and $(x, X) \longmapsto (q^{-1}x, A(q^{-1}x)^{-1}X)$, both defined over $U \times \mathbb{C}^\nu$, where U is an open subset of $\mathbb{P}^1_{\mathbb{C}}$, and with values in $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{C}^\nu$. The latter is deduced from the functional system $Y(q^{-1}x) = A(q^{-1}x)^{-1}Y(x)$, which is equivalent to (0.1).

An early Galois theory for q-difference equations, which we may call the "classical" Picard-Vessiot theory, is based on the construction of abstract solutions (see [**vdPS97**]) and the Galois group is defined as the group of $\mathbb{C}(x)$-automorphisms of the Picard-Vessiot ring, i.e., the "minimal" $\mathbb{C}(x)$-algebra generated by the abstract solutions. A key-point of this approach is that the field of constants $\mathbb{C} = \{c \in \mathbb{C}(x) | \sigma_q(c) = c\}$ is algebraically closed. This assumption allows, among other things, to consider only the $\mathbb{C}$-points of the Galois group, without being obliged to have a schematic point of view.

Other approaches are based on the remark that the system (0.1) determines a fiber bundle over the torus $E := \mathbb{C}^*/q^{\mathbb{Z}}$. The fact that its pull back on $\mathbb{C}^*$ is trivial means that (0.1) has an invertible solution matrix, with entries meromorphic on $\mathbb{C}^*$. See [**Pra86**]. Two Galois theories are based on the existence of these meromorphic solutions. The first one, initiated by Sauloy and Ramis (see [**Sau04b**]), is more analytic, uses the Tannakian formalism and describes the Galois group as a linear algebraic group defined over $\mathbb{C}$. The second one, introduced in [**CHS08**, Definition 2.1], provides a Galois group for (0.1), which is a linear algebraic group scheme defined over the field C_E of meromorphic functions over E. It acts functorially as the group of $C_E(x)$-automorphisms of the "weak" Picard-Vessiot ring, i.e., the $C_E(x)$-algebra generated by the meromorphic solutions of the system.

In all the theories described above, the structure of the Galois group is a mirror of the algebraic relations satisfied by the entries of an invertible solution matrix of (0.1), over the base field. In [**CHS08**], these approaches are compared and proved to coincide up to some field extensions.

In 2008, Hardouin and Singer have developed a Galois theory for parameterized functional equations. Consider a field K of characteristic 0 and an element $q \in K$, $q \neq 0, 1$, not a root of unity. We equip $K(x)$ with a derivation, for instance with the derivation $\partial = x\frac{d}{dx}$. Given a linear q-difference system with coefficients in $K(x)$, the purpose of a parmeterized Galois theory is to produce a group that gives information about the differential algebraic relations between the solutions of the q-difference system, i.e., the algebraic relations satisfied by the solutions and their successive derivatives with respect to ∂. The prototype of the possible applications of a parameterized Galois theory is a Galoisian proof of Holder's theorem, saying that the classical Gamma function cannot be solution of a differential equation with rational coefficients.

In [**HS08**], the authors attached to such a q-difference system a linear differential algebraic group *à la Kolchin*, defined over K. This is a group of matrices defined as the set of zeros of a finite number of algebraic differential equations. In analogy with the constructions of [**vdPS97**], the solutions are abstract and the theory of Hardouin-Singer requires that the field of σ_q-constants is differentially closed with respect to ∂. Other approaches are possible: There are as many parameterized theories as classical theories and, if one considers the trivial derivation, one recovers their classical counterpart.

In this work, we consider the parameterized Galois theories in the special case of q-difference equations and from an arithmetic point of view. Relying on the differential Tannakian formalism (see [**Ovc09**] for instance), we attach to a q-difference system $Y(qx) = A(x)Y(x)$ with $A(x) \in \mathrm{GL}_n(K(x))$ a differential algebraic group scheme, that we call parameterized intrinsic Galois group. Roughly, this differential algebraic group scheme is linked to the differential algebraic relations satisfied by the entries of $A(x)$, in the sense that it only relies on differential algebraic constructions of the associated q-difference module, and therefore on the associated matrix constructions of $A(x)$ and its dynamics. The advantages of considering this group are its intrinsic nature and its arithmetic description (see Chapter 7), which is an analogue of the conjectural description obtained by Katz in [**Kat90**] for the Lie algebra of the intrinsic Galois group of a linear differential system.

Theorem 7.13 below exhibits an arithmetic set of generators of the parameterized intrinsic Galois group. These generators are called the curvatures of the system and are intrinsically defined, since they are obtained specializing conveniently certain sub-sequences of the dynamics $\left(A(q^{n-1}x)\dots A(x)\right)_{n\in\mathbb{N}}$. The proof of Theorem 7.13 relies on a rationality criteria for the solutions of a q-difference system. It extends the main result of [**DV02**], in which the assumption that K is a number field, and hence that q is algebraic, is crucial. Here we only assume K to be a finitely generated $\mathbb{Q}$-algebra and q can be any number, algebraic or transcendental. We state here Theorem 6.12 in the particular case $K = \mathbb{Q}(q)$ and under the assumption that q is a transcendental number:

THEOREM 1. *Let $A(x) \in \mathrm{GL}_\nu(\mathbb{Q}(q,x))$. The q-difference system $Y(qx) = A(x)Y(x)$ admits a full set of solutions in $\mathbb{Q}(q,x)$ if and only for almost all $n \in \mathbb{N}$ there exists an n-th primitive root of unity ζ_n such that $A(q^{n-1}x)\dots A(x)$ specializes to the identity matrix at $q = \zeta_n$.*

Unlike the case of linear differential systems, the computation of the curvatures of a q-difference system relies only on matrix multiplication. Thus, one may hope to develop fast algorithms to compute the curvatures and perhaps also the parameterized intrinsic Galois group in terms of differential polynomial equations annihilated by the curvatures. See [**BS09**] in the differential case. Notice that the arithmetic description of the parameterized intrinsic Galois group provides an arithmetic answer to problem of the rationality of the solutions of the q-difference systems as well as the control of their differential dependencies with respect to parameters (see for instance [**AR13**] for some algorithms that tackle these questions).

In Part 4 we compare the parameterized intrinsic Galois group with all the Galois groups detailed above (see Proposition 8.10), proving that all these differential algebraic groups become isomorphic over a suitable field extension. This result has many consequences. First of all, it shows that the theory of [**HS08**] descends to the field of coefficients of the initial q-difference system, without any assumption on the field of σ_q-constants (see [**Wib12b**] or [**DVH11**] for a descent over an algebraically closed field of σ_q-constants). Secondly, the differential algebraic relations satisfied by meromorphic solutions are encoded by the curvatures of the system (see Corollary 8.13). In fact the group of [**HS08**] acts on some abstract solutions of the q-difference system and one cannot apply the results to special solutions, without some preliminary work. Knowing that all the groups in the literature are forms of the same group allows to encompass this difficulty.

Inspired by the work of André ([**And01**]), we study the behavior of the parameterized intrinsic Galois group when q varies and especially when q goes to 1. We prove that the specialization of the parameterized intrinsic Galois group of a q-difference equation $Y(qx) = A(q,x)Y(x)$ with coefficients in a field $k(q,x)$ such that $[k : \mathbb{Q}] < \infty$ at $q = a$ for any a in the algebraic closure of k, contains the parameterized intrinsic Galois group of the specialized equation. If k is a number field, this holds also if we reduce the equations in positive characteristic, so that q reduces to a parameter in positive characteristic. So if we have a q-difference equation $Y(qx) = A(q,x)Y(x)$, we can either reduce it in positive characteristic and then specialize q, or specialize q and then reduce in positive characteristic. In particular, for $q = 1$ we obtain from

$$\frac{Y(qx) - Y(x)}{(q-1)x} = \frac{A(q,x) - 1}{(q-1)x} Y(x)$$

a differential system. The phenomenon is explicitly described in the case of hypergeometric functions (see Chapter 9 and, in particular, Corollary 9.18).

Finally, the description of the parameterized intrinsic Galois group in terms of curvatures allows us to understand the link between the linear and non-linear Galois theory of q-difference systems. In [**Gra**], A. Granier introduces a Galois D-groupoid for non-linear q-difference equations, in the spirit of Malgrange's work. In Corollary 11.10, we show, using once more the curvature characterization of the parameterized intrinsic Galois group, that the Malgrange-Granier D-groupoid generalizes the parameterized intrinsic Galois group to the non-linear case. Thanks to our comparison results, we are able to compare the Malgrange-Granier D-groupoid to the differential Galois group of Hardouin-Singer. This answers a question of Malgrange ([**Mal09**, page 2]) on the relation among D-groupoids and Kolchin's differential algebraic groups.

Description of the main results

The paper being relatively long, we give here a quite detailed description of the content. Part 1 is an introduction to q-difference equations and explains some preliminaries results.

Grothendieck conjecture for q-difference equations

In [**DV02**], the first author proved a q-difference analogue of the Grothendieck conjecture on p-curvatures, under the assumption that q is an algebraic number and that the field of constants is a number field. In this paper, we generalize this result in two different directions.

Consider a field of rational functions $K(x)$, a transcendental element $q \in K$, such that K is itself a field of rational functions in q of the form $k(q)$, and a q-difference system $Y(qx) = A(x)Y(x)$, with $A(x) \in \mathrm{GL}(K(x))$. We prove the following result (see Theorem 4.2 for a more general and intrinsic result):

THEOREM 2. *A q-difference system $Y(qx) = A(x)Y(x)$, with $A(x) \in \mathrm{GL}_\nu(K(x))$, has a solution matrix in $\mathrm{GL}_\nu(K(x))$ if and only if for almost all positive integer n there exists a primitive n-th root of unity ζ_n such that*

$$\left[A(q^{n-1}x)\cdots A(qx)A(x)\right]_{q=\zeta_n} = \textit{identity matrix}.$$

In the present article we work under more general assumptions. Namely, we assume that k is a perfect field, of any characteristic, and that K is a finite extension of $k(q)$. Replacing k by its perfect closure, the theorem above covers all the possible cases in which q is transcendental over the prime field.

Suppose now that q is algebraic over the prime field, and that the characteristic of K is zero. We consider again the q-difference system $Y(qx) = A(x)Y(x)$, with $A(x) \in \mathrm{GL}(K(x))$. We can always suppose that K is actually finitely generated over $\mathbb{Q}$. For the sake of simplicity, we assume in this introduction that $K = \mathbb{Q}(\alpha)$ is a purely transcendental extension and that $q \in \mathbb{Q}$, $q \neq 0, 1, -1$. For almost all rational primes p the image of q in $\mathbb{F}_p$ is well defined and non-zero, so that there exists a minimal positive number κ_p such that $q^{\kappa_p} \equiv 1$ modulo p. Let ℓ_p be a positive integer such that $1 - q^{\kappa_p} = p^{\ell_p}\frac{h}{g}$, with $h, g \in \mathbb{Z}$ prime to p. We have (see Theorem 3.6):

THEOREM 3. *A q-difference system $Y(qx) = A(x)Y(x)$, with $A(x) \in \mathrm{GL}_\nu(K(x))$, has a solution matrix in $\mathrm{GL}_\nu(K(x))$ if and only if for almost all prime p we have*

$$A(q^{\kappa_p-1}x)\cdots A(qx)A(x) \equiv \textit{identity matrix modulo } p^{\ell_p}.$$

The statement above is a little bit imprecise, since we should have introduced a $\mathbb{Z}$-algebra contained in $K(x)$ that would have given a precise sense to the reduction modulo p^{ℓ_p}, for almost all p. The reader will find a more formal statement in Part 2, where the result above is proved under the assumption that K is any finitely generated extension of $\mathbb{Q}$ and that q is an algebraic number, not a root of unity. As already pointed out, the first author proves in [**DV02**, Thm.7.1.1] the statement above under the assumption that K is a number field. Our proof relies on [**DV02**, Thm.7.1.1], in the sense that we consider a transcendence basis of K over $\mathbb{Q}$ as a set of parameters varying in the algebraic closure of $\mathbb{Q}$ and therefore we make a non-trivial reduction to the situation considered in [**DV02**], for sufficiently many special values of the parameters.

Notice that if one starts with a q-difference system over $\mathbb{C}(x)$ and a complex number q, which is not a root of unity, then it is always possible to reduce to one of the two situations above.

Intrinsic Galois groups

Once again, let K be a field of characteristic zero and q a non-zero element of K, which is not a root of unity. We will denote by σ_q the q-difference operator $f(x) \mapsto f(qx)$. A q-difference module $\mathcal{M}_{K(x)} = (M_{K(x)}, \Sigma_q)$ over $K(x)$ is a $K(x)$-vector space of finite dimension ν equipped with a σ_q-semilinear bijective operator Σ_q:

$$\Sigma_q(fm) = \sigma_q(f)\Sigma_q(m), \text{ for any } m \in M \text{ and } f \in K(x).$$

The coordinates of a vector fixed by Σ_q with respect to a given basis are solution of a linear q-difference system of the form

$$(\mathcal{S}_q) \qquad Y(qx) = A(x)Y(x), \text{ with } A(x) \in \mathrm{GL}_\nu(K(x)).$$

We consider the collection $Constr(\mathcal{M}_{K(x)})$ of $K(x)$-linear algebraic constructions of $\mathcal{M}_{K(x)}$ (direct sums, tensor product, symmetric and antisymmetric product, dual).

The operator Σ_q induces a q-difference operator on every element of $Constr(\mathcal{M}_{K(x)})$, that we will still call Σ_q. Then the intrinsic Galois group of $\mathcal{M}_{K(x)}$ is defined as:

$$Gal(\mathcal{M}_{K(x)},\eta_{K(x)})=\{\varphi\in \mathrm{GL}(M_{K(x)}):\varphi \text{ stabilizes}$$

$$\text{every subset stabilized by } \Sigma_q, \text{ in any construction}\}.$$

Of course, one can give a Tannakian description of $Gal(\mathcal{M}_{K(x)},\eta_{K(x)})$. As in [**Kat82**], Theorem 2 and Theorem 3 are equivalent to the the following descriptions of the intrinsic Galois group:

THEOREM 4. *In the notation of Theorem 2 (resp. Theorem 3), the intrinsic Galois group $Gal(\mathcal{M}_{K(x)},\eta_{K(x)})$ is the smallest algebraic subgroup of* $\mathrm{GL}(M_{K(x)})$*, whose specialization at ζ_n contains the specialization of the operator Σ_q^n at ζ_n, for almost all positive integer n and for a choice of a primitive n-th root of unity ζ_n (resp. whose reduction modulo p^{ℓ_p} contains the reduction of the operator $\Sigma_q^{\kappa_p}$ modulo p^{ℓ_p}, for almost all prime p).*

The statement is a little bit informal. The reader will find a precise statement in Chapter 6.

As the notion of intrinsic Galois group is deeply related to the notion of tannakian category, the notion of differential intrinsic Galois group is related to the notion of differential tannakian category developed by A. Ovchinnikov in [**Ovc09**]. We show in this paper how the category of q-difference modules over $K(x)$ may be endowed with a prolongation functor F and thus turns out to be a differential tannakian category. Intuitively, if $\mathcal{M}$ is a q-difference module, associated with a q-difference system $\sigma_q(Y)=AY$, the q-difference module $F(\mathcal{M})$ is attached to the q-difference system

$$\sigma_q(Z)=\begin{pmatrix} A & \partial A\\ 0 & A\end{pmatrix}Z.$$

Notice that if Y verifies $\sigma_q(Y)=AY$, then $Z=\begin{pmatrix} Y & \partial(Y)\\ 0 & Y\end{pmatrix}$ is solution of the system above. We consider the family $Constr^{\partial}(\mathcal{M}_K(x))$ of constructions of differential algebra of $\mathcal{M}_{K(x)}$, that is the smallest family containing $\mathcal{M}_{K(x)}$ and closed with respect to all algebraic constructions (direct sums, tensor product, symmetric and antisymmetric product, dual) plus the prolongation functor F. Then the differential intrinsic Galois group of $\mathcal{M}_{K(x)}$ is defined as:

$$Gal^{\partial}(\mathcal{M}_{K(x)},\eta_{K(x)})=\{\varphi\in \mathrm{GL}(M_{K(x)}):\ \varphi \text{ stabilizes every } \Sigma_q\text{-stable subset}$$

$$\text{in any construction of differential algebra}\}.$$

The group $Gal^{\partial}(\mathcal{M}_{K(x)},\eta_{K(x)})$ is endowed with a structure of linear differential algebraic group (*cf.* [**Kol73**]). Theorem 2 and Theorem 3 are equivalent to the the following descriptions of the intrinsic Galois group:

THEOREM 5. *In the notation of Theorem 2 (resp. Theorem 3), the parameterized intrinsic Galois group $Gal^{\partial}(\mathcal{M}_{K(x)},\eta_{K(x)})$ is the smallest differential subgroup of* $\mathrm{GL}(M_{K(x)})$*, whose specialization at ζ_n contains the specialization of the operator Σ_q^n at ζ_n, for almost all positive integer n and for a choice of a primitive n-th root of unity ζ_n (resp. whose reduction modulo p^{ℓ_p} contains the reduction of the operator $\Sigma_q^{\kappa_p}$ modulo p^{ℓ_p}, for almost all prime p).*

This implies, for instance, (*cf.* Theorem 4 above and Corollary 7.16 in the text below):

COROLLARY 6. *The differential intrinsic Galois group $Gal^\partial(\mathcal{M}_{K(x)},\eta_{K(x)})$ is a Zariski dense subset of the algebraic intrinsic Galois group $Gal(\mathcal{M}_{K(x)},\eta_{K(x)})$.*

Comparisons with the other Galois theories for linear differential equations

In Part 4, we relate the intrinsic Galois groups, both algebraic and differential, with the more classical notions of Galois groups. In Corollary 8.13, we prove that the differential dimension of $Gal^\partial(\mathcal{M}_{K(x)},\eta_{K(x)})$ as a differential algebraic group is equal to the differential transcendence degree of the field generated by the meromorphic solutions of $Y(qx)=A(x)Y(x)$ over the differential closure of the field of elliptic functions. This means that the differential relations among the solutions can already be determined from the curvatures.

To study the specializations of the intrinsic Galois groups, differential and algebraic, we use the language of generalized differential rings and modules, introduced by Y. André (*cf.* [**And01**]), that allows to treat differential and difference modules in the same setting. It is therefore adapted to our situation where the reductions of $\mathcal{M}_{K(x)}$ can be either q-difference modules or differential modules. We prove that, for all finite places v of K, the specialization of $Gal(\mathcal{M}_{K(x)},\eta_{K(x)})$ (resp. $Gal^\partial(\mathcal{M}_{K(x)},\eta_{K(x)})$) at v gives an upper bound for the intrinsic (resp. differential) Galois group of the reduction of $\mathcal{M}_{K(x)}$ modulo v. Here we are considering the case $K=k(q)$, with q transcendental.

When we specialize q to 1, we find a differential module. Going backwards, i.e., deforming a differential module, we can deduce from the results above a description of an upper bound of its intrinsic Galois group, defined in [**Kat82**]. In fact, given a $k(x)/k$-differential module (M,∇), we can fix a basis $\underline{e}$ of M such that

$$\nabla(\underline{e})=\underline{e}G(x),$$

so that the horizontal vectors of ∇ are solutions of the system $Y'(x)=-G(x)Y(x)$. Then $M_{k(q,x)}:=M\otimes_{k(x)}k(q,x)$ has a natural structure of q-difference module define by $\Sigma_q\underline{e}=\underline{e}(1+(q-1)xG(x))$. This the most naïve q-deformation of a differential module and more sophisticated choices are possible. We have (see Corollary 9.18):

COROLLARY 7. *The intrinsic Galois group of (M,∇) is contained in the "specialization at $q=1$" of the smallest algebraic subgroup G of $\mathrm{GL}(M_{k(q,x)})$ that contains almost all the specialization of the operators $\Lambda_n: M_{k(q,x)}\to M_{k(q,x)}$, defined by:*

$$\Lambda_n\underline{e}=\underline{e}\prod_{i=0}^{n-1}\left(1+(q-1)q^ixG(q^ix)\right),$$

at a primitive n-th root of unity ζ_n, for almost all integer n.

Comparisons with Malgrange-Granier Galois theory for non-linear differential equations

A. Granier has defined a Galois D-groupoid for nonlinear q-difference equations, in the wake of Malgrange's work. In the particular case of a linear system $Y(qx)=A(x)Y(x)$, with $A(x)\in \mathrm{GL}_\nu(\mathbb{C}(x))$, the Malgrange-Granier D-groupoid is the D-envelop of the dynamics, i.e., it encodes all the partial differential equations over $\mathbb{P}^1_{\mathbb{C}}\times\mathbb{C}^\nu$ with analytic coefficients, satisfied by local diffeomorphisms of the form $(x,X)\mapsto(q^kx,A_k(x)X)$ for all $k\in\mathbb{Z}$, where $A_k(x)\in\mathrm{GL}_\nu(\mathbb{C}(x))$ is the matrix obtained by iterating the system $Y(qx)=A(x)Y(x)$ so that:

$$Y(q^kx)=A_k(x)Y(x).$$

Notice that:

$$A_k(x) := A(q^{k-1}x)\dots A(qx)A(x) \text{ for all } k \in \mathbb{Z},\ k > 0;$$
$$A_0(x) = Id_\nu;$$
$$A_k(x) := A(q^k x)^{-1}A(q^{k+1}x)^{-1}\dots A(q^{-1}x)^{-1} \text{ for all } k \in \mathbb{Z},\ k < 0.$$

Using Theorem 10, we relate this analytic D-groupoid with the more algebraic notion of differential intrinsic Galois group. We prove that the solutions in a neighborhood of $\{x_0\} \times \mathbb{C}^\nu$ of the sub-D-groupoid of the Malgrange-Garnier D-groupoid, which fixes the transversals, are precisely the points of the differential intrinsic Galois group in the ring $\mathbb{C}\{x - x_0\}$ of germs of analytic functions at x_0.

For systems with constant coefficients, we retrieve the result of A. Granier (*cf.* [**Gra**, Thm. 2.4]), i.e., the evaluation in $x = x_0$ of the solutions of the transversal D-groupoid is the usual Galois group. Notice that in this case algebraic and differential Galois groups coincide. The analogous result for differential equations is proved in [**Mal01**]. B. Malgrange, in the differential case, and A. Granier, in the q-difference constant case, establish a link between the Galois D-groupoid and the usual Galois group: This is compatible with our results since in those cases the algebraic intrinsic and differential Galois groups, as well as the usual Galois groups, coincide (*cf.* §11.4 below).

Part 1

Introduction to q-difference equations and their Galois theory

CHAPTER 1

Generalities on q-difference modules

We quickly recall some notations and a few basic results about q-difference algebras and q-difference modules. For a more detailed introduction to q-difference modules see [**vdPS97**, Chapter 12], [**DV02**, Part I] or [**DVRSZ03**].

1.1. Basic definitions

Let K be a field and $q \neq 0, 1$ be a fixed element of K. The field $K(x)$ is naturally a q-difference field, i.e., it is equipped with the q-difference operator

$$\begin{array}{rccc} \sigma_q : & K(x) & \longrightarrow & K(x) \\ & f(x) & \longmapsto & f(qx) \end{array}.$$

We can associate to σ_q a non-commutative derivation, that we will call q-derivation, defined by

$$d_q(f)(x) = \frac{f(qx) - f(x)}{(q-1)x},$$

and satisfying a q-Leibniz formula:

$$d_q(fg)(x) = f(qx)d_q(g)(x) + d_q(f)(x)g(x), \text{ for any } f, g \in K(x).$$

Notice that, if we set $[n]_q = \frac{q^n-1}{q-1}$, $[n]_q^! = [n]_q[n-1]_q \cdots [1]_q$, for any $n \geq 1$, $[0]_q^! = 1$, then

$$d_q^s x^n = \frac{[n]_q^!}{[n-s]_q^!} x^{n-s}, \text{ for any pair of positive integers } s, n, \text{ such that } n \geq s.$$

Therefore we define the q-binomial $\binom{n}{s}_q = \frac{[n]_q^!}{[n-s]_q^! [s]_q^!}$, so that $\frac{d_q^s}{[s]_q^!} x^n = \binom{n}{s}_q x^{n-s}$. When q is a root of unity of order κ, d_q^κ and all its iterations are equal to 0. Nonetheless, the q-binomials $\binom{n}{s\kappa}_q$ and the operators $\frac{d_q^{s\kappa}}{[s\kappa]_q^!}$ are well defined and non-zero for every positive integer s.

More generally, we will consider a q-difference extension $\mathcal{F}$ of $K(x)$, i.e., a field extension $\mathcal{F}$ of $K(x)$ equipped with a field automorphism extending the action of σ_q, which we will also call q-difference operator and denote σ_q. Of course, $\mathcal{F}$ is also equipped with the skew derivation $d_q := \frac{\sigma_q - 1}{(q-1)x}$. We denote by $\mathcal{F}^{\sigma_q}$ the field of constant of $\mathcal{F}$, i.e., the subfield of $\mathcal{F}$ of all elements fixed by σ_q.

Typical examples of q-difference extensions of $K(x)$ are the fields $K((x))$ or $K(x^{1/r})$, for $r \in \mathbb{Z}_{>1}$. In the latter case, one sets $\sigma_q(x^{1/r}) = \tilde{q}x^{1/r}$, for a given r-th root $\tilde{q}$ of q. If $K = \mathbb{C}$, one can naturally consider also the fields of meromorphic functions over $\mathbb{C}$, over $\mathbb{C}^* = \mathbb{C} \smallsetminus \{0\}$ or over any domain invariant under the action of σ_q.

Definition 1.1. A q-difference module $\mathcal{M}_\mathcal{F} = (M_\mathcal{F}, \Sigma_q)$ (of rank ν) over $\mathcal{F}$ is a finite dimensional $\mathcal{F}$-vector space $M_\mathcal{F}$ (of dimension ν) equipped with an invertible σ_q-semilinear operator $\Sigma_q : M_\mathcal{F} \to M_\mathcal{F}$, i.e., a bijective additive map from $M_\mathcal{F}$ to itself such that

$$\Sigma_q(fm) = \sigma_q(f)\Sigma_q(m), \text{ for any } f \in \mathcal{F} \text{ and } m \in M_\mathcal{F}.$$

We will call Σ_q a q-difference operator over $M_{\mathcal{F}}$ or the q-difference operator of $\mathcal{M}_{\mathcal{F}}$.

A morphism of q-difference modules (over $\mathcal{F}$) is a morphism of $\mathcal{F}$-vector spaces, commuting with the q-difference operators. We denote by $Diff(\mathcal{F},\sigma_q)$ the category of q-difference modules over $\mathcal{F}$.

1.1.1. Construction of linear algebra. Let $\mathcal{M}_{\mathcal{F}}=(M_{\mathcal{F}},\Sigma_{q,M})$ and $\mathcal{N}_{\mathcal{F}}=(N_{\mathcal{F}},\Sigma_{q,N})$ be two q-difference modules over $\mathcal{F}$. The direct sum $\mathcal{M}_{\mathcal{F}}\oplus\mathcal{N}_{\mathcal{F}}$ of $\mathcal{M}_{\mathcal{F}}$ and $\mathcal{N}_{\mathcal{F}}$ is the q-difference module such that:

- the underline $\mathcal{F}$-vector space is $M_{\mathcal{F}}\oplus N_{\mathcal{F}}$;
- the q-difference operator is a σ_q-semilinear bijection defined by $m\oplus n\mapsto \Sigma_{q,M}(m)\oplus\Sigma_{q,N}(n)$.

The tensor product $\mathcal{M}_{\mathcal{F}}\otimes_{\mathcal{F}}\mathcal{N}_{\mathcal{F}}$ of $\mathcal{M}_{\mathcal{F}}$ and $\mathcal{N}_{\mathcal{F}}$ over $\mathcal{F}$ is the q-difference module such that:

- the underline $\mathcal{F}$-vector space is $M_{\mathcal{F}}\otimes_{\mathcal{F}}N_{\mathcal{F}}$;
- the q-difference operator is a σ_q-semilinear bijection defined by $m\otimes n\mapsto \Sigma_{q,M}(m)\otimes\Sigma_{q,N}(n)$.

The dual q-difference module $\mathcal{M}^*_{\mathcal{F}}=(M^*_{\mathcal{F}},\Sigma^*_{q,M})$ of $\mathcal{M}_{\mathcal{F}}$ is the q-difference module defined as follows:

- the underline $\mathcal{F}$-vector space $M^*_{\mathcal{F}}$ is the dual $\mathcal{F}$-vector space of $M_{\mathcal{F}}$;
- $\Sigma^*_{q,M}:\varphi\mapsto\sigma_q^{-1}\circ\varphi\circ\Sigma_{q,M}$, i.e., for any $m\in M_{\mathcal{F}}$ and any $\varphi\in M^*_{\mathcal{F}}$ we have $\langle\Sigma^*_{q,M}(\varphi),m\rangle=\sigma_q^{-1}\langle\varphi,\Sigma_{q,M}(m)\rangle$.

We say that a q-difference module $\mathcal{N}_{\mathcal{F}}$ over $\mathcal{F}$ is a construction of linear algebra of $\mathcal{M}_{\mathcal{F}}$ if $\mathcal{N}_{\mathcal{F}}$ can be deduced from $\mathcal{M}_{\mathcal{F}}$ by direct sums, duals, tensor products, symmetric and antisymmetric products. The latter constructions can be deduce from the ones defined above in the usual way.

1.1.2. Basis. Let $\mathcal{M}_{\mathcal{F}}=(M_{\mathcal{F}},\Sigma_q)$ be a q-difference module over $\mathcal{F}$ of rank ν. We fix a basis $\underline{e}$ of $M_{\mathcal{F}}$ over $\mathcal{F}$. Let $A\in\mathrm{GL}_\nu(\mathcal{F})$ be such that:

$$\Sigma_q\underline{e}=\underline{e}A.$$

If $\underline{f}$ is another basis of $M_{\mathcal{F}}$, such that $\underline{f}=\underline{e}F$, with $F\in\mathrm{GL}_\nu(\mathcal{F})$, then $\Sigma_q\underline{f}=\underline{f}B$, with $B=F^{-1}A\sigma_q(F)$.

PROPOSITION 1.2. *Let K be a field as above, $\mathcal{M}_{K(x)}$ a q-difference module over $K(x)$ and let $k=\mathbb{Q}$ or $\mathbb{F}_p$, according that the field K has characteristic zero or $p>0$, respectively. For any q-difference module $\mathcal{M}_{K(x)}$ there exists a finite generated extension $\widetilde{K}\subset K$ of k, containing q, and a q-difference module $\mathcal{M}_{\widetilde{K}(x)}$ such that $\mathcal{M}_{K(x)}=\mathcal{M}_{\widetilde{K}(x)}\otimes_{\widetilde{K}(x)}K(x)$.*

PROOF. To prove the lemma, it suffices to fix a basis $\underline{e}$ of $\mathcal{M}_{\mathcal{F}}$ and to consider a field $\widetilde{K}$ generated over k by q and all the entries of the matrix of Σ_q with respect to the basis $\underline{e}$. □

REMARK 1.3. We will always denote with the same letter, but with different subscripts, q-difference modules that become isomorphic after an extension of the base field, as in the statement above.

1.1.3. Horizontal vectors. A horizontal vector of $\mathcal{M}_{\mathcal{F}}$ is an element $m\in M_{\mathcal{F}}$ such that $\Sigma_q(m)=m$. We denote by $\mathcal{M}_{\mathcal{F}}^{\Sigma_q}$ the set of horizontal vectors of $\mathcal{M}_{\mathcal{F}}$. One proves easily that it is a $\mathcal{F}^{\sigma_q}$-vector space. The dimension of $\mathcal{M}_{\mathcal{F}}^{\Sigma_q}$ is invariant by extension of the constants:

PROPOSITION 1.4. *Let $\mathcal{F}$ be a q-difference field and with $K=\mathcal{F}^{\sigma_q}$ and let K' be a σ_q-constant field extension of K. Let $\mathcal{M}_{\mathcal{F}}$ be a q-difference module over $\mathcal{F}$ and*

$\mathcal{M}_{\mathcal{F}(K')} = \mathcal{M}_{\mathcal{F}} \otimes_{\mathcal{F}} \mathcal{F}(K')$ *the q-difference module over* $\mathcal{F}(K')$ *obtained by scalar extension. Then* $\left(\mathcal{M}_{\mathcal{F}(K')}\right)^{\Sigma_q} = \mathcal{M}_{\mathcal{F}}^{\Sigma_q} \otimes_K K'$.

Proof. First of all notice that $\mathcal{F}(K')^{\sigma_q} = K'$. We have a natural injective map

$$K' \otimes_K \mathcal{M}_{\mathcal{F}}^{\Sigma_q} \longrightarrow \left(\mathcal{M}_{\mathcal{F}(K')}\right)^{\Sigma_q}.$$

We have to show that it is also surjective. Let $\underline{e}$ be a basis of $M_{\mathcal{F}}$ over $\mathcal{F}$ such that $\Sigma_q\underline{e} = \underline{e}A$, with $A \in \mathrm{GL}_\nu(\mathcal{F})$. Let $z \in \left(\mathcal{M}_{\mathcal{F}(K')}\right)^{\Sigma_q}$ and let us write $z = \underline{e}Z$, where $Z \in \mathcal{F}(K')^\nu$. The set

$$\mathfrak{a} := \{r \in K' \otimes_K \mathcal{F} \text{ s.t. } rZ \in (K' \otimes_K \mathcal{F})^\nu\}$$

is a non-zero ideal of $K' \otimes_K \mathcal{F}$ stable by σ_q. Indeed, if $r \in \mathfrak{a}$ then $\Sigma_q(rz) = \underline{e}A\sigma_q(rZ)$ and $A\sigma_q(rZ) \in (K' \otimes_K \mathcal{F})^\nu$. Since $\sigma_q(r)z = \Sigma_q(rz)$, we find that $\sigma_q(r) \in \mathfrak{a}$. By [**vdPS97**, Lemma 1.11], the algebra $K' \otimes_K \mathcal{F}$ has no non trivial ideal stable under σ_q. Thus 1 belongs to the ideal $\mathfrak{a}$, which implies that $Z \in (K' \otimes_K \mathcal{F})^\nu$.

Let $\{\lambda_i\}_i \subset K'$ be a (maybe, infinite) basis of K'/K. We can write $z = \sum_i \lambda_i \otimes \underline{e}\vec{y}_i$, for some $\vec{y}_i \in \mathcal{F}^\nu$, not all zero. Since $\Sigma_q(z) = z$, we obtain:

$$\sum_i \lambda_i \otimes \underline{e}\vec{y}_i = \sum_i \lambda_i \otimes \underline{e}A\sigma_q(\vec{y}_i),$$

where σ_q acts on vectors componentwise. We conclude that $\vec{y}_i = A\sigma_q(\vec{y}_i)$ for all i and therefore that $\underline{e}\vec{y}_i \in \mathcal{M}_{\mathcal{F}}^{\Sigma_q}$, for all i. This ends the proof. □

1.1.4. q-difference modules over a ring. In the sequel, we will deal with q-difference modules over rings. We do not want to be too formal on this point, since notations and definitions are quite intuitive.

Let $\mathcal{O}$ be a subring of K containing q. Then $\mathcal{O}[x]$ is stable by σ_q and therefore is a q-difference algebra. Let $\mathcal{A}$ be a q-difference algebra over $\mathcal{O}[x]$, meaning an algebra over $\mathcal{O}[x]$, stable by a natural extension of σ_q. For instance, we will consider algebras of the form

$$\mathcal{O}\left[x, \frac{1}{P(x)}, \frac{1}{P(qx)}, \frac{1}{P(q^2x)}, \dots\right],$$

for some $P(x) \in \mathcal{O}[x]$.

A q-difference module $\mathcal{M} = (M, \Sigma_q)$ over $\mathcal{A}$ will be a free $\mathcal{A}$-module M of finite rank, equipped with a semilinear invertible operator[1] Σ_q. All the notions introduced above generalize intuitively to this case.

If $\mathcal{A}$ is a domain and $\mathcal{F}$ is the fraction field of $\mathcal{A}$, then

$$\mathcal{M}_{\mathcal{F}} = (M_{\mathcal{F}} := M \otimes_{\mathcal{A}} \mathcal{F}, \Sigma_q \otimes \sigma_q)$$

is a q-difference module over $\mathcal{F}$. Notice that any q-difference module over $\mathcal{F}$ comes from a q-difference module over $\mathcal{A}$, for a convenient choice of $\mathcal{A} \subset \mathcal{F}$.

1.2. q-difference modules, systems and equations

Let $\mathcal{M}_{\mathcal{F}} = (M_{\mathcal{F}}, \Sigma_q)$ be a q-difference module of rank ν over a q-difference field $\mathcal{F}$. We fix a basis $\underline{e}$ of $M_{\mathcal{F}}$ over $\mathcal{F}$, such that:

$$\Sigma_q\underline{e} = \underline{e}A,$$

with $A \in \mathrm{GL}_\nu(\mathcal{F})$.

[1] We could have asked that Σ_q is only injective, but then, enlarging the scalars to a q-difference algebra $\mathcal{A}'/\mathcal{A}$, constructed inverting some elements, we would have obtained an invertible operator. For our purpose, the assumption that Σ_q is invertible is not restrictive.

DEFINITION 1.5. We call

$$\sigma_q(Y) = A^{-1}Y, \tag{1.1}$$

the (q-difference) system (of order ν) associated to $\mathcal{M}_{\mathcal{F}}$, with respect to the basis $\underline{e}$.

If $\vec{y} \in \mathcal{F}^{\nu}$ are the coordinates of a horizontal vector $m \in M_{\mathcal{F}}$ with respect to the basis $\underline{e}$, then $\vec{y}$ verifies $\Sigma_q(\underline{e}\vec{y}) = \underline{e}\vec{y}$, i.e., $\vec{y} = A\sigma_q(\vec{y})$. This means that $\vec{y}$ is a solution vector of the q-difference system (1.1). On the other hand, a solution vector of (1.1) always represents a horizontal vector of $\mathcal{M}_{\mathcal{F}}$ in the basis $\underline{e}$.

Two systems are said to be equivalent by gauge transformation if they are associated to the same q-difference module, with respect to two different basis. Of course, one associates a q-difference module, with underlying $\mathcal{F}$-vector space $\mathcal{F}^{\nu}$, to any q-difference system of order ν.

To a given linear q-difference equation

$$a_0y + a_1\sigma_q y + \cdots + a_\nu\sigma_q^\nu y = 0, \quad \text{with } a_1, \dots, a_\nu \in \mathcal{F} \text{ and } a_0a_\nu \neq 0, \tag{1.2}$$

one naturally associates a linear q-difference system

$$\sigma_q(Y) = \left(\begin{array}{c|ccc} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \\ \hline -a_0/a_\nu & -a_1/a_\nu & \dots & -a_{\nu-1}/a_\nu \end{array}\right) Y. \tag{1.3}$$

If z is a solution of (1.2) in some q-difference extension of $\mathcal{F}$, then the vector ${}^t(z, \sigma_q(z), \dots, \sigma_q^{\nu-1}(z))$ is a solution column of (1.3). The equation (1.2) has at most ν solutions in a q-difference extension $\mathcal{G}$ of $\mathcal{F}$, which are linearly independent over the field $\mathcal{G}^{\sigma_q}$ of σ_q-invariant elements of $\mathcal{G}$. If $z_1, \dots, z_\nu$ are those solutions, then the q-analog of the Wronskian Lemma says that the matrix

$$\begin{pmatrix} z_1 & \dots & z_\nu \\ \sigma_q(z_1) & \dots & \sigma_q(z_\nu) \\ \vdots & \dots & \vdots \\ \sigma_q^{\nu-1}(z_1) & \dots & \sigma_q^{\nu-1}(z_\nu) \end{pmatrix}$$

is an invertible solution of (1.3).

Given a q-difference module $(M_{\mathcal{F}}, \Sigma_q)$ of rank ν over $\mathcal{F}$, such that q is not a root of unity of order smaller than ν, the Cyclic Vector Lemma (see for instance [**DV02**, §1.3]) allows to find an element m of $M_{\mathcal{F}}$, called cyclic element, such that $m, \Sigma_q(m), \dots, \Sigma_q^{\nu-1}(m)$ is a basis of $M_{\mathcal{F}}$.

1.3. Some remarks on solutions

Let $\sigma_q(Y) = BY$ be a q-difference system, with $B \in \mathrm{GL}_\nu(\mathcal{F})$.

DEFINITION 1.6. Let $\mathcal{G}$ be a q-difference field extension of $\mathcal{F}$. A fundamental solution matrix of $\sigma_q(Y) = BY$ in $\mathcal{G}$ is an invertible matrix F, with entries in $\mathcal{G}$, such that $\sigma_q(F) = BF$.

Recursively, we obtain from $\sigma_q(Y) = BY$ a family of higher order q-difference systems:

$$\sigma_q^n(Y) = B_nY \text{ and } d_q^nY = G_nY,$$

with $B_n \in \mathrm{GL}_\nu(\mathcal{F})$ and $G_n \in M_\nu(\mathcal{F})$, for any positive integer n. Notice that $B_1 := B$ and:

$$B_{n+1} = \sigma_q(B_n)B_1,\ G_1 = \frac{B_1 - 1}{(q-1)x} \text{ and } G_{n+1} = \sigma_q(G_n)G_1(x) + d_qG_n.$$

It is convenient to set $B_0 = G_0 = 1$ and $G_{[n]} = \frac{G_n}{[n]_q^!}$ for any $n \geq 0$. Notice that $G_{[n]}$ is well defined even if q is a root of unity.

PROPOSITION 1.7. *Let $\mathcal{F} = K(x)$ and suppose that the matrix G_1 does not have a pole at 0 (or equivalently that B does not have a pole at 0 and that $B(0)$ is the identity matrix), then $W(x) = \sum_{n\geq 0} G_{[n]}(0)x^n$ is a fundamental solution matrix (in $K((x))$) of the system $\sigma_q(Y) = BY$. Moreover, it is the only fundamental solution matrix with coefficients in $K[[x]]$, whose constant term is the identity.*

If K is a field equipped with a norm such that $|q| \neq 1$, then $\sum_{n\geq 0} G_{[n]}(0)x^n$ has a non-zero radius of convergence and, hence, an infinite radius of meromorphy.[2]

The proof of the proposition above is similar to the proof of the resolvent in the differential case. Proposition 1.7 has a multiplicative avatar:

PROPOSITION 1.8. *Let K be a field, $|\ |$ a norm (archimedean or ultrametric) over K and q an element of K, such that $|q| > 1$. We consider a q-difference system $Y(qx) = B(x)Y(x)$ such that $B(x) \in \mathrm{GL}_\nu(K(x))$, zero is not a pole of $B(x)$ and such that $B(0)$ is the identity matrix. Then the infinite product*

$$Z(x) = \Big(B(q^{-1}x)B(q^{-2}x)B(q^{-3}x)\dots\Big)$$

is the germ of the analytic fundamental solution matrix at zero such that $Z(0)$ is the identity, and has infinite radius of meromorphy.

PROOF. If $|q| > 1$, the infinite product defining $Z(x)$ is convergent in the neighborhood of zero and it is a solution of $Y(qx) = B(x)Y(x)$, such that $Z(0)$ is the identity matrix. The fact that $Z(x)$ is a meromorphic function with infinite radius of meromorphy follows from the fact that the functional equation $Y(qx) = B(x)Y(x)$ "propagates" meromorphy. □

REMARK 1.9. Notice that, independently of the characteristic of K, if q is not a root of unity, then we can always find a norm over K such that $|q| > 1$. Of course, the norm does not need to be archimedean.

Moreover, in Proposition 1.8, if $|q| < 1$ then one has to consider the product $\prod_{n\geq 0} B(q^n x)^{-1}$.

1.4. Trivial q-difference modules

The purpose of the second part of this work is to give an arithmetic characterization of trivial q-difference modules, where trivial means:

DEFINITION 1.10. We say that the q-difference module $\mathcal{M} = (M, \Sigma_q)$ of rank ν over a q-difference algebra $\mathcal{A}$ is trivial if there exists a basis $\underline{f}$ of M over $\mathcal{F}$ such that $\Sigma_q \underline{f} = \underline{f}$.

The definition applies in particular to the case of a q-difference module over a field. For further reference, we state some properties of trivial q-difference modules.

PROPOSITION 1.11. *Let $\mathcal{F}$ be a q-difference field as above and $\mathcal{M}_{\mathcal{F}}$ be a q-difference module over $\mathcal{F}$. The following statements are equivalent:*

1. *The q-difference module $\mathcal{M}_{\mathcal{F}}$ is trivial.*
2. *There exists a basis $\underline{e}$ of $\mathcal{M}_{\mathcal{F}}$ such that the q-difference system associated to $\mathcal{M}_{\mathcal{F}}$ with respect to the basis $\underline{e}$ has an invertible solution matrix in $\mathrm{GL}_\nu(\mathcal{F})$.*
3. *For any basis $\underline{e}$ of $\mathcal{M}_{\mathcal{F}}$, the q-difference system associated to $\mathcal{M}_{\mathcal{F}}$ with respect to the basis $\underline{e}$ has an invertible solution matrix in $\mathrm{GL}_\nu(\mathcal{F})$.*

[2] In the sense that its entries are quotient of two entire analytic functions with respect to $|\ |$.

(4) $\dim_{\mathcal{F}^{\sigma_q}} \mathcal{M}_{\mathcal{F}}^{\Sigma_q} = \dim_{\mathcal{F}} M_{\mathcal{F}}$.

PROOF. Let $\underline{e}$ be a basis of $\mathcal{M}_{\mathcal{F}}$, such that $\Sigma_q \underline{e} = \underline{e}A(x)$, and $\underline{f}$ be a basis of $\mathcal{M}_{\mathcal{F}}$, such that $\underline{f} = \underline{e}F(x)$, with $F(x) \in \mathrm{GL}_\nu(\mathcal{F})$. Then $\Sigma_q \underline{f} = \underline{f}$ if and only if

$$\underline{f} = \Sigma_q(\underline{e}F(x)) = \underline{e}A(x)F(qx) = \underline{f}F(x)^{-1}A(x)F(qx),$$

therefore if and only if $F(qx) = A(x)^{-1}F(x)$. This proves the equivalence among (1), (2) and (3). The equivalence between (1) and (4) follows from the fact that $\underline{f}$ is both a basis of $M_{\mathcal{F}}$ over $\mathcal{F}$ and of $\mathcal{M}_{\mathcal{F}}^{\Sigma_q}$ over $\mathcal{F}^{\sigma_q}$. □

The following statement is a corollary of the proposition above and of Proposition 1.4:

COROLLARY 1.12. *Let K be a field, $q \neq 0, 1$ be an elementof K, and $\mathcal{M}_{K(x)}$ be a q-difference module over $K(x)$. Let K' be an extension of K, on which σ_q acts as the identity, and let $\mathcal{M}_{K'(x)} = \mathcal{M}_{K(x)} \otimes_{K(x)} K'(x)$. Then $\mathcal{M}_{K(x)}$ is trivial if and only if $\mathcal{M}_{K'(x)}$ is trivial.*

PROOF. It follows from Proposition 1.4 that $\mathcal{M}_{K'(x)}^{\Sigma_q} = \mathcal{M}_{K(x)}^{\Sigma_q} \otimes_K K'$. □

Finally we consider the case of a q-difference module whose associated system has a algebraic solution over the base field $K(x)$.

PROPOSITION 1.13. *Let K be a field and q be an element of K which is not a root of unity. We suppose that there exists a norm $|\ |$ over K, such that $|q| \neq 1$, and we consider a linear q-difference equation*

$$a_\nu(x)y(q^\nu x) + a_{\nu-1}(x)y(q^{\nu-1}x) + \cdots + a_0(x)y(x) = 0 \tag{1.4}$$

with coefficients in $K(x)$. If there exists an algebraic q-difference extension $\mathcal{F}$ of $K(x)$ containing a solution f of (1.4), then f is contained in an extension of $K(x)$ isomorphic to $L(\widetilde{q}, t)$, with $\widetilde{q}^r = q, t^r = x$ and $L|K$ is a finite field extension.

PROOF. Let us look at (1.4) as an equation with coefficients in $K((x))$. Then the algebraic solution f of (1.4) can be identified to a Laurent series in $\overline{K}((t))$, where $\overline{K}$ is the algebraic closure of K and $t^r = x$, for a convenient positive integer r. Let $\widetilde{q}$ be an element of $\overline{K}$ such that $\widetilde{q}^r = q$ and that $\sigma_q(f) = f(\widetilde{q}t)$. We can look at (1.4) as a $\widetilde{q}$-difference equation with coefficients in $K(\widetilde{q}, t)$. Then the recurrence relation induced by (1.4) over the coefficients of a formal solution shows that there exist $f_1, \dots, f_s$ solutions of (1.4) in $K(\widetilde{q})((t))$ such that $f \in \sum_i \overline{K} f_i$. It follows that there exists a finite extension $\widetilde{K}$ of $K(\widetilde{q})$ such that $f \in \widetilde{K}((t))$.

We fix an extension of $|\ |$ to $\widetilde{K}$, that we still call $|\ |$. Since f is algebraic, it is a germ of meromorphic function at 0. Since $|\widetilde{q}| \neq 1$, the functional equation (1.4) itself allows to show that f is actually a meromorphic function with infinite radius of meromorphy. Finally, if we chosen r big enough, f can have at worst a pole at $t = \infty$, since it is an algebraic function, which actually implies that f is the Laurent expansion of a rational function in $\widetilde{K}(\widetilde{q}, t)$. □

We recall the following properties of q-difference fields (see [**CS12**, Lemma A.4] for the case of characteristic zero):

COROLLARY 1.14. *Let K be a field, $q \in K$ be not a root of unity and $\mathcal{M}_{K(x)}$ a q-difference module over $K(x)$. If there exists a finite q-difference extension $\mathcal{F}$ of $K(x)$ such that $\mathcal{M}_{\mathcal{F}} = \mathcal{M}_{K(x)} \otimes_{K(x)} \mathcal{F}$ is trivial, then there exists a positive integer r such that $\mathcal{F} \subset L(x^{1/r})$, where $L|K$ is a finite σ_q-constant field extension.*

PROOF. It is enough to apply the previous proposition to the entries of a fundamental solution matrix of the q-difference system associated to a cyclic basis of $\mathcal{M}_{K(x)}$. □

1.5. Regularity

Let $\mathcal{A}$ be a sub-q-difference algebra of $K((x))$. We recall the following basic definition (see for instance [**vdPS97**] or [**Sau00**]).

DEFINITION 1.15. A q-difference module (M, Σ_q) over $\mathcal{A}$ is said to be regular singular at 0 , if there exists a basis $\underline{e}$ of $(M \otimes_{\mathcal{A}} K((x)), \Sigma_q \otimes \sigma_q)$ over $K((x))$ such that the action of $\Sigma_q \otimes \sigma_q$ over $\underline{e}$ is represented by a constant matrix $A \in \mathrm{GL}_\nu(K)$.

It follows from the Frobenius algorithm[3], that a q-difference module $M_{K(x)}$ over $K(x)$ is regular singular if and only if there exists a basis $\underline{e}$ such that $\Sigma_q \underline{e} = \underline{e}A(x)$ with $A(x) \in \mathrm{GL}_\nu(K(x)) \cap \mathrm{GL}_n(K[[x]])$. The eigenvalues of $A(0)$ are called the exponents of $\mathcal{M}$ at 0. They are well defined modulo $q^{\mathbb{Z}}$. The q-difference module $\mathcal{M}$ is said to be regular singular tout court if it is regular singular both at 0 and at ∞, i.e., after a variable change of the form $x = 1/t$.

For further reference, we explicitly state the following lemma, which is a consequence of the Frobenius algorithm:

PROPOSITION 1.16. *Let $\mathcal{M} = (M, \Sigma_q)$ be a q-difference module over a sub-q-difference ring $\mathcal{A}$ of $K((x))$. We suppose that q is not a root of unity. The following statements are equivalent:*

1. *There exists a basis $\underline{e}$ such that $\Sigma_q \underline{e} = \underline{e}A(x)$, with $A(x) \in \mathrm{GL}_\nu(K(x)) \cap \mathrm{GL}_n(K[[x]])$, and such that $A(0)$ is a diagonal matrix with eigenvalues in q^Z (i.e., $\mathcal{M}$ has a regular singularity at 0, with integral exponents and no logarithmic singularity at 0).*
2. *The q-difference module $\mathcal{M}_{K((x))}$ is trivial.*

Singular regularity can be characterized with the help of a Newton polygon. Namely, regular singular q-difference modules are the ones whose Newton polygon has only one finite slope equal to 0 (see [**Sau04c**, Page 200]). We are not going to define or to list the properties of Newton polygons. We only point out that they are the keys to the proof of the statements below.

Let $\mathcal{M}_{K(x)}$ be a q-difference module of rank ν and let $r \in \mathbb{N}$ be a positive integer. We consider a finite extension L of K containing an element $\widetilde{q}$ such that $\widetilde{q}^r = q$. We consider the field extension $K(x) \hookrightarrow L(t)$, $x \mapsto t^r$. The field $L(t)$ has a natural structure of $\widetilde{q}$-difference field extending the q-difference structure of $K(x)$. If follows from [**Sau04c**, §1.1.4] that:

PROPOSITION 1.17. *The q-difference module $\mathcal{M}$ is regular singular at $x = 0$ if and only if the $\widetilde{q}$-difference module $\mathcal{M}_{L(t)} := (M \otimes_{\mathcal{A}} L(t), \Sigma_{\widetilde{q}} := \Sigma_q \otimes \sigma_{\widetilde{q}})$ over $L(t)$ is regular singular at $t = 0$.*

1.6. Irregularity

Next statement gives the structure of general q-difference modules. It can be deduced from the formal classification of q-difference modules (see [**Pra83**, Corollary 9 and §9, 3)], [**Sau04c**, Theorem 3.1.6]):

PROPOSITION 1.18. *We suppose that q is not a root of unity. Let $\mathcal{M}_{K(x)}$ be a q-difference module of rank ν over $K(x)$. Then there exists a positive integer r and a finite extension $L(t)$ of $K(x)$, with $t^r = x$, $r|\nu!$, and $\widetilde{q} \in L$, with $(\widetilde{q})^r = q$ such that $\mathcal{M}_{K(x)} \otimes L((t))$ is a direct sum of $\widetilde{q}$-difference modules $\mathcal{N}_i$. For any i there exists a basis $\underline{e}_i$ of $\mathcal{N}_i$ and a positive integer r_i such that $\Sigma_{\widetilde{q}} \underline{e}_i = \underline{e}_i \frac{B_i}{t^{r_i}}$, with B_i an invertible matrix with coefficients in L.*

[3] *cf.* [**vdPS97**] or [**Sau00**, §1.1]. The algorithm is briefly summarized also in [**Sau04b**, §1.2.2] and [**DVRSZ03**].

COROLLARY 1.19. *There exist an extension $L(t)/K(x)$ as above, a basis $\underline{f}$ of the $\widetilde{q}$-difference module $\mathcal{M}_{L(t)}$ and an integer ℓ such that $\Sigma_{\widetilde{q}}\underline{f} = \underline{f}B(t)$, with $B(t) \in \mathrm{GL}_\nu(L(t))$ of the following form:*

$$\left\{ \begin{array}{l} B(t) = \dfrac{B_\ell}{t^\ell} + \dfrac{B_{\ell-1}}{t^{\ell-1}} + \dots, \textit{ as an element of } \mathrm{GL}_\nu(L((t))); \\ B_\ell \textit{ is a constant non-nilpotent matrix.} \end{array} \right. \tag{1.5}$$

CHAPTER 2

Differential tannakian formalism

This chapter is preliminary to Part 4. It may give a better insight on Part 3, that uses some notions introduced below in a few very specific points.

In [**HS08**], the authors developed a parametrized Galois theory of q-difference systems, that takes into account the action of an auxiliary derivation. In this Galois theory, the groups are linear differential algebraic groups in the sense of Kolchin ([**Kol73**]), that is, zero sets of differential algebraic equations. The theory of [**HS08**] is part of a more general framework, known as differential Tannakian formalism. Initially developed in [**Ovc09**] and [**Kam10**], it is nowadays generalized to encompass any kind of auxiliary operators (see [**Kam12**]). The differential Tannakian formalism extends the natural ideas of the classical Tannakian formalism in the following sense. A Tannakian category is equivalent to the category of representations of an affine group scheme, in other terms, to the category of comodules over the coordinate ring of an affine group scheme. By Morita equivalence, any morphism on the coordinate ring gives birth to a natural transformation on the category and vice versa (for instance, the existence of tensor products in the category corresponds to the multiplication law in the coordinate ring whereas the existence of dual objects corresponds to the inversion map in the Hopf algebra structure of the coordinate ring). Through this dictionary, one should be able to understand the action of an auxiliary operator on the coordinate ring. For instance, a derivation on the coordinate ring corresponds to what we call prolongation functor in §2.2.

After an introduction to some notions of differential algebra and to some basic facts about differential Tannakian categories, we introduce the parametrized Galois theory, following [**HS08**], and explain its connection with the differential Tannakian formalism.

2.1. Differential algebra

In this section, we quickly recall some basic facts of differential algebra as well as some very basic notions of differential algebraic geometry, mainly in the affine case. We largely use standard notations of differential algebra as can be found in [**Kol73**].

2.1.1. Differential algebra. A differential ring (or ∂-ring for short) is a ring R together with a derivation $\partial : R \rightarrow R$, i.e., a map $\partial : R \rightarrow R$ satisfying the Leibniz rule $\partial(ab) = \partial(a)b + a\partial(b)$, for all $(a,b) \in R^2$. The ring of ∂-constants of R is $R^\partial = \{r \in R |\ \partial(r) = 0\}$. All rings considered in this work are commutative with identity and all *differential* rings contain the ring of integer numbers. In particular, all fields are of characteristic zero.

Given two ∂-rings (R,∂) and (R',∂'), a morphism $\psi : R \rightarrow R'$ of ∂-rings is a morphism of rings such that $\psi\partial = \partial'\psi$.

A ∂-ideal $\mathfrak{I}$ of a ∂-ring R is an ideal of R that is invariant under the action of ∂. A ∂-ring R is said to be ∂-simple if it does not contain any non-zero proper ∂-ideals.

A ∂-field k is a field that is also a ∂-ring. A ∂-k-algebra R is a k-algebra and a ∂-ring such that the morphism $k \to R$ is a morphism of ∂-rings. Given two ∂-k-algebras (R, ∂) and (R', ∂'), a morphism $\psi : R \to R'$ of ∂-k-algebras is a morphism of k-algebras such that $\psi\partial = \partial'\psi$. If, moreover, R is a ∂-field and a ∂-k-algebra, we say that $R|k$ is a ∂-field extension.

Let k be a ∂-field and R a ∂-k-algebra. If B is a subset of R, then $k\{B\}_\partial$ denotes the smallest ∂-k-subalgebra of R that contains B. If $R = k\{B\}_\partial$ for some finite subset B of R, we say that R is finitely ∂-generated over k. If $K|k$ is an extension of ∂-fields and $B \subset K$, then $k\langle B\rangle_\partial$ denotes the smallest ∂-field extension of k inside K that contains B. The ∂-k-algebra $k\{x\}_\partial = k\{x_1, \dots, x_n\}_\partial$ of ∂-polynomials over k in the ∂-variables $x_1, \dots, x_n$ is the polynomial ring over k in the countable set of algebraically independent variables $x_1, \dots, x_n, \partial(x_1), \dots, \partial(x_n), \dots$, with an action of ∂ as suggested by the names of the variables. Of course, for any ∂-field extension $L|k$ and any $f := (f_1, \dots, f_n) \in L^n$, one has a ∂-k-morphism $k\{x\}_\partial$ to L, which assigns x_i to f_i, for all $i = 1, \dots, n$. We say that f is a solution of the differential algebraic equation $P(x) = 0$, for some $P \in k\{x\}_\partial$, if P lies in the kernel of the specialization morphism above.

The differential closure of a ∂-field k is a ∂-field extension $\widetilde{k}$ of k, with the property that any system of differential algebraic equations with coefficients in k, having a solution in some differential field extension of k, has a solution in $\widetilde{k}$. If k coincides with its differential closure, it is said to be differentially closed or ∂-closed, for short.

Definition 2.1. Let $L|K$ be a ∂-field extension. Elements $a_1, \dots, a_n \in L$ are called differentially (or ∂-algebraically) independent over K if the elements $a_1, \dots, a_n, \partial(a_1), \dots, \partial(a_n), \dots$ are algebraically independent over K. Otherwise, they are called differentially dependent over K. A ∂-transcendence basis of L over K is a maximal differentially independent set over K, subset of L. Any two ∂-transcendence basis of $L|K$ have the same cardinality and so we can define the ∂-transcendence degree of $L|K$ (or differential transcendence degree of $L|K$, when the choice of ∂ is clear, or also ∂-trdeg$(L|K)$, for short) as the cardinality of any ∂-transcendence basis of L over K.

Finally, we introduce the notion of (σ_q, ∂)-algebra. As in §1, let K be a field and $q \neq 0, 1$ be a fixed element of K. We endow $K(x)$ with the q-difference operator $\sigma_q(x) := qx$. Let $\mathcal{F}$ be a q-difference field extension of $K(x)$. We assume moreover that $\mathcal{F}$ is a q-difference differential field, i.e., a q-difference field endowed with a derivation ∂ that commutes with σ_q. For instance, endowed with the derivation $\partial := x\frac{d}{dx}$, the field $K(x)$ is a (σ_q, ∂)-field. Since we don't want to bother the reader with many similar definitions, we recall the basic conventions: Algebraic attributes always refer to the underlying ring whereas the operator suffix means that the algebraic attributes commutes with the operator. For instance, a σ_q-ideal is an ideal stable by σ_q, a (σ_q, ∂)-morphism is a ring morphism which commutes with σ_q and ∂.

2.1.2. Differential algebraic geometry. In this paper, we work with the formalism of affine differential group schemes, as can be found in [**Kov02**]. In this section, we fix a ∂-field k of characteristic zero, not necessarily ∂-closed. We define an affine differential k-scheme as follows:

Definition 2.2. An affine ∂-k-scheme (or ∂-scheme over k) is a (covariant) functor from the category of ∂-k-algebras to the category of sets which is representable.

The definition above means that a functor X from the category of ∂-k-algebras to the category of sets is a ∂-k-scheme if and only if there exists a ∂-k-algebra $k\{X\}$ and an isomorphism of functors $X \simeq \mathrm{Alg}_k^{\partial}(k\{X\}, -)$, where $\mathrm{Alg}_k^{\partial}$ stands for morphism of ∂-k-algebras. By the Yoneda lemma, the ∂-k-algebra $k\{X\}$ is uniquely determined up to unique ∂-k-isomorphisms. We call it the ring of ∂-coordinates of X. A ∂-k-scheme X is called ∂-algebraic (over k) if $k\{X\}$ is finitely ∂-generated over k. We say that a ∂-k-scheme X is reduced if $k\{X\}$ has no non-zero nilpotent elements.

Let X be a ∂-k-scheme. By a closed ∂-k-subscheme $Y \subset X$ we mean a subfunctor Y of X which is represented by $k\{X\}/\mathbb{I}(Y)$ for some ∂-ideal $\mathbb{I}(Y)$ of $k\{X\}$. The ideal $\mathbb{I}(Y)$ of $k\{X\}$ is uniquely determined by Y and vice versa. We call it the vanishing ideal of Y in X.

A morphism of ∂-k-schemes is a morphism of functors. If $\phi\colon X \to Y$ is a morphism of ∂-k-schemes, we denote the dual morphism of ∂-k-algebras with $\phi^*\colon k\{Y\} \to k\{X\}$.

If a functor (resp. ∂-algebraic functor) X factors through the category of group, we say that X is a differential (resp. differential algebraic) group k-scheme. By a ∂-subgroup H of G, we mean a group ∂-k-subscheme H of G. We call H normal if $H(S)$ is a normal subgroup of $G(S)$ for every ∂-k-algebra S. As in the classical setting, Yoneda lemma implies that, for a differential group k-scheme G, the algebra $k\{G\}$ is a ∂-k-Hopf algebra, i.e., a ∂-k-algebra equipped with the structure of a Hopf algebra over k such that the Hopf algebra structure maps are morphisms of ∂-rings. It also follows immediately that the category of differential group k-schemes is anti-equivalent to the category of ∂-k-Hopf algebras. Then, since Hopf algebras over fields of characteristic zero are reduced by [**Wat79b**, Cartier's Theorem in §11.4], we get that any differential group k-scheme is automatically reduced. Reduced differential schemes correspond to differential varieties in the sense of Kolchin (see for instance [**Kol73**]), for whom it suffices to focus on the solution set of a system of differential equations with value in a sufficiently big field, i.e., a ∂-closed field.

The differential schemes considered in this paper are all reduced. Thus, we only define the differential dimension of a reduced differential scheme. So let V be a reduced differential algebraic scheme defined over k. We can write $k\{V\} = k\{x_1, \dots, x_n\}_\partial/\mathfrak{q}$ for some positive integer n and some radical ∂-ideal $\mathfrak{q} \subset k\{x_1, \dots, x_n\}_\partial$. Since $\mathfrak{q}$ is radical, by [**Kap57**, Theorem 7.5] there exists finitely many prime ∂-ideals $\mathfrak{p}_i$ such that $\mathfrak{q} = \cap \mathfrak{p}_i$. Now, we can define the differential dimension of V over k, denoted by $\partial\text{-}\dim(V|k)$ as the maximum of the $\partial\text{-}\mathrm{trdeg}(L_i|k)$ where L_i denotes the fraction field of $k\{x_1, \dots, x_n\}_\partial/\mathfrak{p}_i$. In [**Kol73**, III.§6.Proposition 3], Kolchin proved that if $k \subset k'$ is an extension of ∂-field and if V is a reduced differential algebraic scheme defined k, then $\partial\text{-}\dim(V|k) = \partial\text{-}\dim(V_{k'}|k')$, where $V_{k'}$ is the base extension of V to k'.

Let V be an affine k-scheme, i.e., a (covariant) functor from the category of k-algebras to the category of sets which is representable by a k-algebra $k[V]$. We call $k[V]$ the ring of coordinates of V. In [**Gil02**], the author shows that the forgetful functor

$$\eta:\ \partial\text{-}k\text{-algebras} \ \to\ k\text{-algebras},$$

that associates to any ∂-k-algebra its underlying k-algebra, has a left adjoint denoted by D. This implies that the functor $\mathbf{V}$ from the category of ∂-k-algebras to the category of Sets, defined by the composition of V with the forgetful functor η is a differential k-scheme, whose ring of ∂-coordinates is precisely $D(k[V])$. We call $\mathbf{V}$, the differential scheme attached to V. The simple idea behind this construction is that polynomial equations are ∂-polynomials. More precisely if $V \subset \mathbf{A}_k^n$, the affine space of dimension n over k, and if $I(V) \subset k[x_1, \dots, x_n]$ is the vanishing ideal of V as subscheme of $\mathbf{A}_k^n$ then the vanishing ideal of $\mathbf{V}$ as ∂-subscheme of $\mathbf{A}_k^n$ is nothing else than the ∂-ideal generated by $I(V)$ in $k\{x_1, \ldots, x_n\}_\partial$. Finally, Kolchin irreducibility theorem states that if $k[V]$ is a finitely generated integral k-algebra, then $D(k[V])$ is a finitely ∂-generated integral ∂-k-algebra and the dimension of V as affine scheme coincides with the ∂-dimension of $\mathbf{V}$ ([**Gil02**, §2]).

Conversely, given a ∂-subscheme $\mathbf{V}$ of some $\mathbf{A}_k^n$, we can attach to $\mathbf{V}$ an affine subscheme of $\mathbf{A}_k^n$ as follows. Let $\mathbb{I}(\mathbf{V}) \subset k\{x_1, \ldots, x_n\}_\partial$ be the vanishing ideal of $\mathbf{V}$ in $\mathbf{A}_k^n$. Let $\mathbf{V}^Z$ be the affine subscheme of $\mathbf{A}_k^n$ defined by the ideal $\mathbb{I}(V) \cap k[x_1, \ldots, x_n]$. We say that $\mathbf{V}^Z$ is the Zariski closure of $\mathbf{V}$ inside $\mathbf{A}_k^n$. The idea is simply to throw away all the differential algebraic equations of $\mathbf{V}$ that contain a derivation and keep the polynomial ones.

2.2. Fiber and forgetful functor

The differential formalism was simultaneously developed in [**Ovc09**] and [**Kam10**] and later generalized by Kamensky (see [**Kam12**]) to include all type of auxiliary action on the Tannakian category. In this section, we apply this formalism to the category of q-difference modules over a q-difference differential field $(\mathcal{F}, \sigma_q, \partial)$. As detailed for instance in [**Sau04b**], the category $Diff(\mathcal{F}, \sigma_q)$ is a Tannakian category in the sense of [**DMOS82**]. Up to certain field extension k of $\mathcal{F}^{\sigma_q}$, we know by usual Tannakian equivalence that this category is equivalent to the category of comodules over the coordinate ring $k[G]$ of an affine group k-scheme. We show in the sequel how $Diff(\mathcal{F}, \sigma_q)$ can be endowed with an endofunctor, called prolongation functor, that will translate, by Morita equivalence, into a derivation on $k[G]$. This derivation will give to G the structure of a ∂-group scheme over k.

So, let $(\mathcal{F}, \sigma_q, \partial)$ be a (σ_q, ∂)-field and let $Diff(\mathcal{F}, \sigma_q)$ be the category of q-difference module over $\mathcal{F}$. This category is a tensor category and we denote by $\mathbf{1} = (\mathcal{F}, \sigma_q)$ the unit object for the tensor product. $Diff(\mathcal{F}, \sigma_q)$ is a rigid category, i.e., it posses internal Homs and each object is canonically isomorphic to its bidual.

We define below the prolongation functor in the general framework of projective modules over a ∂-k-algebra. In Chapter 7, we will give a more explicit description of this notion in the case of the category $Diff(\mathcal{F}, \sigma_q)$ of q-difference modules, using the associated q-difference system. Let (k, ∂) be a ∂-field and $\mathcal{S}$ be a ∂-k-algebra. We can endow the category $Proj_{\mathcal{S}}$ of finitely generated projective modules over $\mathcal{S}$ with an endofunctor F_∂, called prolongation functor, as follows. For M an object of $Proj_{\mathcal{S}}$, we define $F_\partial(M) := \mathcal{S}[\partial]_{\leq 1} \otimes_{\mathcal{S}} M$ where $\mathcal{S}[\partial]_{\leq 1}$ is the set of differential operators of order less than or equal to 1. In agreement with the Leibniz rule, the right $\mathcal{S}$-module structure of $\mathcal{S}[\partial]$ is given by $\partial.a = a.\partial + \partial(a)$. Then, the left S-module structure of $F_\partial(M)$ satisfies $\lambda\partial \otimes v = \partial \otimes \lambda v - \partial(\lambda) \otimes v$, for all $\lambda \in \mathcal{S}$ and $v \in M$. If $f \in Hom(M, N)$, we define $F_\partial(f) : F_\partial(M) \to F_\partial(N)$ as $F_\partial(f)(\partial^i \otimes m) = \partial^i \otimes f(m)$, for $i = 0, 1$, where we have used the convention that ∂^0 is the identity map. One can remark that, if ∂ is the trivial derivation, then $F_\partial(M)$ coincides with the direct sum $M \oplus M$. Now, if we see $Diff(\mathcal{F}, \sigma_q)$ as a subcategory of $Proj_{\mathcal{F}}$, we point out that, given an object $\mathcal{M}_{\mathcal{F}} = (M_{\mathcal{F}}, \Sigma_q)$, we are able to extend the action of Σ_q to $F_\partial(M_{\mathcal{F}})$ via $\Sigma_q(\partial^i(m)) := \partial^i(\Sigma_q(m))$, for $i = 0, 1$ and $m \in M_{\mathcal{F}}$. This shows that F_∂ restricts to an endofunctor of $Diff(\mathcal{F}, \sigma_q)$. Together with this additional structure, $(Diff(\mathcal{F}, \sigma_q), F_\partial)$ is a differential Tannakian category over $\mathcal{F}^{\sigma_q}$ as defined in [**GGO13**, §4.4], i.e., a $\mathcal{F}^{\sigma_q}$-linear, tensor, rigid category together with a prolongation functor, satisfying precise commutative diagrams.

Now, following [**GGO13**, Definition 4.9], we define the notion of differential fiber functors as follows:

Definition 2.3. Let $\mathcal{S}$ be a ∂-$\mathcal{F}^{\sigma_q}$-algebra. We say that a functor $\omega : Diff(\mathcal{F}, \sigma_q) \to Proj_{\mathcal{S}}$ is a differential fiber functor over $\mathcal{S}$ if it is

- exact,
- faithful,
- $\mathcal{F}^{\sigma_q}$-linear,
- tensor-compatible

and it commutes to F_∂, i.e., $F_\partial \circ \omega = \omega \circ F_\partial$. [1] We say moreover that ω is a neutral differential fiber functor if $\mathcal{S} = \mathcal{F}^{\sigma_q}$.

REMARK 2.4. • If ∂ is the trivial derivation, a differential fiber functor is a fiber functor in the sense of the classical Tannakian theory [**DMOS82**, p. 148].
• The forgetful functor $\eta_{\mathcal{F}} : Diff(\mathcal{F}, \sigma_q) \to Vect_{\mathcal{F}}$, which assigns to any q-difference module its underlying $\mathcal{F}$-vector space, is a differential fiber functor over $\mathcal{F}$.

Since one of our main purposes is to compare distinct fiber functors, we introduce the functor of differential tensor morphisms between two differential fiber functors.

DEFINITION 2.5. Let $\omega_1, \omega_2 : Diff(\mathcal{F}, \sigma_q) \to Proj_{\mathcal{S}}$ be two differential fiber functors. For any $\mathcal{S}$-algebra $\mathcal{R}$, we define $Hom^{\otimes}(\omega_1, \omega_2)(\mathcal{R})$ as the set of all sequences of the form

$$\{\lambda_{\mathcal{X}_{\mathcal{F}}} | \mathcal{X}_{\mathcal{F}} \text{ object of } Diff(\mathcal{F}, \sigma_q)\},$$

such that

- $\lambda_{\mathcal{X}_{\mathcal{F}}}$ is an $\mathcal{R}$-linear homomorphism from $\omega_1(\mathcal{X}_{\mathcal{F}}) \otimes \mathcal{R}$ to $\omega_2(\mathcal{X}_{\mathcal{F}}) \otimes \mathcal{R}$,
- $\lambda_{\mathbf{1}}$ is the identity on $\mathbf{1} \otimes \mathcal{R}$,
- for every $\alpha \in Hom(\mathcal{X}_{\mathcal{F}}, \mathcal{Y}_{\mathcal{F}})$, we have

$$\lambda_{\mathcal{Y}_{\mathcal{F}}} \circ (\alpha \otimes id_{\mathcal{R}}) = (\alpha \otimes id_{\mathcal{R}}) \circ \lambda_{\mathcal{X}_{\mathcal{F}}},$$

- $\lambda_{\mathcal{X}_{\mathcal{F}}} \otimes \lambda_{\mathcal{Y}_{\mathcal{F}}} = \lambda_{\mathcal{X}_{\mathcal{F}} \otimes \mathcal{Y}_{\mathcal{F}}}$.

For $\mathcal{R}$ a ∂-$\mathcal{S}$-algebra, we define $Hom^{\otimes,\partial}(\omega_1, \omega_2)(\mathcal{R})$ as the subset of $Hom^{\otimes}(\omega_1, \omega_2)(\mathcal{R})$ of all sequences such that:

- $F_\partial(\lambda_{\mathcal{X}_{\mathcal{F}}}) = \lambda_{F_\partial(\mathcal{X}_{\mathcal{F}})}$. [2]

The functor $Hom^{\otimes,\partial}(\omega_1, \omega_2)$ is a subfunctor of $Hom^{\otimes}(\omega_1, \omega_2)$, composed with the forgetful functor from ∂-$\mathcal{S}$-algebras to $\mathcal{S}$-algebras. If ∂ is the trivial derivation, these two functors coincides and $Hom^{\otimes}(\omega_1, \omega_2)$ is representable by an affine $\mathcal{F}$-scheme (see [**DMOS82**, p.117]). Since morphisms of tensor functors are isomorphisms, by [**DMOS82**, Proposition 1.13], the same holds for differential morphisms of differential tensor functors. Thus, we will now write $Isom^{\otimes,\partial}(\omega_1, \omega_2)$ (resp. $Isom^{\otimes}(\omega_1, \omega_2)$) instead of $Hom^{\otimes,\partial}(\omega_1, \omega_2)$ (resp. $Hom^{\otimes}(\omega_1, \omega_2)$) and, when $\omega_1 = \omega_2 = \omega$, we write $Aut^{\otimes,\partial}(\omega)$ (resp. $Aut^{\otimes}(\omega)$). In that special case, it occurs that the functor $Aut^{\otimes,\partial}(\omega)$ (resp. $Aut^{\otimes}(\omega)$) is a group functor, where the composition is given by the composition of morphisms.

Finally, [**GGO13**, Proposition 4.25] gives in our context:

PROPOSITION 2.6. *Let $\mathcal{S}$ be a ∂-$\mathcal{F}^{\sigma_q}$-algebra and let $\omega : Diff(\mathcal{F}, \sigma_q) \to Proj_{\mathcal{S}}$ be a differential fiber functor. Let A be the $\mathcal{S}$-Hopf algebra that represents the functor $Aut^{\otimes}(\omega)$ (see* [**Del90**, Proposition 6.6]*). Then, A has a canonical structure of ∂-$\mathcal{S}$-Hopf algebra and represents the functor $Aut^{\otimes,\partial}(\omega)$.*

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[1] This last equality has to be understood as a natural isomorphism.

[2] The first prolongation is to be understood inside $Proj_{\mathcal{R}}$ whereas the second one is the prolongation in $Diff(\mathcal{F}, \sigma_q)$.

This paper is concerned with the Galois group of a given q-difference module rather than with the differential affine group scheme attached to the whole category $Diff(\mathcal{F}, \sigma_q)$. Thus, from now on, we will restrict ourselves to the strictly full differential Tannakian subcategory generated inside $Diff(\mathcal{F}, \sigma_q)$ by a single q-difference module $\mathcal{M}_{\mathcal{F}}$. To do this, we need to introduce some notations. Given a q-difference module $\mathcal{N}_{\mathcal{F}}$, we consider the following categories:

- $\langle \mathcal{N}_{\mathcal{F}} \rangle^{\oplus}$ the strictly full subcategory of $Diff(\mathcal{F}, \sigma_q)$ formed by the subquotients of finite direct sums of copies of $\mathcal{N}_{\mathcal{F}}$, i.e., the abelian subcategory generated by $\mathcal{N}_{\mathcal{F}}$,
- $\langle \mathcal{N}_{\mathcal{F}} \rangle^{\otimes}$ the strictly full Tannakian category generated by $\mathcal{N}_{\mathcal{F}}$,
- $\langle \mathcal{N}_{\mathcal{F}} \rangle^{\otimes,\partial}$ the strictly full differential Tannakian category generated by $\mathcal{N}_{\mathcal{F}}$.

The differential Tannakian category generated by a single q-difference module $\mathcal{M}_{\mathcal{F}}$ admits a very simple description. We consider the constructions of linear (resp. linear differential) algebra of $\mathcal{M}_{\mathcal{F}}$, i.e., the list of q-difference modules

$$\bigoplus \mathcal{M}_{\mathcal{F}}^{\otimes i} \otimes \mathcal{M}_{\mathcal{F}}^{*\,\otimes j} \quad \left(\text{resp. } \bigoplus \mathcal{M}_{\mathcal{F}}^{\otimes i} \otimes \mathcal{M}_{\mathcal{F}}^{*\,\otimes j} \otimes F_{\partial}^{l}(\mathcal{M}_{\mathcal{F}}^{\otimes r} \otimes \mathcal{M}_{\mathcal{F}}^{*\,\otimes s})\right),$$

where i, j (resp. i, j, l, r) are non-negative integers and $\mathcal{M}_{\mathcal{F}}^{*}$ denotes the dual of $\mathcal{M}_{\mathcal{F}}$ (resp. $\mathcal{M}_{\mathcal{F}}^{*}$ denotes the dual of $\mathcal{M}_{\mathcal{F}}$ and F_{∂}^{l} the l-th iterate of the prolongation functor). If we order the sub-objects of the constructions of linear (resp. linear differential) algebra of $\mathcal{M}_{\mathcal{F}}$ by the relation "be a direct summand" then $\langle \mathcal{M}_{\mathcal{F}} \rangle^{\otimes}$ (resp. $\langle \mathcal{M}_{\mathcal{F}} \rangle^{\otimes,\partial}$) is the filtering union of the abelian categories $\langle \mathcal{N}_{\mathcal{F}} \rangle^{\oplus}$, where $\mathcal{N}_{\mathcal{F}}$ runs through the sub-objects of a construction of linear (resp. linear differential) algebra of $\mathcal{M}_{\mathcal{F}}$. These inductive description allows to see Tannakian as well as differential Tannakian equivalence as an inductive limit of Morita equivalences (see [**DMOS82**, Lemma 2.13]).

Now, we restrict ourselves to $\langle \mathcal{M}_{\mathcal{F}} \rangle^{\otimes,\partial}$. Let $\omega : \langle \mathcal{M}_{\mathcal{F}} \rangle^{\otimes,\partial} \rightarrow Vect_{\mathcal{F}^{\sigma_q}}$ be a neutral differential fiber functor. We denote, once again, by $\eta_{\mathcal{F}} : \langle \mathcal{M}_{\mathcal{F}} \rangle^{\otimes,\partial} \rightarrow Vect_{\mathcal{F}}$ the forgetful functor. As a direct application of Proposition 2.6, we find that $Aut^{\otimes,\partial}(\omega)$ (resp. $Aut^{\otimes,\partial}(\eta_{\mathcal{F}})$) is a differential algebraic group defined over $\mathcal{F}^{\sigma_q}$ (resp. $\mathcal{F}$). Moreover, $Aut^{\otimes,\partial}(\eta_{\mathcal{F}})$ coincides with the parameterized intrinsic Galois group $Gal^{\partial}(\mathcal{M}_{\mathcal{F}}, \eta_{\mathcal{F}})$, as defined in Definition 7.3. See Proposition 6.2 and Proposition 7.6 below.

2.3. Fiber functor and parametrized Picard-Vessiot extensions

There is a one to one correspondence between the neutral fiber functors on a category of differential (resp. difference) modules and and the Picard-Vessiot extensions, which are sort of "minimal rings of solutions". (See [**Del90**, §9] for differential equations and [**And01**, §3.4] for a larger class of functional equations.) In [**GGO13**, Theorem 5.5] following the ideas of Deligne, the authors proved among other things that this correspondence still holds for differential equations with differential parameters. We have no doubt that the correspondence established by Deligne holds for arbitrary differential Tannakian categories and especially for q-difference modules with a differential parameter. Anyway this result appear nowhere and we have decided to avoid this point, which is not necessary to our exposition.

In this section, we introduce some of the several known notions of parametrized Picard-Vessiot rings attached to a q-difference equation. We show in Proposition 2.9 how they yield to neutral differential fiber functors. Let $(\mathcal{F}, \sigma_q, \partial)$ be a q-difference differential field and

$$\sigma_q(Y) = AY, \tag{2.1}$$

with $A \in \mathrm{GL}_{\nu}(\mathcal{F})$, a q-difference system. In [**HS08**], the authors define the notion of minimal ∂-$\mathcal{F}$-algebra containing the solutions of (2.1) as follows:

DEFINITION 2.7. A (σ_q, ∂)-$\mathcal{F}$-algebra R is a *parametrized Picard-Vessiot ring* for equations (2.1) if

(1) R is a simple (σ_q, ∂)-$\mathcal{F}$-algebra, i.e., there are no non-trivial ideal stable under σ_q and ∂,
(2) there exists a $Z \in \mathrm{GL}_\nu(R)$ such that $\sigma_q(Z) = AZ$ and
(3) $R = k\{Z, \frac{1}{\det A}\}_\partial$, that is R is generated a ∂-ring by the entries of Z and the inverse of the determinant of Z.

Such a ring always exists. A basic construction is to consider the ring of differential polynomials $S = \mathcal{F}\{Y, \frac{1}{detY}\}_\partial$, where Y is a matrix of differential indeterminates over $\mathcal{F}$ of order ν , and to endow it with a q-difference operator compatible with the differential structure, i.e., such that

$$\sigma_q(Y) = AY,\ \sigma_q(\partial Y) = A\partial Y + \partial AY, \ldots.$$

Any quotient of the ring S by a maximal (σ_q, ∂)-ideal is a (σ_q, ∂)-Picard-Vessiot ring. By [**HS08**, Lemma 6.8], a parametrized Picard-Vessiot ring and more generally any simple (σ_q, ∂)-$\mathcal{F}$-algebra R, finitely differentially generated over $\mathcal{F}$, posses the following structure. There exist a positive integer t and $e_0, \ldots, e_{t-1}$ idempotents of R such that

(1) $R = R_0 \oplus \ldots R_{t-1},\ R_i = e_i R$,
(2) σ_q permutes transitively the set $\{R_0, \ldots, R_{t-1}\}$ and σ_q^t leaves each R_i invariant, and
(3) each R_i is a domain and a simple (σ_q^t, ∂)-$\mathcal{F}$-algebra.

Following [**Wib12a**], we call (σ_q, ∂)-$\mathcal{F}$-pseudoalgebras, the (σ_q, ∂)-$\mathcal{F}$-algebras having the above structure.

Now, we introduce the notion of weak parametrized Picard-Vessiot ring. It is the parametrized analogue of [**CHS08**, Definition 2.1].

DEFINITION 2.8. Let R be a (σ_q, ∂)-$\mathcal{F}$-pseudoalgebra. We say that R is a weak parametrized Picard-Vessiot ring for (2.1) if

(1) $R = \mathcal{F}\{Z, \frac{1}{det(Z)}\}_\partial$ where $Z \in \mathrm{GL}_\nu(R)$ and $\sigma_q(Z) = AZ$,
(2) $R^{\sigma_q} = \mathcal{F}^{\sigma_q}$.

In [**HS08**, Proposition 6.14], the authors shows that if one assume $\mathcal{F}^{\sigma_q}$ to be a ∂-closed field, then the σ_q-constants of any parametrized Picard-Vessiot ring coincide with $\mathcal{F}^{\sigma_q}$. In other words, they show that, starting with a differential closed field of σ_q-constants, a parametrized Picard-Vessiot ring is a weak parametrized Picard-Vessiot ring. Following an idea of M. Wibmer, one can show that, if $\mathcal{F}^{\sigma_q}$ is algebraically closed, there exists a weak Picard-Vessiot ring, which is moreover σ_q-simple, i.e., has no non-trivial σ_q-ideals (see [**Wib12b**] and [**DVH11**]). However, unicity is assured only if one extends the constants to the differential closure of $\mathcal{F}^{\sigma_q}$.

PROPOSITION 2.9. *Let $\mathcal{M}_\mathcal{F}$ be a q-difference module over $\mathcal{F}$ and let R be a weak parametrized Picard-Vessiot ring for a q-difference system $\sigma_q(Y) = AY$ attached to $\mathcal{M}_\mathcal{F}$. Then,*

$$\begin{array}{rccc} \omega_R: & \langle \mathcal{M}_\mathcal{F} \rangle^{\otimes,\partial} & \rightarrow & Vect_{\mathcal{F}^{\sigma_q}}, \\ & \mathcal{N}_\mathcal{F} & \mapsto & Ker(\Sigma_q - id, \mathcal{N}_\mathcal{F} \otimes_\mathcal{F} R) \end{array}$$

is a neutral differential fiber functor.

PROOF. Let i be a positive integer. Since $R = \mathcal{F}\{Z, \frac{1}{det(Z)}\}_\partial$, where $Z \in \mathrm{GL}_\nu(R)$ and $\sigma_q(Z) = AZ$, the prolongation of order i of $\mathcal{M}_\mathcal{F}$ is trivialized by R, i.e., possess a fundamental solution matrix with coefficients in R. Indeed, a q-difference system attached to $F_\partial(\mathcal{M}_\mathcal{F})$ is given by $\sigma_q(Y) = \begin{pmatrix} A & \partial(A) \\ 0 & A \end{pmatrix} Y$ and a fundamental matrix is $\begin{pmatrix} Z & \partial(Z) \\ 0 & Z \end{pmatrix}$. Then R trivializes any construction $\mathcal{N}_{\mathcal{F}}$ of linear differential algebra of $\mathcal{M}_{\mathcal{F}}$. This comes from the fact that a q-difference system (resp. fundamental solution matrix) attached to $\mathcal{X}_{\mathcal{F}}$ is obtained from A (resp. Z) by the same construction of linear differential algebra. Then, it is clear that any sub-object $\mathcal{N}_{\mathcal{F}}$ of $\mathcal{X}_{\mathcal{F}}$ possess a fundamental solution matrix with coefficients in R. Thereby, for any object $\mathcal{N}_{\mathcal{F}}$ in $\langle \mathcal{M}_{\mathcal{F}} \rangle^{\otimes, \partial}$, we find a functorial isomorphism between $\mathcal{N}_{\mathcal{F}} \otimes_{\mathcal{F}} R$ and $\omega_R(\mathcal{N}_{\mathcal{F}}) \otimes_{\mathcal{F}^{\sigma_q}} R$. We deduce from this fact that ω_R is a faithful, exact, $\mathcal{F}^{\sigma_q}$-linear tensor functor. It is neutral because $R^{\sigma_q} = \mathcal{F}^{\sigma_q}$ The fact that ω_R intertwines with F_∂ corresponds exactly to the fact that a fundamental solution matrix attached to $F_\partial(\mathcal{M}_{\mathcal{F}})$ is given by the prolongation of a fundamental solution matrix attached to $\mathcal{M}_{\mathcal{F}}$, as explained above. □

To conclude this chapter, we choose to introduce the differential algebraic group attached to a weak parametrized Picard-Vessiot ring R, i.e., the group of functorial (σ_q, ∂)-$\mathcal{F}$-automorphism of R. Then, we show that this latter group scheme corresponds to the group of differential tensor automorphism of the neutral differential fiber functor ω_R, corresponding to R by Proposition 2.9. This incarnates the differential Tannakian group of a q-difference module, as the group of automorphisms of the solutions preserving the differential algebraic relations between the solutions.

Definition 2.10. Let $\mathcal{M}_{\mathcal{F}}$ be a q-difference module over $\mathcal{F}$. Let R be a weak parametrized Picard-Vessiot ring for a q-difference system attached to $\mathcal{M}_{\mathcal{F}}$. We define the functor of (σ_q, ∂)-automorphisms of R as follows

$$\begin{array}{cccc} G_R^\partial : & \{\partial\text{-}\mathcal{F}^{\sigma_q}\text{-algebras}\} & \rightarrow & \{\text{Groups}\} \\ & \mathcal{S} & \mapsto & Aut_{\mathcal{F}\otimes\mathcal{S}}^{(\sigma_q,\partial)}(R \otimes \mathcal{S}), \end{array}$$

where $Aut_{\mathcal{F}\otimes\mathcal{S}}^{(\sigma_q,\partial)}(R\otimes\mathcal{S})$ stands for the group of (σ_q, ∂)-$\mathcal{F}\otimes\mathcal{S}$-automorphism of $R\otimes\mathcal{S}$.

Remark 2.11. If ∂ is the trivial derivation, this group corresponds to the group G_R as defined in [**CHS08**, Proposition 2.2].

Proposition 2.12. *Let $\mathcal{M}_{\mathcal{F}}$ be a q-difference module over $\mathcal{F}$. Let R be a weak parametrized Picard-Vessiot ring for a q-difference system attached to $\mathcal{M}_{\mathcal{F}}$. Then, G_R^∂ is representable by a linear differential algebraic group defined over $\mathcal{F}^{\sigma_q}$.*

Proof. We omit this proof which is a straightforward differential analogous of [**CHS08**, Proposition 2.2]. □

Proposition 2.13. *Let $\mathcal{M}_{\mathcal{F}}$ be a q-difference module over $\mathcal{F}$. Let R be a weak parametrized Picard-Vessiot ring for a q-difference system attached to $\mathcal{M}_{\mathcal{F}}$. Then, the linear differential algebraic groups $Aut^{\otimes,\partial}(\omega_R)$ and G_R^∂ are isomorphic over $\mathcal{F}^{\sigma_q}$.*

Remark 2.14. The statement above is the parameterized anologue of [**vdPS97**, Theorem 1.32.2)].

Proof. Let $\mathcal{S}$ be a ∂-$\mathcal{F}^{\sigma_q}$-algebra. An element $\gamma_{\mathcal{S}} \in Aut^{\otimes,\partial}(\omega_R)(\mathcal{S})$ acts by $\mathcal{S}$-linearity on the linear forms on the differential symmetric algebra of $\omega(\mathcal{M}_{\mathcal{F}})^\nu \otimes \mathcal{S}$. Thus, $\gamma_{\mathcal{S}}$ defines an $\mathcal{S}$-automorphism on the differential polynomial algebra $\mathcal{F}\{X, \frac{1}{det(X)}\}_\partial \otimes \mathcal{S} := \mathcal{F}\{(X_{i,j})_{1\leq i,j\leq \nu}, \frac{1}{det(X)}\}_\partial \otimes \mathcal{S}$. If we let σ_q acts on $\mathcal{F}\{X, \frac{1}{det(X)}\}_\partial$ with $\sigma_q(X) = AX$ then $\gamma_{\mathcal{S}}$ commutes with σ_q and ∂. This a consequence of the fact that $\gamma_{\mathcal{S}}$ is a differential tensor automorphism of ω. Now, $\gamma_{\mathcal{S}}$ stabilizes every $\omega(\mathcal{N}_{\mathcal{F}}) \otimes \mathcal{S}$ for any q-difference module $\mathcal{N}_{\mathcal{F}}$ contained in a differential algebraic construction of $\mathcal{M}_{\mathcal{F}}$. It follows that $\gamma_{\mathcal{S}}$ stabilizes in $\mathcal{F}\{X, \frac{1}{det(X)}\}_\partial$ the ideal of differential algebraic relations $\mathfrak{I}$ satisfied by a fundamental solution matrix Z over $\mathcal{F}$.

Indeed, a differential algebraic relation for Z can be seen as a $\mathcal{F}$-linear form that annihilates on a construction $\mathcal{N}_{\mathcal{F}}$ of linear differential algebra of $\mathcal{M}_{\mathcal{F}}$. Since the set of $\mathcal{F}$-linear forms that annihilate on $\mathcal{N}_{\mathcal{F}}$ is a q-difference submodule of $\mathcal{N}_{\mathcal{F}}^*$, it must be stabilized by γ. This proves that $\gamma_{\mathcal{S}}$ acts $\mathcal{S}$-linearly on $\mathcal{F}\{X, \frac{1}{det(X)}\}_\partial / I \otimes \mathcal{S} = R \otimes \mathcal{S}$. This gives an embedding of $Aut^{\otimes,\partial}(\omega_R)$ in G_R^∂.

Conversely, any $\tau_{\mathcal{S}} \in G_R^\partial(\mathcal{S})$ acts on $\mathcal{N}_{\mathcal{F}} \otimes R \otimes \mathcal{S}$ via $id \otimes \tau$, for any $\mathcal{N}_{\mathcal{F}}$ contained in a construction of linear differential algebra of $\mathcal{M}_{\mathcal{F}}$. Since this action commutes with Σ_q, we find an action of $\tau_{\mathcal{S}}$ on $\omega(\mathcal{N}_{\mathcal{F}}) \otimes \mathcal{S}$. This gives us the inverse embedding of G_R^∂ into $Aut^{\otimes,\partial}(\omega_R)$. $\square$

Part 2

Triviality of q-difference equations with rational coefficients

CHAPTER 3

Rationality of solutions, when q is an algebraic number

Let K be a field and $q \neq 0, 1$ be an element of K. We are concerned with the problem of finding a necessary and sufficient condition for a q-difference module $\mathcal{M}_{K(x)} = (M_{K(x)}, \Sigma_q)$ over $K(x)$ to be trivial (see Definition 1.10). This is equivalent to the problem of finding a necessary and sufficient condition for a linear q-difference system with coefficients in $K(x)$ to have a fundamental solution matrix with entries in $K(x)$.

Notice that we are not making any assumption on the characteristic of K. We have to consider the following cases:

(1) q is a root of unity;
(2) q is algebraic over the prime field, but is not a root of unity;
(3) q is transcendental over the prime field.

These six cases (three cases for the characteristic zero, and three cases for the positive one) actually boil down to three. In fact, we will first consider the (trivial) situation in which q is a root of unity: If K has positive characteristic this includes both (1) and (2) above. Then we will consider the case in which K has characteristic zero and q is algebraic over $\mathbb{Q}$. Finally, in the next chapter, we will consider the case in which q is transcendental over the prime field, $\mathbb{Q}$ or $\mathbb{F}_p$, regardless of the characteristic.

It is not difficult to prove that:

Proposition 3.1 ([**Hen96**] or [**DV02**, Proposition 2.1.2]). *If q is a primitive root of unity of order κ, a q-difference module $\mathcal{M}_{K(x)}$ over $K(x)$ is trivial if and only if Σ_q^κ is the identity.*

The proposition above completes the study of the triviality of q-difference modules when q is a root of unity, at least as far as the problem we are considering here is regarded. We refer to [**Har10**] for a more sophisticated approach.

3.1. The case of q algebraic, not a root of unity

If q is algebraic, but not a root of unity, we are necessarily in characteristic zero. The example below gives the guidelines for the whole chapter.

Example 3.2. Let $K = \mathbb{Q}(a)$ be a purely transcendental extension of degree 1 and let $q \in \mathbb{Q} \smallsetminus \{0, 1, -1\}$. We consider a q-difference module $\mathcal{M}_{K(x)} = (M_{K(x)}, \Sigma_q)$ over $K(x)$. Let us choose a basis $\underline{e}$ of $M_{K(x)}$ and let $Y(qx) = B(a, x)Y(x)$ be the associated q-difference system. One can construct by hand a $\mathbb{Z}$-algebra stable by σ_q, of the form:

$$\mathcal{A} = \mathbb{Z}\left[a, x, \frac{1}{P(x)}, \frac{1}{P(qx)}, \dots\right],$$

for a convenient choice of $P(x) \in \mathbb{Z}[a, x]$, such that $q \in \mathcal{A}$ and $B(a, x)$ and $B(a, x)^{-1}$ are both matrices with coefficients in $\mathcal{A}$. For almost all primes p in $\mathbb{Z}$, one can reduce both q and $\mathcal{A}$ modulo p, and hence the coefficients of $B(a, x)$. In particular, for all such p's there exist a minimal positive integer κ_p and a positive integer ℓ_p, such that $q^{\kappa_p} \equiv 1$ modulo p and $q^{\kappa_p} - 1 = p^{\ell_p}\frac{r}{s}$, with r, s prime to p. The main result of this chapter (see Theorem 3.6 below) is that the system $Y(qx) = B(a,x)Y(x)$ has a fundamental solution with coefficients in $K(x)$ if and only if for almost all p we have

$$B(a, q^{\kappa_p - 1}x)B(a, q^{\kappa_p - 2}x) \cdots B(a,x) \equiv 1 \text{ modulo } p^{\ell_p}, \text{ i.e., in } \mathcal{A}/p^{\ell_p}\mathcal{A}. \tag{3.1}$$

This last condition is equivalent to the fact that the reduction modulo p^{ℓ_p} of the operator $\Sigma_q^{\kappa_p}$ is the identity, and is verified, in particular, if the reduction of the system $Y(qx) = B(a,x)Y(x)$ modulo p^{ℓ_p} has a fundamental solution matrix with coefficients in $\mathcal{A}/p^{\ell_p}\mathcal{A}$. We will proceed as follows: We will first prove that the system $Y(qx) = B(a,x)Y(x)$ has a fundamental solution with coefficients in $K(x)$ if and only if, for all α in a dense subset of the algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$, the system $Y(qx) = B(\alpha,x)Y(x)$ has a fundamental solution with coefficients in $\overline{\mathbb{Q}}(x)$. As a consequence of [**DV02**, Theorem 7.1.1], we will show that this last condition, holding for all α in a dense subset of $\overline{\mathbb{Q}}$, is equivalent to (3.1).

First of all we need to introduce some notation, that generalizes the one in the previous example to the case of a number field. Notice that we can always suppose, and we will, that K is finitely generated over $\mathbb{Q}$ (see Proposition 1.2). Let Q be the algebraic closure of $\mathbb{Q}$ inside K. Then the field K has the form $Q(\underline{a}, b)$, where $\underline{a} = (a_1, \dots, a_r)$ is a transcendence basis of K/Q and b is a primitive element of the algebraic extension $K/Q(\underline{a})$. We call $\mathcal{O}_Q$ the ring of integers of Q, v a finite place of Q and π_v a v-adic uniformizer in $\mathcal{O}_Q$.

We fix an element $q \in K$ which is algebraic over $\mathbb{Q}$ and not a root of unity, i.e., an element $q \in Q$ which is not a root of unity. For almost all v,

- the order κ_v of q modulo v, as a root of unity,
- the positive integer power ϕ_v of π_v, such that $\phi_v^{-1}(1 - q^{\kappa_v})$ is a unit of $\mathcal{O}_Q$,

are well defined.

We consider a q-difference module $\mathcal{M}_{K(x)} = (M_{K(x)}, \Sigma_q)$ over $K(x)$, of finite rank ν. Choosing conveniently the set of generators $\underline{a}, b$ of K/Q, we can always find a q-difference algebra $\mathcal{A}$ of the form:

$$\mathcal{A} = \mathcal{O}_Q\left[\underline{a}, b, x, \frac{1}{P(x)}, \frac{1}{P(qx)}, \dots\right], \tag{3.2}$$

for some $P(x) \in \mathcal{O}_Q[\underline{a}, b, x]$, and a Σ_q-stable $\mathcal{A}$-lattice M of $\mathcal{M}_{K(x)}$ such that the restriction of Σ_q to M is invertible. According to the definition in §1.1.4, the pair $\mathcal{M} = (M, \Sigma_q)$ is a q-difference module over the ring $\mathcal{A}$.

NOTATION 3.3. For a given q-difference module $\mathcal{M}_{K(x)} = (M_{K(x)}, \Sigma_q)$ over $K(x)$, the pair $\mathcal{M} = (M, \Sigma_q)$ will always denote a q-difference module over a ring $\mathcal{A}$ as above, such that $\mathcal{M} \otimes_{\mathcal{A}} K(x) := (M \otimes_{\mathcal{A}} K(x), \Sigma_q \otimes_{\mathcal{A}} \sigma_q) \cong \mathcal{M}_{K(x)}$. The notation may appear ambiguous, but it is actually convenient and there will be no confusion.

DEFINITION 3.4. We say that a q-difference module $\mathcal{M} = (M, \Sigma_q)$ over a q-difference $\mathcal{O}_Q$-algebra $\mathcal{A}$, as above, has zero v-curvature modulo ϕ_v if the linear operator

$$\Sigma_q^{\kappa_v} : M \otimes_{\mathcal{A}} \mathcal{A}/(\phi_v) \longrightarrow M \otimes_{\mathcal{A}} \mathcal{A}/(\phi_v)$$

is the identity. By abuse of language we will say that the q-difference module $\mathcal{M}_{K(x)} = \mathcal{M} \otimes_{\mathcal{A}} K(x)$ has zero v-curvature modulo ϕ_v, if $\mathcal{M}$ does.

REMARK 3.5. First of all, the definition is justified by the fact that $\Sigma_q^{\kappa_v}$ induces the identity modulo ϕ_v if and only if $(\Delta_q)^{\kappa_v}$, where $\Delta_q = \frac{\Sigma_q - 1}{(q-1)x}$, is zero modulo ϕ_v. Therefore the terminology is inspired by the classical termonology for differential equations, [**Kat70**].

Secondly, we point out that the quotient $\mathcal{O}_Q/(\phi_v)$ is not an integral domain in general. Nonetheless the following implication is always true. If $M \otimes_{\mathcal{A}} \mathcal{A}/(\phi_v)$, equipped with the operator induced by Σ_q, is trivial as a q-difference module over $\mathcal{A}/(\phi_v)$, then $\Sigma_q^{\kappa_v}$ induces the identity modulo ϕ_v. The converse is not true in such generality (see [**DV02**, Proposition 2.1.2]).

Notice that the reduction modulo ϕ_v of $\Sigma_q^{\kappa_v}$ is well-defined, for almost all finite places v of Q. Moreover, given two q-difference module $\mathcal{M}$ over $\mathcal{A}$ and $\mathcal{M}'$ over $\mathcal{A}'$, such that $\mathcal{M} \otimes_{\mathcal{A}} K(x) \cong \mathcal{M}' \otimes_{\mathcal{A}} K(x)$, the reduction modulo ϕ_v of the first one has zero v-curvature if and only if also the other does, provided that ϕ_v is not invertible in both $\mathcal{A}$ and $\mathcal{A}'$.

Our first result is the following:

THEOREM 3.6. *A q-difference module $\mathcal{M}$ over $\mathcal{A}$ has zero v-curvature modulo ϕ_v, for almost all finite places v of Q, if and only if $\mathcal{M}_{K(x)}$ is trivial.*

REMARK 3.7. The theorem above is proved in [**DV02**] under the assumption that K is a number field, i.e., that $Q = K$. Here K is only a finitely generated extension of $\mathbb{Q}$. Notice the proof below relies crucially on [**DV02**], but is not a generalization of the arguments in [**DV02**].

If the q-difference module $\mathcal{M}_{K(x)}$ over $K(x)$ is trivial, it is not difficult to show that $\mathcal{M}$ has zero v-curvature modulo ϕ_v, for almost all finite places v of Q, for any choice of $\mathcal{A}$ and $\mathcal{M}$, such that $\mathcal{M} \otimes_{\mathcal{A}} K(x) \cong \mathcal{M}_{K(x)}$. So we only have to prove the inverse implication.

We are actually going to prove a stronger result:

THEOREM 3.8. *A q-difference module $\mathcal{M}$ over $\mathcal{A}$ has zero v-curvature modulo ϕ_v, for all places v in a set S of finite places of Q of Dirichlet density* 1 *if and only if $\mathcal{M}_{K(x)}$ is trivial.*

We recall that a subset S of the set of finite places $\mathcal{C}$ of Q has Dirichlet density 1 if

$$\limsup_{s \to 1^+} \frac{\sum_{v \in S, v|p} p^{-s f_v}}{\sum_{v \in \mathcal{C}, v|p} p^{-s f_v}} = 1, \tag{3.3}$$

where f_v is the degree of the residue field of v over $\mathbb{F}_p$.

3.2. Global nilpotence.

We start proving a result of regularity (see §1.5 for the definition), inspired by [**Kat70**].

DEFINITION 3.9. We say that a q-difference module $\mathcal{M} = (M, \Sigma_q)$ over a q-difference $\mathcal{O}_Q$-algebra $\mathcal{A}$, as above, has nilpotent v-curvature modulo π_v, or simply that it has nilpotent reduction modulo π_v, if the linear operator

$$\Sigma_q^{\kappa_v} : M \otimes_{\mathcal{A}} \mathcal{A}/(\pi_v) \longrightarrow M \otimes_{\mathcal{A}} \mathcal{A}/(\pi_v)$$

is unipotent (or equivalently, if the linear operator induced by $\Delta_q^{\kappa_v}$ is nilpotent. See [**DV02**, §2]).

PROPOSITION 3.10. *Let $\mathcal{M} = (M, \Sigma_q)$ be a q-difference module over a q-difference $\mathcal{O}_Q$-algebra $\mathcal{A}$ of the form* (3.2).

(1) *If $\mathcal{M}$ has nilpotent v-curvature modulo π_v, for infinitely many finite places v of Q, then the q-difference module $\mathcal{M}_{K(x)}$ is regular singular.*
(2) *If there exists a set S of finite places v of Q of Dirichlet density 1 such that $\mathcal{M}$ has nilpotent v-curvature modulo π_v, for all $v \in S$, then $\mathcal{M}_{K((x))}$ is trivial.*

The proof of Proposition 3.10 is almost the same as [**DV02**, Theorem 6.2.2 and Proposition 6.2.3]. The last sentence of the proof of 1) in *loc. cit.* needs to be rectified, so that we prefer to repeat the proof here. We recall the following key-proposition:

PROPOSITION 3.11 ([**DV02**, Proposition 6.1.1]). *Let S be a set of finite places of Q of Dirichlet density equal to 1. If a and b are two non-zero elements of Q, not roots of unity, such that*
(1) for all $v \in S$, the reduction of a et b modulo π_v is well defined and non-zero;
(2) for all $v \in S$, the reduction modulo π_v of b belongs to the cyclic group generated by the reduction modulo π_v of a.
Then $b \in a^{\mathbb{Z}}$.

PROOF OF PROPOSITION 3.10. To prove assertion (1), it is enough to prove that 0 is a regular singular point for $\mathcal{M}$, the proof at ∞ being completely analogous.

In the notation of Corollary 1.19, we consider the extension $L(t)$ of $K(x)$, the $\widetilde{q}$-difference module $\mathcal{M}_{L(t)}$ obtain by scalar extension and the basis $\underline{f}$ such that $\Sigma_{\widetilde{q}}\underline{f} = \underline{f}B(t)$, with $B(t)$ as in (1.5). Let $\widetilde{Q}$ be the algebraic closure of $\mathbb{Q}$ in L and $\mathcal{B} \subset L(t)$ be a $\widetilde{q}$-difference algebra over the ring of integers $\mathcal{O}_{\widetilde{Q}}$ of $\widetilde{Q}$, of the same form as (3.2), containing the entries of $B(t)$ and the inverse of its determinant. Let w be a finite place of $\widetilde{Q}$ and $\pi_w \in \widetilde{Q}$ be the uniformizer of w. Then there exists a $\widetilde{q}$-difference module $\mathcal{N}$ over $\mathcal{B}$ such that $\mathcal{N} \otimes_{\mathcal{B}} L(t) \cong \mathcal{M}_{L(t)}$, having the following properties:
1. $\mathcal{N}$ has nilpotent w-curvature modulo π_w, for infinitely many finite places w of $\widetilde{Q}$;
2. there exists a basis $\underline{f}$ of $\mathcal{N}$ over $\mathcal{B}$ such that $\Sigma_{\widetilde{q}}\underline{f} = \underline{f}B(t)$ and $B(t)$ verifies (1.5).

Iterating the operator $\Sigma_{\widetilde{q}}$ we obtain:

$$\Sigma_{\widetilde{q}}^m(\underline{f}) = \underline{f}B(t)B(\widetilde{q}t)\cdots B(\widetilde{q}^{m-1}t) = \underline{f}\left(\frac{B_\ell^m}{\widetilde{q}^{\frac{\ell m(\ell m-1)}{2}}t^{m\ell}} + h.o.t.\right).$$

We know that, for infinitely many finite places w of $\widetilde{Q}$, the matrix $B(t)$ verifies

$$\left(B(t)B(\widetilde{q}t)\cdots B(\widetilde{q}^{\kappa_w-1}t) - 1\right)^{n(w)} \equiv 0 \bmod \pi_w, \tag{3.4}$$

where κ_w is the order $\widetilde{q}$ modulo π_w and $n(w)$ is a convenient positive integer. Suppose that $\ell \neq 0$. Then $B_\ell^{\kappa_w n(w)} \equiv 0$ modulo π_w, , for infinitely many w, and hence B_ℓ is a nilpotent matrix, in contradiction with Corollary 1.19. So necessarily $\ell = 0$.

Finally we have $\Sigma_{\widetilde{q}}(\underline{f}) = \underline{f}\,(B_0 + h.o.t)$. It follows from (3.4) that B_0 is actually invertible, which implies that $\mathcal{M}_{L(t)}$ is regular singular at 0. Proposition 1.17 allows to end the proof of (1).

Let us prove the second part of Proposition 3.10. We have already proved that 0 is a regular singularity for $\mathcal{M}$. This means that there exists a basis $\underline{e}$ of $\mathcal{M}_{K(x)}$ over $K(x)$ such that $\Sigma_q\underline{e} = \underline{e}A(x)$, with $A(x) \in \mathrm{GL}_\nu(K[[x]]) \cap \mathrm{GL}_\nu(K(x))$.

The Frobenius algorithm (*cf.* [**Sau00**, §1.1.1]) implies that there exists a shearing transformation $S \in \mathrm{GL}_\nu(K[x, 1/x])$, such that $S(qx)A(x)S(x)^{-1} \in \mathrm{GL}_\nu(K[[x]]) \cap \mathrm{GL}_\nu(K(x))$ and that the constant term A_0 of $S(x)^{-1}A(x)S(qx)$ has the following properties: if α and β are eigenvalues of A_0 and $\alpha\beta^{-1} \in q^{\mathbb{Z}}$, then $\alpha = \beta$. So choosing the basis $\underline{e}S(x)$ instead of $\underline{e}$, we can assume that $A_0 = A(0)$ has this last property.

Always following the Frobenius algorithm (*cf.* [**Sau00**, §1.1.3]), one constructs recursively the entries of a matrix $F(x) \in \mathrm{GL}_\nu(K[[x]])$, with $F(0) = 1$, such that we have $F(x)^{-1}A(x)F(qx) = A_0$. This means that there exists a basis $\underline{f}$ of $\mathcal{M}_{K((x))}$ such that $\Sigma_q \underline{f} = \underline{f} A_0$.

The matrix A_0 can be written as the product of a semi-simple matrix and a unipotent matrix. Since $\mathcal{M}$ has nilpotent reduction modulo π_v, we deduce that the reduction of $A_0^{\kappa_v}$ modulo π_v is the identity matrix, for any $v \in S$. First of all, this implies that A_0 is diagonalisable. Let $\widetilde{K}$ be a finite extension of K in which we can find all the eigenvalues of A_0. Then any eigenvalue $\alpha \in \widetilde{K}$ of A_0 has the property that $\alpha^{\kappa_v} = 1$ modulo π_w, for all w finite place of the algebraic closure of Q in $\widetilde{K}$ such that $w|v$ and $v \in S$. In other words, the reduction modulo w of an eigenvalue α of A_0 belongs to the multiplicative cyclic group generated by the reduction of q modulo the uniformizer π_w of w. Proposition 3.11 implies that $\alpha \in q^{\mathbb{Z}}$. We conclude appling Proposition 1.16. □

3.3. Proof of Theorem 3.6 and 3.8

The proof is divided into steps. We remaind that, if K is finite over $\mathbb{Q}$, the statement is proved in [**DV02**].

Step 0. Reduction to a purely transcendental extension K/Q. Let $\underline{a}$ be a transcendence basis of K/Q and b is a primitive element of $K/Q(\underline{a})$, so that $K = Q(\underline{a}, b)$. By restriction of scalars, the module $\mathcal{M}_{K(x)}$ is also a q-difference module of finite rank over $Q(\underline{a})(x)$. Since the field $K(x)$ is a trivial q-difference module over $Q(\underline{a})(x)$, we have:

- the module $\mathcal{M}_{K(x)}$ is trivial over $K(x)$ if and only if it is trivial over $Q(\underline{a})(x)$ (see Corollary 1.12);
- under the present assumptions, there exist an algebra $\mathcal{A}'$ of the form

$$\mathcal{A}' = \mathcal{O}_Q\left[\underline{a}, x, \frac{1}{R(x)}, \frac{1}{R(qx)}, \ldots.\right], \text{ with } R(x) \in \mathcal{O}_Q[\underline{a}, x], \tag{3.5}$$

  and a $\mathcal{A}'$-lattice $\mathcal{M}_{\mathcal{A}'}$ of q-difference module $\mathcal{M}_{K(x)}$ over $Q(\underline{a})(x)$, such that $\mathcal{M}_{\mathcal{A}'} \otimes_{\mathcal{A}'} Q(\underline{a}, x) = \mathcal{M}_{K(x)}$, as a q-difference module over $Q(\underline{a}, x)$, and $\Sigma_q^{\kappa_v}$ induces the identity on $\mathcal{M}_{\mathcal{A}'} \otimes_{\mathcal{A}'} \mathcal{A}'/(\phi_v)$, for all places $v \in S$.

For this reason, we can actually assume that K is a purely transcendental extension of Q of degree $d > 0$ and that $\mathcal{A} = \mathcal{A}'$. We fix an immersion of $Q \hookrightarrow \overline{\mathbb{Q}}$, so that we will think to the transcendental basis $\underline{a}$ as a set of parameter generically varying in $\overline{\mathbb{Q}}^d$. □

Step 0bis. Initial data. Let $K = Q(\underline{a})$ and q be a non-zero element of Q, which is not a root of unity. We are given a q-difference module $\mathcal{M}$ over a convenient algebra $\mathcal{A}$ as above, such that $K(x)$ is the field of fraction of $\mathcal{A}$ and such that $\Sigma_q^{\kappa_v}$ induces the identity on $M \otimes_{\mathcal{A}} \mathcal{A}/(\phi_v)$, for all finite places $v \in S$. We fix a basis $\underline{e}$ of $\mathcal{M}$, such that $\Sigma_q \underline{e} = \underline{e}A^{-1}(x)$, with $A(x) \in \mathrm{GL}_\nu(\mathcal{A})$. We will rather work with the associated q-difference system:

$$Y(qx) = A(x)Y(x). \tag{3.6}$$

It follows from Proposition 3.10 that $\mathcal{M}_{K(x)}$ is regular singular, with no logarithmic singularities, and that its exponents are in $q^{\mathbb{Z}}$ (see also Proposition 1.16). Enlarging a little bit the algebra $\mathcal{A}$ (more precisely replacing the polynomial R by a multiple of R), we can suppose that both 0 and ∞ are not poles of $A(x)$ and that $A(0), A(\infty)$ are diagonal matrices with eigenvalues in $q^{\mathbb{Z}}$ (see [**Sau00**, Theoreme §2.1]). □

STEP 1. CONSTRUCTION OF A FUNDAMENTAL SOLUTION AT 0. We construct a fundamental matrix of solutions, applying the Frobenius algorithm to this particular situation. There exists a shearing transformation $S_0(x) \in \mathrm{GL}_\nu(K[x, x^{-1}])$ such that

$$S_0^{-1}(qx)A(x)S_0(x) = A_0(x)$$

and $A_0(0)$ is the identity matrix. In particular, the matrix $S_0(x)$ can be written as a product of invertible constant matrices and diagonal matrix with integer powers of x on the diagonal. Once again, up to a finitely generated extension of the algebra $\mathcal{A}$, obtained inverting a convenient polynomial, we can suppose that $S_0(x) \in \mathrm{GL}_\nu(\mathcal{A})$.

Notice that, since q is not a root of unity, there always exists a norm, non-necessarily archimedean, on Q such that $|q| > 1$. We can always extend such a norm to K, giving an arbitrary value to the elements of a basis of transcendence (see [**Bou64**, §2.4]). As in Proposition 1.8, the system

$$Z(qx) = A_0(x)Z(x) \tag{3.7}$$

has a unique convergent solution $Z_0(x)$, such that $Z_0(0)$ is the identity and $Z_0(x)$ is a germ of a meromorphic function with infinite radius of meromorphy. So we have the following meromorphic solution of $Y(qx) = A(x)Y(x)$:

$$Y_0(x) = \Big(A_0(q^{-1}x)A_0(q^{-2}x)A_0(q^{-3}x)\dots\Big)\, S_0(x).$$

We remind that this infinite product represents a meromophic fundamental solution of $Y(qx) = A(x)Y(x)$ for any norm over K such that $|q| > 1$. □

STEP 2. CONSTRUCTION OF A FUNDAMENTAL SOLUTION AT ∞. In exactly the same way, we can construct a solution at ∞ of the form $Y_\infty(x) = Z_\infty(x)S_\infty(x)$, where the matrix S_∞ belongs to $GL_\nu(K[x, x^{-1}]) \cap \mathrm{GL}_\nu(\mathcal{A})$ and has the same form as $S_0(x)$, and $Z_\infty(x)$ is analytic in a neighborhood of ∞, with $Z_\infty(\infty) = 1$: $Y_\infty(x) = \Big(A_\infty(x)A_\infty(qx)A_\infty(q^2x)\dots\Big)\, S_\infty(x)$. □

STEP 3. THE BIRKHOFF MATRIX. To summarize we have constructed two fundamental solution matrices, $Y_0(x)$ at zero and $Y_\infty(x)$ at ∞, which are meromorphic over $\mathbb{A}^1_K \smallsetminus \{0\}$, for any norm on K such that $|q| > 1$, and such that their set of non-zero poles and zeros is contained in the q-orbits of the set of poles at zeros of $A(x)$ and $A(x)^{-1}$. The Birkhoff matrix

$$B(x) = Y_0^{-1}(x)Y_\infty(x) = S_0(x)^{-1}Z_0(x)^{-1}Z_\infty(x)S_\infty(x)$$

is a meromorphic matrix on $\mathbb{A}^1_K \smallsetminus \{0\}$ with elliptic entries, i.e., $B(qx) = B(x)$. All the zeros and poles of $B(x)$, other than 0 and ∞, are contained in the q-orbits of zeros and poles of the matrices $A(x)$ and $A(x)^{-1}$ (see [**Sau00**, §2.3.1]). □

STEP 4.RATIONALITY OF THE BIRKHOFF MATRIX. Let us choose $\underline{\alpha} = (\alpha_1, \dots, \alpha_r)$, with α_i in the algebraic closure $\overline{\mathbb{Q}}$ of Q, so that we can specialize $\underline{a}$ to $\underline{\alpha}$ in the coefficients of $A(x), A(x)^{-1}, S_0(x), S_\infty(x)$ and that the specialized matrices are still invertible. Then we obtain a q-difference system with coefficients in $Q(\underline{\alpha})$. It follows from Proposition 1.8 that for any norm on $Q(\underline{\alpha})$ such that $|q| > 1$, we can specialize $Y_0(x), Y_\infty(x)$ and, therefore $B(x)$, to matrices with meromorphic entries on $Q(\underline{\alpha})^*$. We will write $A^{(\underline{\alpha})}(x)$, $Y_0^{(\underline{\alpha})}(x)$, etc. for the specialized matrices.

For almost all v, it still makes sense to reduce $A_{\kappa_v}^{(\underline{\alpha})}(x)$ modulo ϕ_v. Moreover, since $A_{\kappa_v}(x)$ is the identity modulo ϕ_v, the same holds for $A_{\kappa_v}^{(\underline{\alpha})}(x)$. Therefore the reduced system has zero v-curvature modulo ϕ_v, for almost all $v \in S$. We know from [**DV02**, Theorem 7.1.1], that $Y_0^{(\underline{\alpha})}(x)$ and $Y_\infty^{(\underline{\alpha})}(x)$ are the germs at zero of rational functions, and therefore that $B^{(\underline{\alpha})}(x)$ is a constant matrix in $\mathrm{GL}_\nu(Q(\underline{\alpha}))$.

As we have already pointed out, $B(x)$ is q-invariant meromorphic matrix on $\mathbb{P}^1_K \smallsetminus \{0,\infty\}$. The set of its poles and zeros is the union of a finite numbers of q-orbits of the forms $\beta q^{\mathbb{Z}}$, such that β is algebraic over K and is a pole or a zero of $A(x)$ or $A(x)^{-1}$. If β is a pole or a zero of an entry $b(x)$ of $B(x)$ and $h_\beta(x), k_\beta(x) \in Q[\underline{a}, x]$ are the minimal polynomials of β and β^{-1} over K, respectively, then we have:

$$b(x) = \lambda \frac{\prod_\gamma \prod_{n\geq 0} h_\gamma(q^{-n}x) \prod_{n\geq 0} k_\gamma(1/q^n x)}{\prod_\delta \prod_{n\geq 0} h_\delta(q^{-n}x) \prod_{n\geq 0} k_\delta(1/q^n x)},$$

where $\lambda \in K$ and γ and δ vary in a system of representatives of the q-orbits of the zeroes and the poles of $b(x)$, respectively. We have proved that there exists a dense subset of $\overline{\mathbb{Q}}^d$ such that the specialization of $b(x)$ at any point of this set is constant. Since the factorization written above must specialize to a convergent factorization of the same form of the corresponding element of $B^{(\underline{\alpha})}(x)$, we conclude that $b(x)$, and therefore $B(x)$, a constant. $\square$

The fact that $B(x) \in \mathrm{GL}_\nu(K)$ implies that the solutions $Y_0(x)$ and $Y_\infty(x)$ glue to a meromorphic solution on $\mathbb{P}^1_K$ and ends the proof of Theorem 3.6.

CHAPTER 4

Rationality of solutions when q is transcendental

In this chapter we consider the case of q transcendental over the prime field.

4.1. Statement of the main result

Let us consider the field of rational function $k(q)$ with coefficients in a perfect field k, of any characteristic. We fix $d \in]0,1[$ and for any irreducible polynomial $v = v(q) \in k[q]$ we set:

$$|f(q)|_v = d^{\deg_q v(q) \cdot \mathrm{ord}_{v(q)} f(q)}, \ \forall f(q) \in k[q].$$

The definition of $|\ |_v$ extends to $k(q)$ by multiplicativity. To this set of norms one has to add the q^{-1}-adic one, defined on $k[q]$ by:

$$|f(q)|_{q^{-1}} = d^{-deg_q f(q)}.$$

Once again, this definition extends by multiplicativity to $k(q)$. Then, the product formula holds:

$$\begin{array}{rcl}\prod_{v\in k[q]\text{ irred.}} \left|\frac{f(q)}{g(q)}\right|_v & = & d^{\sum_v \deg_q v(q)\ \left(\mathrm{ord}_{v(q)} f(q) - \mathrm{ord}_{v(q)} g(q)\right)} \\ & = & d^{deg_q f(q) - \deg_q g(q)} \\ & = & \left|\frac{f(q)}{g(q)}\right|_{q^{-1}}^{-1}.\end{array}$$

For any finite extension K of $k(q)$, we consider the family $\mathcal{P}$ of ultrametric norms, that extends the norms defined above, up to equivalence. We suppose that the norms in $\mathcal{P}$ are normalized so that the product formula still holds. We consider the following partition of $\mathcal{P}$:

- the set $\mathcal{P}_\infty$ of places of K such that the associated norms extend, up to equivalence, either $|\ |_q$ or $|\ |_{q^{-1}}$;
- the set $\mathcal{P}_f$ of places of K such that the associated norms extend, up to equivalence, one of the norms $|\ |_v$ for an irreducible $v = v(q) \in k[q]$, $v(q) \neq q$.[1]

Moreover we consider the set $\mathcal{C}$ of places $v \in \mathcal{P}_f$ such that v divides a valuation of $k(q)$ having as uniformizer a factor of a cyclotomic polynomial, other than $q-1$. Equivalently, $\mathcal{C}$ is the set of places $v \in \mathcal{P}_f$ such that q reduces to a root of unity modulo v of order κ_v strictly greater than 1. We will call $v \in \mathcal{C}$ a cyclotomic place.

Sometimes we will write $\mathcal{P}_K$, $\mathcal{P}_{K,f}$, $\mathcal{P}_{K,\infty}$ and $\mathcal{C}_K$, to stress out the choice of the base field.

In the sequel, we will deal with an arithmetic situation, in the following sense. We consider the ring of integers $\mathcal{O}_K$ of K, i.e., the integral closure of $k[q]$ in K, and

[1]The notation $\mathcal{P}_f$, $\mathcal{P}_\infty$ is only psychological, since all the norms involved here are ultrametric. Nevertheless, there exists a fundamental difference between the two sets, in fact for any $v \in \mathcal{P}_\infty$ one has $|q|_v \neq 1$, while for any $v \in \mathcal{P}_f$ the v-adic norm of q is 1. Therefore, from a v-adic analytic point of view, a q-difference equation has a totally different nature with respect to the norms in the sets $\mathcal{P}_f$ or $\mathcal{P}_\infty$.

a q-difference algebra of the form

$$\mathcal{A} = \mathcal{O}_K \left[x, \frac{1}{P(x)}, \frac{1}{P(qx)}, \frac{1}{P(q^2x)}, \dots \right], \tag{4.1}$$

for some $P(x) \in \mathcal{O}_K[x]$, such that $q \in \mathcal{A}$. Then $\mathcal{A}$ is stable by the action of σ_q and we can consider a q-difference module $\mathcal{M} = (M, \Sigma_q)$ over $\mathcal{A}$. Remember that $\mathcal{M}_{K(x)} = (M_{K(x)} = M \otimes_{\mathcal{A}} K(x), \Sigma_q \otimes \sigma_q)$ is a q-difference module over $K(x)$ and that any q-difference module over $K(x)$ comes from a q-difference module over $\mathcal{A}$, for a convenient choice of $\mathcal{A}$.

We denote by ϕ_v the uniformizer of the cyclotomic place of $k(q)$ induced by $v \in \mathcal{C}_K$. The ring $\mathcal{A} \otimes_{\mathcal{O}_K} \mathcal{O}_K/(\phi_v)$ is not reduced in general, nevertheless it has a q-difference algebra structure and the results in [**DV02**, §2] apply again. Therefore we set:

Definition 4.1. A q-difference module $\mathcal{M}$ over $\mathcal{A}$ has zero v-curvature (modulo ϕ_v) if the operator $\Sigma_q^{\kappa_v}$ induces the identity (or equivalently if the operator $\Delta_q^{\kappa_v}$, with $\Delta_q = \frac{\Sigma_q - 1}{(q-1)x}$, induces the zero operator) on the module $M \otimes_{\mathcal{A}} \mathcal{A}/(\phi_v)$.

Our main result is the following.

Theorem 4.2. *A q-difference module $\mathcal{M}$ over $\mathcal{A}$ has zero v-curvature modulo ϕ_v, for almost all $v \in \mathcal{C}$, if and only if $\mathcal{M}$ becomes trivial over $K(x)$.*

Remark 4.3. As proved in [**DV02**, Proposition 2.1.2], if $\Sigma_q^{\kappa_v}$ is the identity modulo ϕ_v then the q-difference module structure induced on $\mathcal{M} \otimes_{\mathcal{A}} \mathcal{A}/(\phi_v)$ is trivial.

As far as the proof of Theorem 4.2 is regarded, one implication is trivial. We will come back to the proof of the other implication in §4.3.

4.2. Regularity and triviality of the exponents

In this section, we are going to prove that a q-difference module is regular singular and has integral exponents if it has nilpotent reduction for sufficiently many cyclotomic places. We denote by π_v an uniformizer of $v \in \mathcal{C}$.

Definition 4.4. We say that a q-difference module $\mathcal{M} = (M, \Sigma_q)$ over a q-difference $\mathcal{O}_K$-algebra $\mathcal{A}$, as above, has nilpotent v-curvature modulo π_v, or simply that it has nilpotent reduction modulo π_v, if the linear operator $\Sigma_q^{\kappa_v} : M \otimes_{\mathcal{A}} \mathcal{A}/(\pi_v) \longrightarrow M \otimes_{\mathcal{A}} \mathcal{A}/(\pi_v)$ is unipotent (or equivalently if $\Delta_q^{\kappa_v}$ is nilpotent. See [**DV02**, §2]).

We prove the following result:

Proposition 4.5.

(1) *If a q-difference module $\mathcal{M}$ over $\mathcal{A}$ has nilpotent v-curvature modulo π_v, for infinitely many $v \in \mathcal{C}$, then it is regular singular.*
(2) *Let $\mathcal{M}$ be a q-difference module over $\mathcal{A}$. If there exists an infinite set of positive primes $\wp \subset \mathbb{Z}$ such that $\mathcal{M}$ has nilpotent v-curvature modulo π_v, for all $v \in \mathcal{C}$, such that $\kappa_v \in \wp$, then $\mathcal{M}_{K((x))}$ is trivial.*

Proof. The proof of Proposition 3.10 applies word by word to this case, until the the argument showing that A_0 is diagonalisable. To conclude with Proposition 1.16, one has to show that the eigenvalues of A_0 are in $q^{\mathbb{Z}}$. Let $\widetilde{K}$ be a finite extension of K in which we can find all the eigenvalues of A_0. Then any eigenvalue $\alpha \in \widetilde{K}$ of A_0 has the property that $\alpha^{\kappa_v} = 1$ modulo w, for all $w \in \mathcal{C}_{\widetilde{K}}$, $w|v$ and v satisfies the assumptions. In other words, the reduction modulo w of an eigenvalue

α of A_0 belongs to the multiplicative cyclic group generated by the reduction of q modulo π_v.

To end the proof, we are reduced to prove the proposition below. $\square$

PROPOSITION 4.6. *Let k be a perfect field, $K/k(q)$ be a finite extension and $\wp \subset \mathbb{Z}$ be an infinite set of positive primes. For any $v \in \mathcal{C}$, let κ_v be the order of q modulo π_v, as a root of unity.*

If $\alpha \in K$ is such that $\alpha^{\kappa_v} \equiv 1$ modulo π_v, for all $v \in \mathcal{C}$ such that $\kappa_v \in \wp$, then $\alpha \in q^{\mathbb{Z}}$.

REMARK 4.7. Let $K = \mathbb{Q}(\widetilde{q})$, with $\widetilde{q}^r = q$, for some integer $r > 1$. If $\widetilde{q}$ is an eigenvalue of A_0 we would be asking that for infinitely many positive primes $\ell \in \mathbb{Z}$ there exists a primitive root of unity $\zeta_{r\ell}$ of order $r\ell$, which is also a root of unity of order ℓ. Of course, this cannot be true, unless $r = 1$.

4.2.1. Proof of Proposition 4.6. We denote by k_0 either the field of rational numbers $\mathbb{Q}$, if the characteristic of k is zero, or the field with p elements $\mathbb{F}_p$, if the characteristic of k is $p > 0$. First of all, let us suppose that k is a finite perfect extension of k_0 of degree d and fix an embedding $k \hookrightarrow \overline{k}$ of k in its algebraic closure $\overline{k}$. In the case of a rational function $\alpha = f(q) \in k(q)$, Proposition 4.6 is a consequence of the following lemma:

LEMMA 4.8. *Let k be a perfect field, $[k : k_0] = d < \infty$ and let $f(q) \in k(q)$ be non-zero rational function. If there exists an infinite set of positive primes $\wp \subset \mathbb{Z}$ with the following property:*

> *for any $\ell \in \wp$ there exists a primitive root of unity ζ_ℓ of order ℓ*
> *such that $f(\zeta_\ell)$ is a root of unity of order ℓ,*

then $f(q) \in q^{\mathbb{Z}}$.

REMARK 4.9. If $k = \mathbb{C}$ and $y - f(q)$ is irreducible in $\mathbb{C}[q, y]$, the result can be deduced from [**Lan83**, Chapter 8, Theorem 6.1], whose proof uses Bézout theorem. We give here a totally elementary proof, that holds also in positive characteristic.

Proposition 4.6 can be rewritten in the language of rational dynamic. We denote by μ_ℓ the group of root of unity of order ℓ. The following assertions are equivalent:

(1) $f(q) \in k(q)$ satisfies the assumptions of Lemma 4.8.
(2) There exist infinitely many $\ell \in \mathbb{N}$ such that the group μ_ℓ of roots of unity of order ℓ verifies $f(\mu_\ell) \subset \mu_\ell$.
(3) $f(q) \in q^{\mathbb{Z}}$.
(4) The Julia set of f is the unit circle.

As it was pointed out to us by C. Favre, the equivalence between the last two assumptions is a particular case of [**Zdu97**], while the equivalence between the second and the fourth assumption can be deduced from [**FRL06**] or [**Aut01**].

PROOF. Let $f(q) = \frac{P(q)}{Q(q)}$, with $P = \sum_{i=0}^{D} a_i q^i, Q = \sum_{i=0}^{D} b_i q^i \in k[q]$ coprime polynomials of degree less equal to D, and let ℓ be a prime such that:

- $f(\zeta_\ell) \in \mu_\ell$;
- $2D < \ell - 1$.

Moreover, since $\wp$ is infinite, we can chose $\ell >> 0$ so that the extensions k and $k_0(\mu_\ell)$ are linearly disjoint over k_0. Since k is perfect, this implies that the minimal polynomial of the primitive ℓ-th root of unity ζ_ℓ over k is $\chi(X) = 1 + X + ... + X^{\ell-1}$. Now let $\kappa \in \{0, \dots, \ell - 1\}$ be such that $f(\zeta_\ell) = \zeta_\ell^\kappa$, i.e.,

$$\sum_{i=0}^{D} a_i \zeta_\ell^i = \sum_{i=0}^{D} b_i \zeta_\ell^{i+\kappa}.$$

We consider the polynomial $H(q)=\sum_{i=0}^{D} a_i q^i - \sum_{j=\kappa}^{D+\kappa} b_{j-\kappa} q^j$ and distinguish three cases:

(1) If $D+\kappa<\ell-1$, then $H(q)$ has ζ_ℓ as a zero and has degree strictly inferior to $\ell-1$. Necessarily $H(q)=0$. Thus we have

$$a_0=a_1=\ldots=a_{\kappa-1}=b_{D+1-\kappa}=\ldots=b_D=0 \quad \text{and} \quad a_i=b_{i-\kappa} \text{ for } i=\kappa,\ldots,D,$$

which implies $f(q)=q^\kappa$.

(2) If $D+\kappa=\ell-1$ then $H(q)$ is a k-multiple of $\chi(q)$ and therefore all the coefficients of $H(q)$ are all equal. Notice that the inequality $D+\kappa\geq\ell-1$ forces κ to be strictly bigger than D, in fact otherwise one would have $\kappa+D\leq 2D<\ell-1$. For this reason the coefficients of $H(q)$ of the monomials $q^{D+1},\ldots,q^\kappa$ are all equal to zero. Thus

$$a_0=a_1=\ldots=a_D=b_0=\ldots=b_D=0$$

and therefore $f=0$ against the assumptions. So the case $D+\kappa=l-1$ cannot occur.

(3) If $D+\kappa>\ell-1$, then $\kappa>D>D+\kappa-\ell$, since $\kappa>D$ and $\kappa-\ell<0$. In this case we shall rather consider the polynomial $\widetilde{H}(q)$ defined by:

$$\widetilde{H}(q)=\sum_{i=0}^{D} a_i q^i - \sum_{i=\kappa}^{\ell-1} b_{i-\kappa} q^i - \sum_{i=0}^{D+\kappa-\ell} b_{i+\ell-\kappa} q^i.$$

Notice that $H(\zeta_\ell)=\widetilde{H}(\zeta_\ell)=0$ and that $\widetilde{H}(q)$ has degree smaller or equal than $\ell-1$. As in the previous case, $\widetilde{H}(q)$ is a k-multiple of $\chi(q)$. We get

$$b_j=0 \text{ for } j=0,\ldots,\ell-1-\kappa$$

and

$$a_0-b_{\ell-\kappa}=\ldots=a_{D+\kappa-\ell}-b_D=a_{D+\kappa-\ell+1}=\ldots=a_D=0.$$

We conclude that $f(q)=q^{\kappa-\ell}$.

This ends the proof. □

We are going to deduce Proposition 4.6 from Lemma 4.8 in two steps: first of all we are going to show that we can drop the assumption that $[k:k_0]$ is finite and then that one can always reduce to the case of a rational function.

Lemma 4.10. *Lemma 4.8 holds if k/k_0 is a finitely generated (not necessarily algebraic) extension.*

Remark 4.11. Since $f(q)\in k(q)$, replacing k by the field generated by the coefficients of f over k_0, we can always assume that k/k_0 is finitely generated.

Proof. Let $\widetilde{k}$ be the algebraic closure of k_0 in k and let k' be an intermediate field of $k/\widetilde{k}$, such that $f(q)\in k'(q)\subset k(q)$ and that $k'/\widetilde{k}$ has minimal transcendence degree ι. We suppose that $\iota>0$, to avoid the situation of Lemma 4.8. So let $a_1,\ldots,a_\iota$ be transcendence basis of $k'/\widetilde{k}$ and let $k''=\widetilde{k}(a_1,\ldots,a_\iota)$. If $k'/\widetilde{k}$ is purely transcendental, i.e., if $k'=k''$, then $f(q)=P(q)/Q(q)$, where $P(q)$ and $Q(q)$ can be written in the form:

$$P(q)=\sum_i\sum_{\underline{j}} \alpha_{\underline{j}}^{(i)} a_{\underline{j}} q^i \quad \text{and} \quad Q(q)=\sum_i\sum_{\underline{j}} \beta_{\underline{j}}^{(i)} a_{\underline{j}} q^i,$$

with $\underline{j}=(j_1,\ldots,j_\iota)\in\mathbb{Z}_{\geq 0}^\iota$, $a_{\underline{j}}=a_{j_1}\cdots a_{j_\iota}$ and $\alpha_{\underline{j}}^{(i)},\beta_{\underline{j}}^{(i)}\in\widetilde{k}$. If we reorganize the terms of P and Q so that

$$P(q)=\sum_{\underline{j}} a_{\underline{j}} D_{\underline{j}}(q) \quad \text{and} \quad Q(q)=\sum_{\underline{j}} a_{\underline{j}} C_{\underline{j}}(q),$$

we conclude that the assumption $f(\zeta_\ell) \subset \mu_\ell$ for infinitely many primes ℓ implies that $f_{\underline{j}} = \frac{D_{\underline{j}}}{C_{\underline{j}}}$ is a rational function with coefficients in $\widetilde{k}$ satisfying the assumptions of Lemma 4.8. Moreover, since the f_j's take the same values at infinitely many roots of unity, they are all equal. Finally, we conclude that $f_{\underline{j}}(q) = q^d$ for any $\underline{j}$ and hence that $f = q^d \frac{\sum \alpha_{\underline{j}}}{\sum \alpha_{\underline{j}}} = q^d$.

Now let us suppose that $k' = k''(b)$ for some primitive element b, algebraic over k'', of degree e. Then once again we write $f(q) = P(q)/Q(q)$, with:

$$P(q) = \sum_i \sum_{h=0}^{e-1} \alpha_{i,h} b^h q^i \quad \text{and} \quad Q(q) = \sum_i \sum_{h=0}^{e-1} \beta_{i,h} b^h q^i,$$

with $\alpha_{i,h}, \beta_{i,h} \in k''$. Again we conclude that $\frac{\sum_i \alpha_{i,h} q^i}{\sum_i \beta_{i,h} q^i} = q^d$ for any $h = 0, \dots, e-1$, and hence that $f(q) = q^d$. $\square$

End of the proof of Proposition 4.6. Let $\widetilde{K} = k(q, f) \subset K$. If the characteristic of k is p, replacing f by a p^n-th power of f, we can suppose that $\widetilde{K}/k(q)$ is a Galois extension. So we set:

$$y = \prod_{\varphi \in Gal(\widetilde{K}/k(q))} f^\varphi \in k(q).$$

For infinitely many $v \in \mathcal{C}_{k(q)}$ such that κ_v is a prime, we have $f^{\kappa_v} \equiv 1$ modulo w, for any $w|v$. Since $Gal(\widetilde{K}/K)$ acts transitively over the set of places $w \in \mathcal{C}_{\widetilde{K}}$ such that $w|v$, this implies that $y^{\kappa_v} \equiv 1$ modulo π_v. Then Lemmas 4.10 and 4.8 allow us to conclude that $y \in q^{\mathbb{Z}}$. This proves that we are in the following situation: f is an algebraic function such that $|f|_w = 1$ for any $w \in \mathcal{P}_{\widetilde{K},f}$ and that $|f|_w \neq 1$ for any $w \in \mathcal{P}_{\widetilde{K},\infty}$. We conclude that $f = cq^{s/r}$ for some non-zero integers s, r and some constant c in a finite extension of k. Since $f^{\kappa_v} \equiv 1$ modulo w, for all $w \in \mathcal{C}_{\widetilde{K}}$ such that $\kappa_v \in \wp$, we finally obtain that $r = 1$ and $c = 1$. $\square$

4.3. Proof of Theorem 4.2

Under the assumption of Theorem 4.2, Proposition 4.5 implies that the q-difference module $\mathcal{M}$ becomes trivial over $K((x))$. To conclude we need to show the following proposition:

Proposition 4.12. *If a q-difference module $\mathcal{M}$ over $\mathcal{A}$ has zero v-curvature modulo ϕ_v, for almost all $v \in \mathcal{C}$, then there exists a basis $\underline{e}$ of $M_{K(x)}$ over $K(x)$ such that the associated q-difference system has a formal fundamental solution $Y(x) \in \mathrm{GL}_\nu(K((x)))$, which is the Taylor expansion at 0 of a matrix in $\mathrm{GL}_\nu(K(x))$, i.e., $\mathcal{M}$ becomes trivial over $K(x)$.*

Remark 4.13. This is the only part of the proof of Theorem 4.2 where we need to suppose that the v-curvature are zero modulo ϕ_v, for almost all $v \in \mathcal{C}$.

Proof. (*cf.* [**DV02**, Proposition 8.2.1]) Let $\underline{e}$ be a basis of M over $K(x)$. Applying a basis change with coefficients in $K\left[x, \frac{1}{x}\right]$, we can actually suppose that $\Sigma_q \underline{e} = \underline{e} A(x)$, where $A(x) \in \mathrm{GL}_\nu(K(x))$ has no pole at 0 and $A(0)$ is the identity matrix. In the notation of §1.3, the recursive relation defining the matrices $G_n(x)$ implies that they have no pole at 0. This means that $Y(x) := \sum_{n \geq 0} G_{[n]}(0) x^n$ is a fundamental solution of the q-difference system associated to $\mathcal{M}_{K(x)}$ with respect to the basis $\underline{e}$.

We recall the definition of the Gauss norm associated to an ultrametric norm $v \in \mathcal{P}$:

$$\text{for any } \frac{\sum a_i x^i}{\sum b_j x^j} \in K(x), \quad \left|\frac{\sum a_i x^i}{\sum b_j x^j}\right|_{v,Gauss} = \frac{\sup |a_i|_v}{\sup |b_j|_v}.$$

We have:

LEMMA 4.14. *Let $v \in \mathcal{C}_K$. We assume that $|G_1(x)|_{v,Gauss} \leq 1$. Then the following assertions are equivalent:*

(1) *The module $\mathcal{M} = (M, \Sigma_q)$ has zero v-curvature modulo ϕ_v.*
(2) *For any positive integer n, we have $\left|G_{[n]}\right|_{v,Gauss} \leq 1$.*

REMARK 4.15. Let k_v be the residue field of K modulo v and q_v the reduction of q in k_v, which is defined for almost all $v \in \mathcal{C}$. According to [**Har10**, §3], the second assertion of the lemma above can be rewritten as: $\mathcal{M}_{k_v(x)}$ has a natural structure of iterated q_v-difference module.

PROOF OF LEMMA 4.14. The only non-trivial implication is "1 $\Rightarrow$ 2" whose proof is quite similar to [**DV02**, Lemma 5.1.2]. The Leibniz Formula for d_q and Δ_q implies that:

$$G_{(n+1)\kappa_v} = \sum_{i=0}^{\kappa_v} \binom{\kappa_v}{i}_q \sigma_q^{\kappa_v - i}(d_q^i\,(G_{n\kappa_v}))G_{\kappa_v - i}.$$

If $\mathcal{M}$ has zero v-curvature modulo ϕ_v then $|G_{\kappa_v}|_{v,Gauss} \leq |\phi_v|_v$. One obtains recursively that $|G_m|_{v,Gauss} \leq |\phi_v|_v^{\left[\frac{m}{\kappa_v}\right]}$, where we have denoted by $[a]$ the integral part of $a \in \mathbb{R}$, i.e., $[a] = max\{n \in \mathbb{Z} : n \leq a\}$. Since $|[\kappa_v]_q|_v = |\phi_v|_v$ and $|[m]_q^!|_v = |\phi_v|_v^{\left[\frac{m}{\kappa_v}\right]}$, we conclude that:

$$\left|\frac{G_m}{[m]_q^!}\right|_{v,Gauss} \leq 1. \tag{4.2}$$

This ends the proof of the lemma. □

We go back to the proof of Proposition 4.12. The entries of $Y(x) = \sum_{n\geq 0} G_{[n]}(0)x^n$ verify the following properties:

- For any $v \in \mathcal{P}_\infty$, the matrix $Y(x)$ is analytic at 0 and has infinite v-adic radius of meromorphy (see Proposition 1.7).
- Since $|[n]_q|_{v,Gauss} = 1$ for any non-cyclotomic place $v \in \mathcal{P}_f$, we have $\left|G_{[m]}(x)\right|_{v,Gauss} \leq 1$, for almost all $v \in \mathcal{P}_f \setminus \mathcal{C}$. For the finitely many $v \in \mathcal{P}_f$ such that $|G_1(x)|_{v,Gauss} > 1$, there exists a constant $C > 0$ such that $\left|G_{[m]}(x)\right|_{v,Gauss} \leq C^m$, for any positive integer m.
- For almost all $v \in \mathcal{C}$ and all positive integer m, $\left|G_{[m]}(x)\right|_{v,Gauss} \leq 1$ (*cf.* Lemma 4.14), while for the remaining finitely many $v \in \mathcal{C}$ there exists a constant $C > 0$ such that $\left|G_{[m]}(x)\right|_{v,Gauss} \leq C^m$ for any positive integer m.

This implies that:

$$\limsup_{m\to\infty} \frac{1}{m} \sum_{v\in\mathcal{P}} \log^+ \left|G_{[m]}(x)\right|_{v,Gauss} < \infty.$$

To conclude that $Y(x)$ is the expansion at zero of a matrix with rational entries, we apply a simplified form of the Borel-Dwork criteria for function fields, which says exactly that a formal power series having positive radius of convergence for almost all places and infinite radius of meromorphy at one fixed place is the expansion of a rational function. The proof in this case is a slight simplification of [**DV02**, Proposition 8.4.1][2], which is itself a simplification of the more general criteria [**And04**, Theorem 5.4.3]. We are omitting the details. $\square$

4.4. Link with iterative q-difference equations

We denote by k_v the residue field of K with respect to a place $v \in \mathcal{P}$ and by q_v the image of q in k_v, which is defined for all places $v \in \mathcal{P}$. For almost all $v \in \mathcal{P}_f$ we can consider the $k_v(x)$-vector space $M_{k_v(x)} = M \otimes_{\mathcal{A}} k_v(x)$, with the structure induced by Σ_q. In this way, for almost all $v \in \mathcal{P}$, we obtain a q_v-difference module $\mathcal{M}_{k_v(x)} = (M_{k_v(x)}, \Sigma_{q_v})$ over $k_v(x)$,

In the framework of iterative q-difference equations [**Har10**], Theorem 4.2 is equivalent to the following statement, which is a q-analogue of the conjecture stated at the very end of [**MvdP03**]:

COROLLARY 4.16. *For a q-difference module $\mathcal{M}$ over $\mathcal{A}$ the following statement are equivalent:*

(1) *The q-difference module $\mathcal{M}$ over $\mathcal{A}$ becomes trivial over $K(x)$;*
(2) *It induces an iterative q_v-difference structure over $\mathcal{M}_{k_v(x)}$, for almost all $v \in \mathcal{C}$;*
(3) *It induces a trivial iterative q_v-difference structure over $\mathcal{M}_{k_v(x)}$, for almost all $v \in \mathcal{C}$.*

PROOF. The equivalence $1 \Leftrightarrow 2$ is a consequence of Lemma 4.14 and Theorem 4.2, while the implication $3 \Rightarrow 2$ is tautological.

Let us prove that $1 \Rightarrow 3$. If the q-difference module $\mathcal{M}$ becomes trivial over $K(x)$, then there exist an $\mathcal{A}$-algebra $\mathcal{A}'$, of the form (4.1), obtained from $\mathcal{A}$ inverting a polynomial and its q-iterates, and a basis $\underline{e}$ of $M \otimes_{\mathcal{A}} \mathcal{A}'$ over $\mathcal{A}'$, such that the associated q-difference system is $\sigma_q(Y) = Y$. Therefore, for almost all $v \in \mathcal{C}$, $\mathcal{M}$ induces an iterative q_v-difference module $\mathcal{M}_{k_v(x)}$ whose iterative q_v-difference equations are given by $\frac{d_{q_v}^{\kappa_v}}{[\kappa_v]_{q_v}^!}(Y) = 0$ for all $n \in \mathbb{N}$ (*cf.* [**Har10**, Proposition 3.17]). $\square$

[2] The simplification comes from the fact, in this setting, that there are no archimedean norms.

CHAPTER 5

A unified statement

Let K be a field, $q \in K$, $q \neq 0, 1$ be a fixed element. If follows from Proposition 1.4 that we can suppose that K is finitely generated over the prime field. Let $\mathcal{M} = (M_{K(x)}, \Sigma_q)$ be a q-difference module over $K(x)$. We recall the following notations:

($\mathcal{A}$) If *q is algebraic over $\mathbb{Q}$, but not a root of unity*, we are in the following situation. We call Q the algebraic closure of $\mathbb{Q}$ inside K, $\mathcal{O}_Q$ the ring of integer of Q, $\mathcal{C}$ the set of finite places v of Q and $\pi_v \in \mathcal{O}_Q$ a v-adic uniformizer. For almost all finite place v of Q, the following are well defined: the order κ_v, as a root of unity, of the reduction of q modulo π_v and the positive integer power ϕ_v of π_v, such that $\phi_v^{-1}(1 - q^{\kappa_v})$ is a unit of $\mathcal{O}_Q$. The field K has the form $Q(\underline{a}, b)$, where $\underline{a} = (a_1, \dots, a_r)$ is a transcendence basis of K/Q and b is a primitive element of the algebraic extension $K/Q(\underline{a})$. Choosing conveniently the set of generators $\underline{a}, b$ and $P(x) \in \mathcal{O}_Q\left[\underline{a}, b, x\right]$, we can always find a q-difference algebra $\mathcal{A}$ of the form

$$\mathcal{A} = \mathcal{O}_Q \left[\underline{a}, b, x, \frac{1}{P(x)}, \frac{1}{P(qx)}, \dots\right] \tag{5.1}$$

and a Σ_q-stable $\mathcal{A}$-lattice M of $\mathcal{M}_{K(x)}$, so that we can consider the $\mathcal{A}/(\phi_v)$-linear operator

$$\Sigma_q^{\kappa_v} : M \otimes_{\mathcal{A}} \mathcal{A}/(\phi_v) \longrightarrow M \otimes_{\mathcal{A}} \mathcal{A}/(\phi_v),$$

that we have called the v-curvature of $\mathcal{M}_{K(x)}$-modulo ϕ_v. Notice that $\mathcal{O}_Q/(\phi_v)$ is not an integral domain in general.

($\mathcal{T}$) If *q is transcendental over the prime field of K*, then there exists a subfield k of K such that K is a finite extension of $k(q)$. We denote by $\mathcal{C}$ the set of places of K that extend the places of $k(q)$, associated to irreducible polynomials ϕ_v of $k[q]$, that vanish at roots of unity. Let κ_v be the order of the roots of ϕ_v. Let $\mathcal{O}_K$ be the integral closure of $k[q]$ in K. Choosing conveniently $P(x) \in \mathcal{O}_K[x]$, we can always find a q-difference algebra $\mathcal{A}$ of the form:

$$\mathcal{A} = \mathcal{O}_K \left[x, \frac{1}{P(x)}, \frac{1}{P(qx)}, \dots\right] \tag{5.2}$$

and a Σ_q-stable $\mathcal{A}$-lattice M of $\mathcal{M}_{K(x)}$, so that we can consider the $\mathcal{A}/(\phi_v)$-linear operator

$$\Sigma_q^{\kappa_v} : M \otimes_{\mathcal{A}} \mathcal{A}/(\phi_v) \longrightarrow M \otimes_{\mathcal{A}} \mathcal{A}/(\phi_v),$$

that we have also called the v-curvature of $\mathcal{M}_{K(x)}$-modulo ϕ_v. Notice that, once again, $\mathcal{O}_K/(\phi_v)$ is not an integral domain in general.

($\mathcal{R}$) If *q is a primitive root of unity of order κ*, we define $\mathcal{C}$ to be the set containing only the trivial valuation v on K, $\phi_v = 0$ and $\kappa_v = \kappa$. Then there exists a polynomial $P(x) \in K[x]$ such that the algebra $\mathcal{A} = K\left[x, \frac{1}{P(x)}, \frac{1}{P(qx)}, \dots\right]$ is σ_q-stable and there exists a Σ_q-stable $\mathcal{A}$-lattice M of $\mathcal{M}_{K(x)}$, so that we can consider the $\mathcal{A}/(\phi_v)$-linear operator

$$\Sigma_q^{\kappa_v} : M \otimes_{\mathcal{A}} \mathcal{A}/(\phi_v) \longrightarrow M \otimes_{\mathcal{A}} \mathcal{A}/(\phi_v),$$

that we will call the v-curvature of $\mathcal{M}_{K(x)}$-modulo ϕ_v. Notice that this is simply the κ-th iterate of Σ_q, namely $\Sigma_q^\kappa : M \longrightarrow M$.

Then the main result of the first part of this work is:

THEOREM 5.1. *A q-difference module $\mathcal{M}_{K(x)} = (M_{K(x)}, \Sigma_q)$ over $K(x)$ is trivial if and only if there exist an algebra $\mathcal{A}$, as above, and a Σ_q-stable $\mathcal{A}$-lattice M of $M_{K(x)}$ such that the map*

$$\Sigma_q^{\kappa_v} : M \otimes_{\mathcal{A}} \mathcal{A}/(\phi_v) \longrightarrow M \otimes_{\mathcal{A}} \mathcal{A}/(\phi_v),$$

is the identity, for any v in a cofinite non-empty subset of $\mathcal{C}$.

In the case $(\mathcal{A})$ we can take $\mathcal{C}$ to be a set of finite places of Q of density 1, depending on $\mathcal{M}_{K(x)}$.

PROOF. So the statement above coincide with Proposition 3.1 if q is a root of unity, and with Theorem 3.8 if q is algebraic, but not a root of unity. Finally, to deduce the third case from Theorem 4.2, it is enough to remark that we can replace k by its perfect closure. □

Of course, for a given module $\mathcal{M}_{K(x)}$ we can always find a q-difference algebra $\mathcal{A}$ as above and a q-difference module $\mathcal{M}$ over $\mathcal{A}$ such that $\mathcal{M} \otimes_{\mathcal{A}} K(x) \cong \mathcal{M}_{K(x)}$. Also, if the statement above is true for a choice of $\mathcal{A}$ and one q-difference module $\mathcal{M}$ over $\mathcal{A}$, then it is true for all choice of $\mathcal{A}$ and of $\mathcal{M}$. In the following chapters, we will use this fact implicitly.

Part 3

Intrinsic Galois groups

CHAPTER 6

The intrinsic Galois group

6.1. Arithmetic characterization of the intrinsic Galois group

6.1.1. Definition. Let $\mathcal{F}$ be a q-difference field and $\mathcal{M}_{\mathcal{F}} = (M_{\mathcal{F}}, \Sigma_q)$ be a q-difference module of rank ν over $\mathcal{F}$, in the sense of Chapter 1. We can consider the family $Constr_{\mathcal{F}}(\mathcal{M}_{\mathcal{F}})$ of q-difference modules containing $\mathcal{M}_{\mathcal{F}}$ and closed under direct sum, tensor product, dual, symmetric and antisymmetric products (see §1.1.1). We will denote by $Constr_{\mathcal{F}}(M_{\mathcal{F}})$ the collection of constructions of linear algebra of the $\mathcal{F}$-vector space $M_{\mathcal{F}}$, i.e., the collection of underlying vector spaces of the family $Constr_{\mathcal{F}}(\mathcal{M}_{\mathcal{F}})$. Notice that $\mathrm{GL}(M_{\mathcal{F}})$ acts naturally, by functoriality, on any element of $Constr_{\mathcal{F}}(M_{\mathcal{F}})$.

Definition 6.1. The intrinsic Galois group[1] $Gal(\mathcal{M}_{\mathcal{F}}, \eta_{\mathcal{F}})$ of $\mathcal{M}_{\mathcal{F}}$ is the subgroup of $\mathrm{GL}(M_{\mathcal{F}})$ which is the stabilizer of all the q-difference submodules over $\mathcal{F}$ of any object in $Constr_{\mathcal{F}}(\mathcal{M}_{\mathcal{F}})$.

In the definitions above and below, the term "stabilizer" has to be understood in the functorial sense of [**DG70**, II.1.36]. For instance, $Gal(\mathcal{M}_{\mathcal{F}}, \eta_{\mathcal{F}})$ is a functor from the category of $\mathcal{F}$-algebras to the category of groups, that associates to any $\mathcal{F}$-algebra S, the subgroup of $\mathrm{GL}(M_{\mathcal{F}}) \otimes S$ that stabilizes $\mathcal{N}_{\mathcal{F}} \otimes S$, for all the q-difference submodules $\mathcal{N}_{\mathcal{F}}$ over $\mathcal{F}$ of any object in $Constr_{\mathcal{F}}(\mathcal{M}_{\mathcal{F}})$. By [**DG70**, II.1.36], this functor is representable and thus defines an algebraic group scheme over $\mathcal{F}$.

Notice that in positive characteristic p, the group $Gal(\mathcal{M}_{\mathcal{F}}, \eta_{\mathcal{F}})$ is not necessarily reduced. An easy example is given by the equation $y(qx) = q^{1/p} y(x)$, whose intrinsic Galois group is μ_p (*cf.* [**vdPR07**, §7]).

The group $Gal(\mathcal{M}_{\mathcal{F}}, \eta_{\mathcal{F}})$ is a tannakian object. In fact, the full tensor category $\langle \mathcal{M}_{\mathcal{F}} \rangle^{\otimes}$ generated by $\mathcal{M}_{\mathcal{F}}$ in $Diff(\mathcal{F}, \sigma_q)$ is naturally a tannakian category, when equipped with the forgetful functor

$$\eta_{\mathcal{F}} : \langle \mathcal{M}_{\mathcal{F}} \rangle^{\otimes} \longrightarrow \{\mathcal{F}\text{-vector spaces}\}.$$

In the notation of §2.2, the functor $Aut^{\otimes}(\eta_{\mathcal{F}})$ corresponds to the algebraic group $Gal(\mathcal{M}_{\mathcal{F}}, \eta_{\mathcal{F}})$. Moreover, we have the following proposition.

Proposition 6.2. *Let $\omega : \langle \mathcal{M}_{\mathcal{F}} \rangle^{\otimes} \to Vect_{\mathcal{F}^{\sigma_q}}$ be a neutral fiber functor. The algebraic group schemes $Aut^{\otimes}(\omega) \otimes \mathcal{F}$ and $Gal(\mathcal{M}_{\mathcal{F}}, \eta_{\mathcal{F}})$, defined over $\mathcal{F}$, are isomorphic over the algebraic closure $\overline{\mathcal{F}}$ of $\mathcal{F}$.*

Proof. In the notations of §2.2, the affine scheme $Isom^{\otimes}(\omega \otimes_{\mathcal{F}^{\sigma_q}} \mathcal{F}, \eta_{\mathcal{F}})$ is representable by a non-zero finitely generated $\mathcal{F}$-algebra. Since $\overline{\mathcal{F}}$ is algebraically closed, the previous algebraic scheme has a point in $\overline{\mathcal{F}}$. This ends the proof. □

We will come back on the tannakian point of view in Part 4.

Remark 6.3. We recall that the Chevalley theorem, that also holds for non-reduced groups (*cf.* [**DG70**, II, §2, n.3, Corollary 3.5]), ensures that $Gal(\mathcal{M}_{\mathcal{F}}, \eta_{\mathcal{F}})$

[1] In the literature, the intrinsic Galois group is also called the generic Galois group of $\mathcal{M}_{\mathcal{F}}$.

can be defined as the stabilizer of a rank one submodule (which is not necessarily a q-difference module) of a q-difference module contained in an algebraic construction of $\mathcal{M}_{\mathcal{F}}$. Nevertheless, it is possible to find a line that defines $Gal(\mathcal{M}_{\mathcal{F}}, \eta_{\mathcal{F}})$ as the stabilizer and that is also a q-difference module. In fact the noetherianity of $\mathrm{GL}(M_{\mathcal{F}})$ implies that $Gal(\mathcal{M}_{\mathcal{F}}, \eta_{\mathcal{F}})$ is defined as the stabilizer of a finite family of q-difference submodules $\mathcal{W}_{\mathcal{F}}^{(i)} = (W_{\mathcal{F}}^{(i)}, \Sigma_q)$ contained in some objects $\mathcal{M}_{\mathcal{F}}^{(i)}$ of $\langle \mathcal{M}_{\mathcal{F}} \rangle^{\otimes}$. It follows that the line

$$L_{\mathcal{F}} = \wedge^{\dim \oplus_i W_{\mathcal{F}}^{(i)}} \left(\bigoplus_i W_{\mathcal{F}}^{(i)} \right) \subset \wedge^{\dim \oplus_i W_{\mathcal{F}}^{(i)}} \left(\bigoplus_i M_{\mathcal{F}}^{(i)} \right)$$

is a q-difference module and defines $Gal(\mathcal{M}_{\mathcal{F}}, \eta_{\mathcal{F}})$ as a stabilizer (*cf.* [**Kat82**, proof of Proposition 9]).

In the sequel, we will use the notation $Stab(W_{\mathcal{F}}^{(i)}, i)$ to say that a group is the stabilizer of the set of vector spaces $\{W_{\mathcal{F}}^{(i)}\}_i$.

6.1.2. Main result. From now on we consider the particular case $\mathcal{F} = K(x)$, with the notations introduced in Chapter 5. Let G be a closed algebraic subgroup of $\mathrm{GL}(M_{K(x)})$, such that $G = Stab(L_{K(x)})$ for some line $L_{K(x)}$ contained in an object $\mathcal{W}_{K(x)}$ of $\langle \mathcal{M}_{K(x)} \rangle^{\otimes}$. For a q-difference algebra $\mathcal{A}$, a Σ_q-stable $\mathcal{A}$-lattice M of $M_{K(x)}$ determines an $\mathcal{A}$-lattice L of $L_{K(x)}$ and an $\mathcal{A}$-lattice W of $W_{K(x)}$. The latter is the underlying space of a q-difference module $\mathcal{W} = (W, \Sigma_q)$ over $\mathcal{A}$.

Definition 6.4. Let $\widetilde{\mathcal{C}}$ be a cofinite non-empty subset of $\mathcal{C}$ and $(\Lambda_v)_{v \in \widetilde{\mathcal{C}}}$ be a family of $\mathcal{A}/(\phi_v)$-linear operators acting on $M \otimes_{\mathcal{A}} \mathcal{A}/(\phi_v)$. We say that the algebraic group $G \subset \mathrm{GL}(M_{K(x)})$ contains the operators Λ_v modulo ϕ_v for almost all $v \in \mathcal{C}$ if for almost all, and at least one, $v \in \widetilde{\mathcal{C}}$ the operator Λ_v stabilizes $L \otimes_{\mathcal{A}} \mathcal{A}/(\phi_v)$ inside $W \otimes_{\mathcal{A}} \mathcal{A}/(\phi_v)$:

$$\Lambda_v \in Stab_{\mathcal{A}/(\phi_v)}(L \otimes_{\mathcal{A}} \mathcal{A}/(\phi_v)).$$

Remark 6.5. First of all, starting from now, we will always use the phrase "for almost all" to mean "for almost all, and at least one". In this way the statements will be correct even in the case $(\mathcal{R})$ (see Chapter 5).

As in [**DV02**, 10.1.2], one can prove that the definition above is independent of the choice of $\mathcal{A}$, M and $L_{K(x)}$.

The main result of this section is the following:

Theorem 6.6. *The algebraic group $Gal(\mathcal{M}_{K(x)}, \eta_{K(x)})$ is the smallest closed algebraic subgroup of $\mathrm{GL}(M_{K(x)})$, that contains the operators $\Sigma_q^{\kappa_v}$ modulo ϕ_v, for almost all $v \in \mathcal{C}$.*

Remark 6.7.
- The noetherianity of $\mathrm{GL}(M_{K(x)})$ implies that the smallest closed algebraic subgroup of $\mathrm{GL}(M_{K(x)})$ that contains the operators $\Sigma_q^{\kappa_v}$ modulo ϕ_v, for almost all $v \in \mathcal{C}$, is well-defined. Theorem 6.6 has been proved in [**Hen96**, Chapter 6] when q is a root of unity and in [**DV02**] when q is algebraic and K is a number field.
- Under the assumption $(\mathcal{A})$ (see Chapter 5), the statement above is still true if we replace $\mathcal{C}$ by a set of finite places of Q of density 1. This remark applies to all statements in this and the next chapter.

A part of Theorem 6.6 is easy to prove:

Lemma 6.8. *The algebraic group $Gal(\mathcal{M}_{K(x)}, \eta_{K(x)})$ contains the operators $\Sigma_q^{\kappa_v}$ modulo ϕ_v, for almost all $v \in \mathcal{C}$.*

PROOF. The statement follows immediately from the fact that $Gal(\mathcal{M}_{K(x)}, \eta_{K(x)})$ can be defined as the stabilizer of a rank one q-difference module in $\langle\mathcal{M}_{K(x)}\rangle^{\otimes}$, which is *a fortiori* stable by the action of $\Sigma_q^{\kappa_v}$. □

COROLLARY 6.9. *$Gal(\mathcal{M}_{K(x)}, \eta_{K(x)}) = \{1\}$ if and only if $\mathcal{M}_{K(x)}$ is a trivial q-difference module.*

PROOF. Because of the lemma above, if $Gal(\mathcal{M}_{K(x)}, \eta_{K(x)}) = \{1\}$ is the trivial group, then $\Sigma_q^{\kappa_v}$ induces the identity on $M \otimes_{\mathcal{A}} \mathcal{A}/(\phi_v)$, for almost all $v \in \mathcal{C}$. Therefore Theorem 5.1 implies that $\mathcal{M}_{K(x)}$ is trivial. On the other hand, if $\mathcal{M}_{K(x)}$ is trivial, then it is isomorphic to the q-difference module $(K^\nu \otimes_K K(x), 1 \otimes \sigma_q)$. It follows that the intrinsic Galois group $Gal(\mathcal{M}_{K(x)}, \eta_{K(x)})$ is forced to stabilize all the lines generated by vectors of the type $m \otimes 1$, with $m \in K^\nu$. Therefore it is the trivial group. □

Now we are ready to give the proof of Theorem 6.6, whose main ingredient is Theorem 5.1. The argument is inspired by [**Kat82**, §X].

PROOF OF THEOREM 6.6. Lemma 6.8 says that $Gal(\mathcal{M}_{K(x)}, \eta_{K(x)})$ contains the smallest subgroup G of $\mathrm{GL}(M_{K(x)})$, that contains the operator $\Sigma_q^{\kappa_v}$ modulo ϕ_v, for almost all $v \in \mathcal{C}$. Let $L_{K(x)}$ be a line contained in some object of the category $\langle\mathcal{M}_{K(x)}\rangle^{\otimes}$, that defines G as a stabilizer. Then there exists a smaller q-difference module $\mathcal{W}_{K(x)}$ over $K(x)$ that contains $L_{K(x)}$. Let L and $\mathcal{W} = (W, \Sigma_q)$ be the associated $\mathcal{A}$-modules. Any generator m of L as an $\mathcal{A}$-module is a cyclic vector for $\mathcal{W}$ and the operator $\Sigma_q^{\kappa_v}$ acts on $W \otimes_{\mathcal{A}} \mathcal{A}/(\phi_v)$ with respect to the basis induced by the cyclic basis generated by m via a diagonal matrix. Because of the definition of the q-difference structure on the dual module $\mathcal{W}^*$ of $\mathcal{W}$, the group G can be define as the subgroup of $\mathrm{GL}(M_{K(x)})$ that fixes a line L' in $W^* \otimes W$, i.e., such that $\Sigma_q^{\kappa_v}$ acts as the identity on $L' \otimes_{\mathcal{A}} \mathcal{A}/(\phi_v)$, for almost all $v \in \mathcal{C}$. It follows from Theorem 5.1 that the minimal submodule $\mathcal{W}'$ that contains L' becomes trivial over $K(x)$. Since the tensor category generated by $\mathcal{W}'_{K(x)}$ is contained in $\langle\mathcal{M}_{K(x)}\rangle^{\otimes}$, we have a functorial surjective group morphism

$$Gal(\mathcal{M}_{K(x)}, \eta_{K(x)}) \longrightarrow Gal(\mathcal{W}'_{K(x)}, \eta_{K(x)}) = \{1\}.$$

We conclude that $Gal(\mathcal{M}_{K(x)}, \eta_{K(x)})$ acts trivially over $\mathcal{W}'_{K(x)}$, and therefore that $Gal(\mathcal{M}_{K(x)}, \eta_{K(x)})$ is contained in G. □

COROLLARY 6.10. *Theorem 5.1 and Theorem 6.6 are equivalent.*

PROOF. We have seen in the proof above that Theorem 5.1 implies Theorem 6.6. Corollary 6.9 gives the opposite implication. □

6.1.3. Finite intrinsic Galois groups. We deduce from Theorem 6.6 the following description of a finite intrinsic Galois group:

COROLLARY 6.11. *The following facts are equivalent:*

(1) *There exists a positive integer r such that the q-difference module $\mathcal{M} = (M, \Sigma_q)$ becomes trivial as a $\widetilde{q}$-difference module over $K(\widetilde{q}, t)$, with $\widetilde{q}^r = q$, $t^r = x$.*
(2) *There exists a positive integer r such that, for almost all $v \in \mathcal{C}$, the morphism $\Sigma_q^{\kappa_v r}$ induces the identity on $M \otimes_{\mathcal{A}} \mathcal{A}/(\phi_v)$.*
(3) *There exists a q-difference field extension $\mathcal{F}/K(x)$ of finite degree such that $\mathcal{M}$ becomes trivial over $\mathcal{F}$.*
(4) *The (intrinsic) Galois group of $\mathcal{M}$ is finite.*

In particular, if $Gal(\mathcal{M}_{K(x)}, \eta_{K(x)})$ is finite, it is necessarily cyclic (of order r, if one chooses r minimal in the assertions above).

PROOF. The equivalence "1 $\Leftrightarrow$ 2" follows from Theorem 5.1 applied to the $\widetilde{q}$-difference module $(M \otimes K(\widetilde{q}, t), \Sigma_q \otimes \sigma_{\widetilde{q}})$, over the field $K(\widetilde{q}, t)$.

If the intrinsic Galois group is finite, the reduction modulo ϕ_v of $\Sigma_q^{\kappa_v}$ must be a cyclic operator of order dividing the cardinality of $Gal(\mathcal{M}_{K(x)}, \eta_{K(x)})$. So we have proved that "4 $\Rightarrow$ 2". On the other hand, assertion 2 implies, by Theorem 6.6, that there exists a basis of $M_{K(x)}$ such that the representation of $Gal(\mathcal{M}_{K(x)}, \eta_{K(x)})$ is given by the group of diagonal matrices, whose diagonal entries are r-th roots of unity.

Of course, assertion 1 implies assertion 3. The inverse implication follows from the Corollary 1.14, applied to a cyclic basis of $\mathcal{M}_{K(x)}$. $\square$

6.2. Intrinsic Galois group of a q-difference module over $\mathbb{C}(x)$, for $q \neq 0, 1$

We deduce from the previous section a curvature characterization of the intrinsic Galois group of a q-difference module over $\mathbb{C}(x)$, for $q \in \mathbb{C} \smallsetminus \{0, 1\}$.[2]

Let $\mathcal{M}_{\mathbb{C}(x)} = (M_{\mathbb{C}(x)}, \Sigma_q)$ be a q-difference module over $\mathbb{C}(x)$. We can consider a finitely generated extension K of $\mathbb{Q}$ such that there exists a q-difference module $\mathcal{M}_{K(x)} = (M_{K(x)}, \Sigma_q)$ satisfying $\mathcal{M}_{\mathbb{C}(x)} = \mathcal{M}_{K(x)} \otimes_{K(x)} \mathbb{C}(x)$.

With an abuse of language, Theorem 5.1 can be rephrased as:

THEOREM 6.12. *The q-difference module $\mathcal{M}_{\mathbb{C}(x)} = (M_{\mathbb{C}(x)}, \Sigma_q)$ is trivial if and only if there exists a finitely generated extension K of $\mathbb{Q}$, a set of places $\mathcal{C}$ as in Chapter 5 and a q-difference module $\mathcal{M}_{K(x)}$ such that $\mathcal{M}_{\mathbb{C}(x)} \cong \mathcal{M}_{K(x)} \otimes_{K(x)} \mathbb{C}(x)$ and $\mathcal{M}_{K(x)}$ has zero v-curvature, for almost all $v \in \mathcal{C}$.*

We can of course define as in the previous sections an intrinsic Galois group $Gal(\mathcal{M}_{\mathbb{C}(x)}, \eta_{\mathbb{C}(x)})$. A noetherianity argument, that we have already used several times, shows the following:

PROPOSITION 6.13. *In the notation above we have:*

$$Gal(\mathcal{M}_{\mathbb{C}(x)}, \eta_{\mathbb{C}(x)}) \subset Gal(\mathcal{M}_{K(x)}, \eta_{K(x)}) \otimes_{K(x)} \mathbb{C}(x).$$

Moreover there exists a finitely generated extension K' of K such that

$$Gal(\mathcal{M}_{K(x)} \otimes_{K(x)} K'(x), \eta_{K'(x)}) \otimes_{K'(x)} \mathbb{C}(x) \cong Gal(\mathcal{M}_{\mathbb{C}(x)}, \eta_{\mathbb{C}(x)}).$$

Choosing K large enough, we can assume that $K = K'$, which we will do implicitly in the following informal statement. We can deduce from Theorem 6.12:

THEOREM 6.14. *The intrinsic Galois group $Gal(\mathcal{M}_{\mathbb{C}(x)}, \eta_{\mathbb{C}(x)})$ is the smallest algebraic subgroup of $\mathrm{GL}(M_{\mathbb{C}(x)})$ that contains the v-curvature of the q-difference module $\mathcal{M}_{K(x)}$, for K large enough and for almost all $v \in \mathcal{C}$.*

[2]All the statements in this subsection remain true if one replace $\mathbb{C}$ with any field of characteristic zero.

CHAPTER 7

The parametrized intrinsic Galois group

7.1. Parametrized intrinsic Galois groups

We recall some facts from Chapter 2. Let $\mathcal{F}$ be a q-difference-differential field of *characteristic zero*, that is, an extension of $K(x)$ equipped with an extension of the q-difference operator σ_q and a derivation ∂ commuting with σ_q. For instance, the q-difference-differential field $(K(x), \sigma_q, x\frac{d}{dx})$ satisfies these assumptions.

We can define an action of the derivation ∂ on the category $Diff(\mathcal{F}, \sigma_q)$, twisting the q-difference modules with the right $\mathcal{F}$-module $\mathcal{F}[\partial]_{\leq 1}$ of differential operators of order less or equal than one. We recall that the structure of right $\mathcal{F}$-module on $\mathcal{F}[\partial]_{\leq 1}$ is defined via the Leibniz rule, i.e.,

$$\partial.\lambda = \lambda\partial + \partial(\lambda), \text{ for any } \lambda \in \mathcal{F}.$$

Let V be an $\mathcal{F}$-vector space. We denote by $F_\partial(V)$ the tensor product of the right $\mathcal{F}$-module $\mathcal{F}[\partial]_{\leq 1}$ with the left $\mathcal{F}$-module V:

$$F_\partial(V) := \mathcal{F}[\partial]_{\leq 1} \otimes_{\mathcal{F}} V.$$

We will write v for $1 \otimes v \in F_\partial(V)$ and $\partial(v)$ for $\partial \otimes v \in F_\partial(V)$, so that $av + b\partial(v) := (a + b\partial) \otimes v$, for any $v \in V$ and $a + b\partial \in \mathcal{F}[\partial]_{\leq 1}$. We endow $F_\partial(V)$ with a left $\mathcal{F}$-module structure such that if $\lambda \in \mathcal{F}$:

$$\lambda\partial(v) = \partial(\lambda v) - \partial(\lambda)v, \text{ for all } v \in V,$$

which means that $\lambda(\partial \otimes v) = \partial \otimes \lambda v - 1 \otimes \partial(\lambda)v$. This construction comes out of the Leibniz rule $\partial(\lambda v) = \lambda\partial(v) + \partial(\lambda)v$, which justifies the notation introduced above.

Definition 7.1. The prolongation functor F_∂ is defined on the category of $\mathcal{F}$-vector spaces as follows. It associates to any object V the $\mathcal{F}$-vector space $F_\partial(V)$. If $f : V \longrightarrow W$ is a morphism of $\mathcal{F}$-vector space then we define

$$F_\partial(f) : F_\partial(V) \to F_\partial(W),$$

setting $F_\partial(f)(\partial^i(v)) = \partial^i(f(v))$, for any $i = 0, 1$ and any $v \in V$ (using the convention that ∂^0 is the identity).

The prolongation functor F_∂ restricts to a functor from the category $Diff(\mathcal{F}, \sigma_q)$ to itself in the following way:

(1) If $\mathcal{M}_{\mathcal{F}} := (M_{\mathcal{F}}, \Sigma_q)$ is an object of $Diff(\mathcal{F}, \sigma_q)$ then $F_\partial(\mathcal{M}_{\mathcal{F}})$ is the q-difference module, whose underlying $\mathcal{F}$-vector space is $F_\partial(M_{\mathcal{F}}) = \mathcal{F}[\partial]_{\leq 1} \otimes M_{\mathcal{F}}$, as above, equipped with the q-invertible σ_q-semilinear operator defined by $\Sigma_q(\partial^i(m)) := \partial^i(\Sigma_q(m))$ for $i = 0, 1$.
(2) If $f \in Hom(\mathcal{M}_{\mathcal{F}}, \mathcal{N}_{\mathcal{F}})$ then $F_\partial(f)$ is defined in the same way as for $\mathcal{F}$-vector spaces.

Remark 7.2. This formal definition comes from a simple and concrete idea. Let $\mathcal{M}_{\mathcal{F}}$ be an object of $Diff(\mathcal{F}, \sigma_q)$. We fix a basis $\underline{e}$ of $\mathcal{M}_{\mathcal{F}}$ over $\mathcal{F}$ such that

$\Sigma_q\underline{e} = \underline{e}A$. Then $(\underline{e}, \partial(\underline{e}))$ is a basis of $F_\partial(M_{\mathcal{F}})$ and

$$\Sigma_q(\underline{e}, \partial(\underline{e})) = (\underline{e}, \partial(\underline{e})) \begin{pmatrix} A & \partial A \\ 0 & A \end{pmatrix}.$$

In other terms, if $\sigma_q(Y) = A^{-1}Y$ is a q-difference system associated to $\mathcal{M}_{\mathcal{F}}$ with respect to a fixed basis $\underline{e}$, the q-difference system associated to $F_\partial(\mathcal{M}_{\mathcal{F}})$ with respect to the basis $\underline{e}, \partial(\underline{e})$ is:

$$\sigma_q(Z) = \begin{pmatrix} A^{-1} & \partial(A^{-1}) \\ 0 & A^{-1} \end{pmatrix} Z = \begin{pmatrix} A & \partial A \\ 0 & A \end{pmatrix}^{-1} Z.$$

If Y is a solution of $\sigma_q(Y) = A^{-1}Y$ in some q-difference-differential extension of $\mathcal{F}$ then we have:

$$\sigma_q \begin{pmatrix} \partial Y & Y \\ Y & 0 \end{pmatrix} = \begin{pmatrix} A^{-1} & \partial(A^{-1}) \\ 0 & A^{-1} \end{pmatrix} \begin{pmatrix} \partial Y & Y \\ Y & 0 \end{pmatrix},$$

in fact the commutation of σ_q and ∂ implies:

$$\sigma_q(\partial Y) = \partial(\sigma_q Y) = \partial(A^{-1}\, Y) = A^{-1}\, \partial Y + \partial(A^{-1})\, Y.$$

Let V be a finite dimensional $\mathcal{F}$-vector space. We denote by $Constr_{\mathcal{F}}^{\partial}(V)$ the smallest family of finite dimensional $\mathcal{F}$-vector spaces containing V and closed with respect to the constructions of linear algebra (i.e., direct sums, tensor product, symmetric and antisymmetric product, dual. See §1.1.1) and the functor F_∂. We will say that an element $Constr_{\mathcal{F}}^{\partial}(V)$ is a construction of differential linear algebra of V. By functoriality, the linear algebraic group $\mathrm{GL}(V)$ operates on $Constr_{\mathcal{F}}^{\partial}(V)$. For example $g \in \mathrm{GL}(V)$ acts on $F_\partial(V)$ through $g(\partial^i(v)) = \partial^i(g(v))$, for $i = 0, 1$.

If we start with a q-difference module $\mathcal{M}_{\mathcal{F}} = (M_{\mathcal{F}}, \Sigma_q)$ over $\mathcal{F}$, then every object of $Constr_{\mathcal{F}}^{\partial}(M_{\mathcal{F}})$ has a natural structure of q-difference module (see also §1.1.1). We will denote $Constr_{\mathcal{F}}^{\partial}(\mathcal{M}_{\mathcal{F}})$ the family of q-difference modules obtained in this way.

Definition 7.3. We call parametrized intrinsic Galois group of an object $\mathcal{M}_{\mathcal{F}} = (M_{\mathcal{F}}, \Sigma_q)$ of $Diff(\mathcal{F}, \sigma_q)$ the group defined by

$$\begin{aligned} Gal^{\partial}(\mathcal{M}_{\mathcal{F}}, \eta_{\mathcal{F}}) := \Big\{ & g \in \mathrm{GL}(M_{\mathcal{F}}) : g(N_{\mathcal{F}}) \subset N_{\mathcal{F}} \text{ for all sub-}q\text{-difference module} \\ & \mathcal{N}_{\mathcal{F}} = (N_{\mathcal{F}}, \Sigma_q) \text{ contained in an object of } Constr_{\mathcal{F}}^{\partial}(\mathcal{M}_{\mathcal{F}}) \Big\} \subset \mathrm{GL}(M_{\mathcal{F}}). \end{aligned}$$

Similarly to §6, one has to understand the definition above in a functorial sense. More precisely, $Gal^{\partial}(\mathcal{M}_{\mathcal{F}}, \eta_{\mathcal{F}})$ is a functor from the category of ∂-$\mathcal{F}$-algebras to the category of groups, that associates to any $\mathcal{F}$-algebra S, the subgroup of $\mathrm{GL}(M_{\mathcal{F}}) \otimes S$ that stabilizes $N_{\mathcal{F}} \otimes S$, for all the q-difference submodules $\mathcal{N}_{\mathcal{F}}$ over $\mathcal{F}$ of any object in $Constr_{\mathcal{F}}^{\partial}(\mathcal{M}_{\mathcal{F}})$. The proposition below shows that this functor is representable and thus defines a differential algebraic group over $\mathcal{F}$.

Proposition 7.4. *The group $Gal^{\partial}(\mathcal{M}_{\mathcal{F}}, \eta_{\mathcal{F}})$ is a reduced differential $\mathcal{F}$-subgroup of* $\mathrm{GL}(M_{\mathcal{F}})$.

Remark 7.5. We recall that in the notations of §2.1.2, the ring of differential coordinates $\mathcal{F}\left\{Y, \frac{1}{\det Y}\right\}_\partial$ of $\mathrm{GL}(M_{\mathcal{F}}) = \mathrm{GL}_\nu(\mathcal{F})$ for some ν is defined as follows. We denote by $\mathcal{F}\{Y\}_\partial$ the ring of differential polynomials in the ∂-differential indeterminates $Y = \{y_{i,j} : i, j = 1, \ldots, \nu\}$. The differential Hopf-algebra $\mathcal{F}\left\{Y, \frac{1}{\det Y}\right\}_\partial$ of $\mathrm{GL}_\nu(\mathcal{F})$ is obtained from $\mathcal{F}\{Y\}_\partial$ by inverting $\det Y$. Now, Proposition 7.4 says that the functor $Gal^{\partial}(\mathcal{M}_{\mathcal{F}}, \eta_{\mathcal{F}})$ is represented by a ∂-$\mathcal{F}$-algebra, quotient of $\mathcal{F}\left\{Y, \frac{1}{\det Y}\right\}_\partial$ by some radical ∂-ideal.

PROOF. In the notations of §2.1.2, let us denote by $D : \mathcal{F}$-algebras $\rightarrow \partial$-$\mathcal{F}$-algebras the left adjoint of the forgetful functor that attach to any ∂-$\mathcal{F}$-algebras its underlying $\mathcal{F}$-algebra. By [**DG70**, II.1.36], the functor $Gal^{\partial}(\mathcal{M}_{\mathcal{F}}, \eta_{\mathcal{F}}) \circ D$ is representable by a $\mathcal{F}$-algebra $\mathcal{S}$. Then, $D(\mathcal{S})$ represents $Gal^{\partial}(\mathcal{M}_{\mathcal{F}}, \eta_{\mathcal{F}})$. Moreover, since $Gal^{\partial}(\mathcal{M}_{\mathcal{F}}, \eta_{\mathcal{F}})$ is a group functor, $D(\mathcal{S})$ is an Hopf algebra over a field of characteristic zero. Then, $D(\mathcal{S})$ is automatically reduced by Cartier's theorem (see [**Wat79a**, §11.4]). □

Following the approach of Chapter 2, we denote by $\langle \mathcal{M}_{\mathcal{F}} \rangle^{\otimes, \partial}$ the full abelian tensor subcategory of $Diff(\mathcal{F}, \sigma_q)$ generated by $\mathcal{M}_{\mathcal{F}}$ and closed under the prolongation functor F_{∂}. Then $\langle \mathcal{M}_{\mathcal{F}} \rangle^{\otimes, \partial}$ is naturally a differential tannakian category, when equipped with the forgetful functor

$$\eta_{\mathcal{F}} : \langle \mathcal{M}_{K(x)} \rangle^{\otimes, \partial} \longrightarrow \{\mathcal{F}\text{-vector spaces}\}.$$

The functor $Aut^{\otimes, \partial}(\eta_{\mathcal{F}})$ defined over the category of $\mathcal{F}$-algebras coincides with $Gal^{\partial}(\mathcal{M}_{\mathcal{F}}, \eta_{\mathcal{F}})$. Moreover, we have the following result.

PROPOSITION 7.6. *Let $\mathcal{F}$ be a q-difference differential field and let $\mathcal{M}_{\mathcal{F}}$ be a q-difference module over $\mathcal{F}$. Let $\omega : \langle \mathcal{M}_{\mathcal{F}} \rangle^{\otimes, \partial} \rightarrow Vect_{\mathcal{F}^{\sigma_q}}$ be a neutral differential fiber functor. Then, the differential algebraic groups $Aut^{\otimes, \partial}(\omega) \otimes \mathcal{F}$ and $Gal^{\partial}(\mathcal{M}_{\mathcal{F}}, \eta_{\mathcal{F}})$, defined over $\mathcal{F}$, are isomorphic over the differential closure $\widetilde{\mathcal{F}}$ of $\mathcal{F}$.*

PROOF. By [**GGO13**, Proposition 4.28], the affine differential scheme $Isom^{\otimes, \partial}(\omega \otimes_{\mathcal{F}^{\sigma_q}} \mathcal{F}, \eta_{\mathcal{F}})$ is representable by a non-zero differential finitely generated $\mathcal{F}$-algebra. Since $\widetilde{\mathcal{F}}$ is differentially closed, the previous differential algebraic scheme has a point in $\widetilde{\mathcal{F}}$. This ends the proof. □

Once again, we will come back on this point of view in Part 4.

For further reference, we recall (a particular case of) the Ritt-Raudenbush theorem (*cf.* [**Kap57**, Theorem 7.1]):

THEOREM 7.7. *Let $(\mathcal{F}, \partial)$ be a differential field of characteristic zero. If R is a reduced finitely generated ∂-$\mathcal{F}$-algebra then R is ∂-noetherian.*

This means that any ascending chain of *radical* differential ideals (i.e., radical ∂-stable ideals) is stationary or equivalently that every radical ∂-ideal has a finite set of generators as radical ∂-ideal (which in general does not mean that it is a finitely generated ideal). Theorem 7.7 combined with Proposition 7.4 asserts that the parametrized intrinsic Galois group as well as any $\mathrm{GL}_{\nu}(\mathcal{F})$ are ∂-noetherian.

The ∂-noetherianity of $\mathrm{GL}_{\nu}(\mathcal{F})$ implies the following:

COROLLARY 7.8. *The parametrized intrinsic Galois group $Gal^{\partial}(\mathcal{M}_{\mathcal{F}}, \eta_{\mathcal{F}})$ can be defined as the stabilizer of a line in a construction of differential algebra of $\mathcal{M}_{\mathcal{F}}$. This line can be chosen so that it is also a q-difference module in the category $\langle \mathcal{M}_{\mathcal{F}} \rangle^{\otimes, \partial}$.*

PROOF. Since $\mathrm{GL}(M_{\mathcal{F}})$ is ∂-noetherian, any descending chain of reduced differential sub-schemes in $\mathrm{GL}(M_{\mathcal{F}})$ is stationary. Then, let $\left\{\mathcal{W}^{(i)}; i \in I_h\right\}_h$ be an ascending chain of finite sets of q-difference submodules contained in some elements of $Constr^{\partial}(\mathcal{M}_{K(x)})$ so that any q-difference submodule contained in a construction of linear differential algebra is contained in some $\left\{\mathcal{W}^{(i)}; i \in I_h\right\}$. Let $\mathcal{G}_h$ be the differential subgroup of $\mathrm{GL}(M_{\mathcal{F}})$ defined as the stabilizer of $\{\mathcal{W}^{(i)}; i \in I_h\}$. By Cartier's theorem, the $\mathcal{G}_h$ are reduced (see previous proposition). Then, the descending chain of differential algebraic subgroups $\mathcal{G}_h$ of $\mathrm{GL}(M_{\mathcal{F}})$ is stationary. This proves that $Gal^{\partial}(\mathcal{M}_{\mathcal{F}}, \eta_{\mathcal{F}})$ is the stabilizer of a finite number of q-difference submodules $\mathcal{W}^{(i)}$, $i \in I$, contained in some elements of $Constr^{\partial}(\mathcal{M}_{K(x)})$. It follows from a standard argument of linear algebra that $Gal^{\partial}(\mathcal{M}_{\mathcal{F}}, \eta_{\mathcal{F}})$ is the stabilizer of the maximal exterior power of the direct sum of the $\mathcal{W}^{(i)}$'s (see Remark 6.3). $\square$

Let $Gal(\mathcal{M}_{\mathcal{F}}, \eta_{\mathcal{F}})$ be the intrinsic Galois group defined in the previous chapter. We have the following inclusion, that we will characterize in a more precise way in the next pages:

LEMMA 7.9. *Let $\mathcal{M}_{\mathcal{F}}$ be an object of $Diff(\mathcal{F}, \sigma_q)$. The following inclusion of differential algebraic groups holds*

$$Gal^{\partial}(\mathcal{M}_{\mathcal{F}}, \eta_{\mathcal{F}}) \subset Gal(\mathcal{M}_{\mathcal{F}}, \eta_{\mathcal{F}}).$$

REMARK 7.10. The inclusion above means that, for all ∂-$\mathcal{F}$-algebra S, we have $Gal^{\partial}(\mathcal{M}_{\mathcal{F}}, \eta_{\mathcal{F}})(S) \subset Gal(\mathcal{M}_{\mathcal{F}}, \eta_{\mathcal{F}})(\eta(S))$, where $\eta(S)$ is the underlying $\mathcal{F}$-algebra of S. We would like to underline the fact that differential algebraic groups are not algebraic groups, while algebraic groups may be considered as differential algebraic groups (whose defining equations are polynomials-see also §2.1.2). In particular, the parametrized intrinsic Galois group is not an algebraic subgroup of the intrinsic Galois group but only a differential algebraic subgroup. Later, for $\mathcal{F} = K(x)$, we will prove that $Gal^{\partial}(\mathcal{M}_{\mathcal{F}}, \eta_{\mathcal{F}})$ is actually Zariski dense in $Gal(\mathcal{M}_{\mathcal{F}}, \eta_{\mathcal{F}})$.

PROOF. We recall, that the algebraic group $Gal(\mathcal{M}_{\mathcal{F}}, \eta_{\mathcal{F}})$ is defined as the stabilizer in $\mathrm{GL}(M_{\mathcal{F}})$ of all the subobjects contained in a construction of linear algebra of $\mathcal{M}_{\mathcal{F}}$. Because the list of subobjects contained in a construction of differential linear algebra of $\mathcal{M}_{\mathcal{F}}$ includes those contained in a construction of linear algebra of $\mathcal{M}_{\mathcal{F}}$, we get the claimed inclusion. $\square$

7.2. Characterization of the parametrized intrinsic Galois group by curvatures

From now on we focus on the special case $\mathcal{F} = K(x)$, where K is a finitely generated extension of $\mathbb{Q}$. We endow $K(x)$ with the derivation $\partial := x\frac{d}{dx}$, that commutes with σ_q. We refer to Chapter 5 for notations.

Let $\mathcal{M}_{K(x)} = (M_{K(x)}, \Sigma_q)$ be a q-difference module. The differential version of Chevalley's theorem (*cf.* [**Cas72**, Proposition 14], [**MO10**, Theorem 5.1]) implies that any closed differential subgroup G of $\mathrm{GL}(M_{K(x)})$ can be defined as the stabilizer of some line $L_{K(x)}$ contained in an object $\mathcal{W}_{K(x)}$ of $\langle \mathcal{M}_{K(x)} \rangle^{\otimes, \partial}$. Because the derivation ∂ does not modify the set of poles of a rational function, the lattice $\mathcal{M}$ of $\mathcal{M}_{K(x)}$ determines a Σ_q-stable $\mathcal{A}$-lattice of all the objects of $\langle \mathcal{M}_{K(x)} \rangle^{\otimes, \partial}$. In particular, the $\mathcal{A}$-lattice M of $M_{K(x)}$ determines an $\mathcal{A}$-lattice L of $L_{K(x)}$ and an $\mathcal{A}$-lattice W of $W_{K(x)}$. The latter is the underlying space of a q-difference module $\mathcal{W} = (W, \Sigma_q)$ over $\mathcal{A}$.

DEFINITION 7.11. Let $\widetilde{\mathcal{C}}$ be a non-empty cofinite subset of $\mathcal{C}$ and $(\Lambda_v)_{v \in \widetilde{\mathcal{C}}}$ be a family of $\mathcal{A}/(\phi_v)$-linear operators acting on $M \otimes_{\mathcal{A}} \mathcal{A}/(\phi_v)$. We say that a differential algebraic group $G = Stab(L_{K(x)})$ over $K(x)$ contains the operators Λ_v modulo ϕ_v, for almost all $v \in \mathcal{C}$, if for almost all (i.e. for almost all and at least one) $v \in \widetilde{\mathcal{C}}$ the operator Λ_v stabilizes $L \otimes_{\mathcal{A}} \mathcal{A}/(\phi_v)$ inside $W \otimes_{\mathcal{A}} \mathcal{A}/(\phi_v)$:

$$\Lambda_v \in Stab_{\mathcal{A}/(\phi_v)}(L \otimes_{\mathcal{A}} \mathcal{A}/(\phi_v)).$$

REMARK 7.12. The differential Chevalley's theorem and the ∂-noetherianity of $\mathrm{GL}(M_{K(x)})$ imply that the notions of a differential algebraic group containing the operators Λ_v modulo ϕ_v, for almost all $v \in \mathcal{C}$, and the smallest closed differential subgroup of $\mathrm{GL}(M_{K(x)})$ containing the operators Λ_v modulo ϕ_v, for almost all $v \in \mathcal{C}$, are well defined. In particular they are independent of the choice of $\mathcal{A}$, $\mathcal{M}$ and $L_{K(x)}$ (See [**DV02**, 10.1.2] and Remark 6.5).

The main result of this section is the following:

THEOREM 7.13. *The differential algebraic group $Gal^{\partial}(\mathcal{M}_{K(x)}, \eta_{K(x)})$ is the smallest closed differential algebraic subgroup of* $\mathrm{GL}(M_{K(x)})$ *that contains the operators $\Sigma_q^{\kappa_v}$ modulo ϕ_v, for almost all $v \in \mathcal{C}$.*

PROOF. The lemmas below plus the differential Chevalley theorem allow to prove Theorem 7.13 in exactly the same way as Theorem 6.6. □

LEMMA 7.14. *The differential algebraic group $Gal^{\partial}(\mathcal{M}_{K(x)}, \eta_{K(x)})$ contains the operators $\Sigma_q^{\kappa_v}$ modulo ϕ_v, for almost all $v \in \mathcal{C}$.*

PROOF. The statement follows immediately from the fact that $Gal^{\partial}(\mathcal{M}_{K(x)}, \eta_{K(x)})$ can be defined as the stabilizer of a rank one q-difference module in $\langle \mathcal{M}_{K(x)} \rangle^{\otimes, \partial}$, which is *a fortiori* stable under the action of $\Sigma_q^{\kappa_v}$. □

LEMMA 7.15. *$Gal^{\partial}(\mathcal{M}_{K(x)}, \eta_{K(x)}) = \{1\}$ if and only if $\mathcal{M}_{K(x)}$ is a trivial q-difference module.*

PROOF. The proof is analogous to the proof of Corollary 6.9. It suffices to replace $\langle \mathcal{M}_{K(x)} \rangle^{\otimes}$ with $\langle \mathcal{M}_{K(x)} \rangle^{\otimes, \partial}$. □

We obtain the following:

COROLLARY 7.16. *The parametrized intrinsic Galois group $Gal^{\partial}(\mathcal{M}_{K(x)}, \eta_{K(x)})$ is a Zariski dense subset of the algebraic intrinsic Galois group $Gal(\mathcal{M}_{K(x)}, \eta_{K(x)})$.*

PROOF. We have seen in Lemma 7.9 that $Gal^{\partial}(\mathcal{M}_{K(x)}, \eta_{K(x)})$ is a subgroup of $Gal(\mathcal{M}_{K(x)}, \eta_{K(x)})$. By Theorem 7.13 (resp. Theorem 6.6) we have that the intrinsic Galois group $Gal^{\partial}(\mathcal{M}_{K(x)}, \eta_{K(x)})$ (resp. $Gal(\mathcal{M}_{K(x)}, \eta_{K(x)})$) is the smallest closed differential subgroup (resp. closed algebraic group) of $\mathrm{GL}(M_{K(x)})$ that contains the operators $\Sigma_q^{\kappa_v}$ modulo ϕ_v, for almost all $v \in \mathcal{C}$. This immediately implies the Zariski density. □

EXAMPLE 7.17. The logarithm is solution both a q-difference and a differential system:

$$Y(qx) = \begin{pmatrix} 1 & \log q \\ 0 & 1 \end{pmatrix} Y(x), \qquad \partial Y(x) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} Y(x).$$

It is easy to verify that the two systems are integrable in the sense that $\partial \sigma_q Y(x) = \sigma_q \partial Y(x)$ (and therefore that the induced condition on the matrices of the systems is verified).

By iterating the q-difference system for any $n \in \mathbb{Z}_{>0}$ we obtain:

$$Y(q^n x) = \begin{pmatrix} 1 & n \log q \\ 0 & 1 \end{pmatrix} Y(x).$$

This implies that the parametrized intrinsic Galois group is the subgroup of $\mathbb{G}_{a,K(x)}$ defined by the equation $\partial y = 0$. This coincides with the group $\mathbb{G}_{a,K}$, which is coherent with the integrability criteria in [**HS08**]. For more precise comparison results with the theory developed in [**HS08**], we refer to Part 4.

7.3. Parametrized intrinsic Galois group of a q-difference module over $\mathbb{C}(x)$, for $q \neq 0, 1$

We conclude with some remarks on complex q-difference modules. Let $\mathcal{M}_{\mathbb{C}(x)} = (M_{\mathbb{C}(x)}, \Sigma_q)$ be a q-difference module over $\mathbb{C}(x)$. We can consider a finitely generated extension of K of $\mathbb{Q}$ such that there exists a q-difference module $\mathcal{M}_{K(x)} = (M_{K(x)}, \Sigma_q)$ satisfying $\mathcal{M}_{\mathbb{C}(x)} = \mathcal{M}_{K(x)} \otimes_{K(x)} \mathbb{C}(x)$. We can of course define, as above, two parametrized intrinsic Galois groups, $Gal^{\partial}(\mathcal{M}_{K(x)}, \eta_{K(x)})$ and $Gal^{\partial}(\mathcal{M}_{\mathbb{C}(x)}, \eta_{\mathbb{C}(x)})$.

A (differential) noetherianity argument, that we have already used several times, on the submodules stabilized by those groups shows the following:

PROPOSITION 7.18. *In the notations above, we have:*

$$Gal^{\partial}(\mathcal{M}_{\mathbb{C}(x)}, \eta_{\mathbb{C}(x)}) \subset Gal^{\partial}(\mathcal{M}_{K(x)}, \eta_{K(x)}) \otimes_{K(x)} \mathbb{C}(x).$$

Moreover there exists a finitely generated extension K' of K such that

$$Gal^{\partial}(\mathcal{M}_{K(x)} \otimes_{K(x)} K'(x), \eta_{K'(x)}) \otimes_{K'(x)} \mathbb{C}(x) \cong Gal^{\partial}(\mathcal{M}_{\mathbb{C}(x)}, \eta_{\mathbb{C}(x)}).$$

We can informally rephrase Theorem 7.13 in the following way:

THEOREM 7.19. *The parametrized intrinsic Galois group $Gal^{\partial}(\mathcal{M}_{\mathbb{C}(x)}, \eta_{\mathbb{C}(x)})$ is the smallest differential algebraic subgroup of $\mathrm{GL}_{\nu}(M_{\mathbb{C}(x)})$ that contains a non-empty cofinite set of curvatures of the q-difference module $\mathcal{M}_{K'(x)}$.*

7.4. The example of the Jacobi Theta function

Consider the Jacobi Theta function

$$\Theta(x) = \sum_{n \in \mathbb{Z}} q^{-n(n-1)/2} x^n,$$

which is solution of the q-difference equation

$$\Theta(qx) = qx\Theta(x).$$

Iterating the equation, one proves that Θ satisfies $y(q^n x) = q^{n(n+1)/2} x^n y(x)$, for any $n \geq 0$. Therefore we immediately deduce that the intrinsic Galois group of the rank one q-difference module $\mathcal{M}_{\Theta} = (K(x).\Theta, \Sigma_q)$, with

$$\begin{array}{rccc} \Sigma_q & : K(x).\Theta & \longrightarrow & K(x).\Theta \\ & f(x)\Theta & \longmapsto & f(qx)qx\Theta \end{array},$$

is the whole multiplicative group $\mathbb{G}_{m,K(x)}$. As far as the parametrized intrinsic Galois group is concerned we have:

PROPOSITION 7.20. *The parametrized intrinsic Galois group $Gal^{\partial}\left(\mathcal{M}_{\Theta}, \eta_{K(x)}\right)$ is defined by $\partial(\partial(y)/y) = 0$.*

PROOF. For almost any $v \in \mathcal{C}$, the reduction modulo ϕ_v of $q^{\kappa_v(\kappa_v+1)/2} x^{\kappa_v}$ is the monomial x^{κ_v}, which satisfies the equation $\partial\left(\frac{\partial x^{\kappa_v}}{x^{\kappa_v}}\right) = 0$. This means that parametrized intrinsic Galois group $Gal^{\partial}\left(\mathcal{M}_{\Theta}, \eta_{K(x)}\right)$ is a subgroup of the differential algebraic group defined by $\partial\left(\frac{\partial y}{y}\right) = 0$. In other words, the logarithmic derivative

$$\begin{array}{ccc} \mathbb{G}_m & \longrightarrow & \mathbb{G}_a \\ y & \longmapsto & \frac{\partial y}{y} \end{array}$$

sends $Gal^{\partial}\left(\mathcal{M}_{\Theta}, \eta_{K(x)}\right)$ into a subgroup of the additive group $\mathbb{G}_{a,K(x)}$ defined by the equation $\partial z = 0$. This is nothing else that $\mathbb{G}_{a,K}$, whose proper subgroup is only $\{0\}$. If the image by the logarithmic derivative of $Gal^{\partial}\left(\mathcal{M}_{\Theta}, \eta_{K(x)}\right)$ were $\{0\}$, then the curvatures should be constant with respect to ∂. It is not the case, which ends the proof. □

Let us consider a norm $| \ |$ on K such that $|q| \neq 1$. The differential dimension of the subgroup $\partial\left(\frac{\partial y}{y}\right) = 0$ is zero. We will show in Part 4 that this means that Θ is differentially algebraic over the field of rational functions $\widetilde{C}_E(x)$ with coefficients in the differential closure $\widetilde{C}_E$ of the elliptic function over $K^*/q^{\mathbb{Z}}$. In fact, the function Θ satisfies

$$\sigma_q\left(\frac{\partial\Theta}{\Theta}\right)=\frac{\partial\Theta}{\Theta}+1,$$

which implies that $\partial\left(\frac{\partial\Theta}{\Theta}\right)$ is an elliptic function. Since the Weierstrass function is differentially algebraic over $K(x)$, the Jacobi Theta function is also differentially algebraic over $K(x)$.

Notice that, if q is transcendental over $\mathbb{Q}$, the derivation $\frac{d}{dq}$ naturally comes into the picture. Since it intertwines with σ_q in a relatively complicate way, the study of this situation requires a specific approach. See [**DVH11**].

Part 4

Comparison with other Galois theories for linear q-difference equations

CHAPTER 8

Meromorphic solutions and comparison theorems

In the first section of this chapter, we remind that a q-difference system with coefficients in $K(x)$, where K is a finitely generated extension of $\mathbb{Q}$, together with a norm $|.|$, such that $|q| \neq 1$, admits a basis of meromorphic solutions with respect to $|.|$. These solutions are linearly independent over the field of elliptic functions over the torus $K^*/q^{\mathbb{Z}}$, denoted C_E. Letting $\mathcal{M}_{K(x)}$ be a q-difference module corresponding to the initial q-difference system, this allows us to construct a neutral differential fiber functor for $\langle \mathcal{M}_{K(x)} \otimes C_E(x)\rangle^{\otimes,\partial}$ (see §2.2 for notations and definitions). The second section of this chapter is devoted to the comparison of the distinct parametrized Galois groups attached to the q-difference module $\mathcal{M}_{K(x)}$. Corollary 8.9, together with Proposition 8.10, proves the differential analogue of [**CHS08**, Theorem 3.1], i.e., that all the differential algebraic group schemes considered are forms of the same differential algebraic group scheme. As a corollary, we get that the differential algebraic relations satisfied by the meromorphic solutions of the q-difference system are encoded in the differential algebraic relations satisfied by the curvatures (see Corollary 8.13).

8.1. Meromorphic solutions and differential fiber functor

8.1.1. Classical functions as solutions of q-difference equations. For a fixed complex number q, with $|q| \neq 1$, Praagman proves in [**Pra86**] that every linear q-difference equation with meromorphic coefficients over $\mathbb{C}^*$ admits a basis of solutions, meromorphic over $\mathbb{C}^*$, linearly independent over the field C_E of elliptic functions, i.e., the field of meromorphic functions over the elliptic curve $E := \mathbb{C}^*/q^{\mathbb{Z}}$. The reformulation of his theorem in the tannakian language is that the category of q-difference modules over the field of meromorphic functions on the punctured plane $\mathbb{C}^*$ is a neutral tannakian category over C_E, i.e., admits a fiber functor into $Vect_{C_E}$. We give below the intrinsic analogue of this theorem for q-difference modules over $K(x)$ where K is a valued field of characteristic zero.

Let $K(x)$ be a q-difference field, $\partial = x\frac{d}{dx}$, $|\ |$ a norm on K such that $|q| > 1$ and C an algebraically closed field extension of K, complete with respect to $|\ |$. [1] Here are a few examples to keep in mind:

- K is a subfield of $\mathbb{C}$ equipped with the norm induced by $\mathbb{C}$ and $C = \mathbb{C}$;
- K is finite extension of a field of rational functions $k(q)$, with k of characteristic 0, equipped with the q^{-1}-adic norm;
- K is a finitely generated extension of $\mathbb{Q}$ and q is an algebraic number, nor a root of unity: in this case there always exists a norm on the algebraic closure Q of $\mathbb{Q}$ in K such that $|q| > 1$, that can be extended to K. The field C is equal to $\mathbb{C}$ if the norm is archimedean.

[1] What follows is of course valid also for the norms for which $|q| < 1$ and can be deduced by transforming the q-difference system $\sigma_q(Y) = AY$ in the q^{-1}-difference system $\sigma_{q^{-1}}(Y) = \sigma_{q^{-1}}(A^{-1})Y$.

We call holomorphic function over C^* a power series $f = \sum_{n=-\infty}^{\infty} a_n x^n$, with coefficients in C, that satisfies

$$\lim_{n\to\infty} |a_n|\rho^n = 0 \text{ and } \lim_{n\to-\infty} |a_n|\rho^n = 0, \text{ for all } \rho > 0.$$

The holomorphic functions on C^* form a ring $\mathcal{H}ol(C^*)$. Its fraction field $\mathcal{M}er(C^*)$ is the field of meromorphic functions over C^*.

PROPOSITION 8.1. *Every q-difference system $\sigma_q(Y) = AY$, with $A \in \mathrm{GL}_\nu(K(x))$ (and actually also $A \in \mathrm{GL}_\nu(\mathcal{M}er(C^*))$), admits a fundamental solution matrix with coefficients in $\mathcal{M}er(C^*)$, i.e., an invertible matrix $U \in \mathrm{GL}_\nu(\mathcal{M}er(C^*))$, such that $\sigma_q(U) = AU$.*

REMARK 8.2. Notice that the field of σ_q-constants of $\mathcal{M}er(C^*)$ is the field C_E of elliptic functions over the torus $E = C^*/q^{\mathbb{Z}}$. The proposition above is equivalent to the global triviality of the pull back over C^* of the fiber bundles on elliptic curves. A more explicit construction of meromorphic solutions of q-difference equations has been given recently by T. Dreyfus [**Dre14**].

PROOF. We are only sketching the proof. The Jacobi theta function

$$\Theta_q(x) = \sum_{n\in\mathbb{Z}} q^{-n(n-1)/2} x^n,$$

is an element of $\mathcal{M}er(C^*)$. It is solution of the q-difference equation

$$y(qx) = qx\, y(x).$$

We follow [**Sau00**]. Since

- for any $c \in C^*$, the meromorphic function $\Theta(cx)/\Theta_q(x)$ is solution of $y(qx) = cy(x)$;
- the meromorphic function $x\Theta_q'(x)/\Theta_q(x)$ is solution of the equation $y(qx) = y(x) + 1$;

we can write a meromorphic fundamental solution to any regular singular system at 0, and, more generally, of any system whose Newton polygon has only one slope (*cf.* for instance [**Sau00**], [**DVRSZ03**] or [**Sau04b**, §1.2.2]). For the "pieces" of solutions linked to the Stokes phenomenon, all the techniques of q-summation in the case $q \in \mathbb{C}$, $|q| > 1$, apply in a straightforward way to our situation (*cf.* [**Sau04a**, §2, §3]) and give a fundamental solution meromorphic over C^*. □

8.1.2. Differential fiber functors. We consider the q-difference-differential field $\left(C(x), \sigma_q, \partial = x\frac{d}{dx}\right)$, where C is a complete algebraically closed normed extension of $(K, |\ |)$, with $|q| > 1$. Notice that both $\mathcal{H}ol(C^*)$ and $\mathcal{M}er(C^*)$ are stable under the action of σ_q and ∂. Because σ_q and ∂ commute, the derivation ∂ stabilizes C_E inside $\mathcal{M}er(C^*)$, so that C_E is naturally endowed with a structure of q-difference-differential field. Let $\widetilde{C}_E$ be a differential closure of C_E with respect to ∂ (*cf.* [**CS06**, §9.1]). We still denote by ∂ the derivation of $\widetilde{C}_E$ and we extend the action of σ_q to $\widetilde{C}_E$ by setting $\sigma_q|_{\widetilde{C}_E} = id$. Let $C_E(x)$ (resp. $\widetilde{C}_E(x)$) denote the field $C(x)(C_E)$ (resp. $C(x)(\widetilde{C}_E)$). [2]

Let $\mathcal{M}_{K(x)}$ be a q-difference module over $K(x)$. As usual, for any q-difference field extension $\mathcal{F}/K(x)$ we will denote by $\mathcal{M}_{\mathcal{F}}$ the q-difference module over $\mathcal{F}$ obtained from $\mathcal{M}_{K(x)}$ by scalar extension. Thanks to Proposition 8.1, we are able to construct a weak parametrized Picard-Vessiot ring for $\mathcal{M}_{C_E(x)}$ (see Definition 2.8).

[2] Notice that C_E (resp. $\widetilde{C}_E$) and $C(x)$ are linearly disjoint over C. The field $\widetilde{C}_E(x)$ is the intrinsic analogue of the field $\mathcal{G}(x)$ in [**HS08**, p. 340].

LEMMA 8.3. *Let $\mathcal{M}_{C_E(x)}$ be a q-difference module over $C_E(x)$ and let $\sigma_q(Y) = AY$, with $A \in \mathrm{GL}_\nu(C_E(x))$, be a q-difference system attached to $\mathcal{M}_{C_E(x)}$. By Proposition 8.1, let $U \in \mathrm{GL}_\nu(\mathcal{M}er(C^*))$ be a fundamental solution matrix. Then, the ring $R_E := C_E(x)\{U, \frac{1}{det(U)}\}_\partial$ is a weak parametrized Picard-Vessiot ring for $\mathcal{M}_{C_E(x)}$ over $C_E(x)$ and an integral domain.*

PROOF. Notice that $R_M \subset \mathcal{M}er(C^*)$ and that $C_E \subset R_M^{\sigma_q} \subset \mathcal{M}er(C^*)^{\sigma_q} = C_E$. □

8.2. Constructions of the fiber functors

We remind the notations introduced so far and we refer to §2.3 for notions on Picard-Vessiot ring. Let K be a field and $|\ |$ a norm on K such that $|q| > 1$. We will be dealing with groups defined over the following fields:
C = smallest algebraically closed and complete extension of the normed field $(K, |\ |)$;
C_E = field of constants with respect to σ_q of $\mathcal{M}er(C^*)$;
$\overline{C}_E$ = algebraic closure of C_E;
$\widetilde{C}_E$ = differential closure of C_E.
We remind that any q-difference system $Y(qx) = A(x)Y(x)$, with $A(x) \in \mathrm{GL}_\nu(K(x))$, has a fundamental solution in $\mathcal{M}er(C^*)$ (*cf.* Proposition 8.1).

Let $\mathcal{M}_{K(x)}$ be a q-difference module over $K(x)$. By [**DVH11**, Proposition 1.16], one can attach to the q-difference module $\mathcal{M}_{C(x)}$, a weak parametrized Picard-Vessiot ring R, which is also σ_q-simple and satisfies $R^{\sigma_q} = C$ (This will be crucial in Corollary 8.9). By Lemma 8.3, one can also consider the weak parametrized ring R_E, generated by meromorphic solutions of $\mathcal{M}_{C_E(x)}$. Finally, if we denote by $\widetilde{C}_E$ a differential closure of C_E, we can apply the constructions of [**HS08**], to attach to the q-difference module $\mathcal{M}_{\widetilde{C}_E(x)}$ a parametrized Picard-Vessiot ring $\widetilde{R}$. Since $\widetilde{C}_E$ is differentially closed, $\widetilde{R}^{\sigma_q} = \widetilde{C}_E$ by [**HS08**, Corollary 6.15].

By Proposition 2.9, each of these weak parametrized Picard-Vessiot rings yields to a neutral differential fiber functor for $\langle \mathcal{M}_{C(x)} \rangle^{\otimes,\partial}, \langle \mathcal{M}_{C_E(x)} \rangle^{\otimes,\partial}, \langle \mathcal{M}_{\widetilde{C}_E(x)} \rangle^{\otimes,\partial}$. When restricted to the Tannakian category generated by $\mathcal{M}_{C(x)}, \mathcal{M}_{C_E(x)}, \mathcal{M}_{\widetilde{C}_E(x)}$, these differential fiber functors induce neutral fiber functors in the classical sense of [**Del90**]. We keep the notations of Proposition 2.9.

WARNING 8.4. We want to compare the behaviour of the group of differential tensor automorphisms of a given module $\mathcal{M}_{K(x)}$ with respect to field extensions. In order to avoid any confusion, we will change a little bit the notation of Chapter 2, namely we will denote by $Aut^{\otimes,\partial}(\mathcal{M}, \omega)$ what was previously denoted $Aut^{\otimes,\partial}(\omega)$.

We remind first of all the neutral fiber functors defined above:

$$\omega_R : \langle \mathcal{M}_{C(x)} \rangle^{\otimes} \longrightarrow Vect_C, \qquad \mathcal{N} \mapsto \ker(\Sigma_q - Id, R \otimes_{C(x)} \mathcal{N}); \tag{8.1}$$

$$\omega_{R_E} : \langle \mathcal{M}_{C_E(x)} \rangle^{\otimes} \longrightarrow Vect_{C_E} \qquad \mathcal{N} \mapsto \ker(\Sigma_q - Id, R_E \otimes_{C_E(x)} \mathcal{N}); \tag{8.2}$$

$$\omega_{\widetilde{R}} : \langle \mathcal{M}_{\widetilde{C}_E(x)} \rangle^{\otimes} \longrightarrow Vect_{\widetilde{C}_E} \qquad \mathcal{N} \mapsto \ker(\Sigma_q - Id, \widetilde{R} \otimes_{\widetilde{C}_E(x)} \mathcal{N}); \tag{8.3}$$

and the three neutral differential fiber functors extending them:

$$\omega_R : \langle \mathcal{M}_{C(x)} \rangle^{\otimes,\partial} \longrightarrow Vect_C; \tag{8.4}$$

$$\omega_{R_E} : \langle \mathcal{M}_{C_E(x)} \rangle^{\otimes,\partial} \longrightarrow Vect_{C_E}; \tag{8.5}$$

$$\omega_{\widetilde{R}} : \langle \mathcal{M}_{\widetilde{C}_E(x)} \rangle^{\otimes,\partial} \longrightarrow Vect_{\widetilde{C}_E}. \tag{8.6}$$

We also have four forgetful functors:

(8.7) $\eta_{K(x)} : \langle \mathcal{M}_{K(x)} \rangle^{\otimes} \longrightarrow Vect_{K(x)}$ and its extension to $\langle \mathcal{M}_{K(x)} \rangle^{\otimes,\partial}$;

(8.8) $\eta_{C(x)} : \langle \mathcal{M}_{C(x)} \rangle^{\otimes} \longrightarrow Vect_{C(x)}$ and its extension to $\langle \mathcal{M}_{C(x)} \rangle^{\otimes,\partial}$;

(8.9) $\eta_{C_E(x)} : \langle \mathcal{M}_{C_E(x)} \rangle^{\otimes} \longrightarrow Vect_{C_E(x)}$ and its extension to $\langle \mathcal{M}_{C_E(x)} \rangle^{\otimes,\partial}$;

(8.10) $\eta_{\widetilde{C}_E(x)} : \langle \mathcal{M}_{\widetilde{C}_E(x)} \rangle^{\otimes} \longrightarrow Vect_{\widetilde{C}_E}$ and its extension to $\langle \mathcal{M}_{\widetilde{C}_E(x)} \rangle^{\otimes,\partial}$.

8.3. Comparison of "classical" Galois groups

The group of tensor automorphisms of ω_R corresponds to the "classical" Picard-Vessiot group of a q-difference equation attached to $\mathcal{M}_{K(x)}$, defined in [**vdPS97**, §1.2]. It can be identified to the group of ring automorphims of the subring S of R generated over $C(x)$ by a fundamental solution matrix and the inverse of its determinant, stabilizing $C(x)$ and commuting with σ_q. It is a linear algebraic group over C and its dimension is equal to the transcendence degree of the total ring of quotients of S over $C(x)$, i.e., it measures the algebraic relations between the formal solutions introduced in [**DVH11**].

The group of tensor automorphisms of ω_{R_E} corresponds to another Picard-Vessiot group attached to $\mathcal{M}_{K(x)}$. Its dimension as a linear algebraic group is equal to the transcendence degree of the field F generated over $C_E(x)$ by an invertible matrix of meromorphic solutions in $\mathrm{GL}_\nu(R_E)$. In other words, $Aut^{\otimes}(\mathcal{M}_{C_E(x)}, \omega_{R_E})$ measures the algebraic relations between the meromorphic solutions, introduced in §8.1.2. One of the main results of [**CHS08**, §3] is:

THEOREM 8.5. *The linear algebraic groups $Aut^{\otimes}(\mathcal{M}_{C(x)}, \omega_R)$, $Aut^{\otimes}(\mathcal{M}_{C_E(x)}, \omega_{R_E})$, $Aut^{\otimes}(\mathcal{M}_{C_E(x)}, \omega_{\widetilde{R}})$ become isomorphic over $\widetilde{C}_E$.*

REMARK 8.6. In [**Sau04b**], Sauloy constructs a $\mathbb{C}$-linear fiber functor for q-difference modules over $\mathbb{C}(x)$, using a basis of meromorphic solutions. Since $\mathbb{C}$ is algebraically closed, it follows from the classical general theory of tannakian categories, that such a fiber functor gives rise to a group that is isomorphic to the Picard-Vessiot group of [**vdPS97**] over $\mathcal{F} = \mathbb{C}(x)$. We won't consider Sauloy's point of view in this paper.

8.4. Comparison of parametrized Galois groups

The goal of this section is to compare the differential algebraic groups attached to the differential fiber functors defined above (see Definition 2.5). For ∂ the trivial derivation, we retrieve of course the study of [**CHS08**, §3] (see §8.3).

In this section, we adapt the techniques of [**CHS08**, §2] to a differential framework, in order to compare the distinct parametrized Picard-Vessiot rings, attached to $\mathcal{M}_{K(x)}$ over C, C_E and $\widetilde{C}_E$. For a model theoretic approach of these questions, we refer to [**PN09**].

The following proposition compare formal and meromorphic solutions. It is a differential analogue of [**CHS08**, Proposition 2.4]. We keep the notations of the previous sections. In particular, let R be the parametrized Picard-Vessiot ring attached to the system as in [**DVH11**, Proposition 1.16]. We remind that R can be written in the form $R = C(x)\{Y, \frac{1}{det(Y)}\}_\partial/\mathfrak{q}$, where Y is an invertible matrix satisfying the system $\sigma_q(Y) := AY$ and $\mathfrak{q}$ is not only a maximal (σ_q, ∂)-ideal but also a maximal σ_q-ideal. We have:

PROPOSITION 8.7. *Let $\mathcal{M}_{C(x)}$ be a q-difference module over $C(x)$ and let $\sigma_q(Y) = AY$ be a q-difference system attached to $\mathcal{M}_{C(x)}$. Let $\mathcal{F}$ be a q-difference differential field extension of $C(x)$ such that $\mathcal{F} = C(x)(\mathcal{F}^{\sigma_q})$. Then, $S := \mathcal{F}\{Y, \frac{1}{\det(Y)}\}_\partial/\mathfrak{q}\mathcal{F}$ is a parametrized Picard-Vessiot ring for $\mathcal{M}_{\mathcal{F}}$ and $S^{\sigma_q} = \mathcal{F}^{\sigma_q}$.*

PROOF. First note that $\mathfrak{q}\mathcal{F} \subsetneq \mathcal{F}\{Y, \frac{1}{det(Y)}\}_\partial$. Then, consider the map $\phi : R \otimes \mathcal{F}^{\sigma_q} \to S$. Let $\mathfrak{I} \subset R \otimes \mathcal{F}^{\sigma_q}$ be a σ_q-ideal. Since $R^{\sigma_q} = C$ and R is σ_q-simple, [**vdPS97**, Lemma 1.11] implies that the σ_q-ideal $\mathfrak{I}$ in $R\otimes\mathcal{F}^{\sigma_q}$ is generated by $\mathfrak{I}\cap R$. Since R is σ_q-simple, we get that $R\otimes\mathcal{F}^{\sigma_q}$ is σ_q-simple and that ϕ injective. Now let $R' = \phi(R\otimes\mathcal{F}^{\sigma_q})$. Since, for all $x \in S$, there exists $a \in R'$ such that $ax \in R'$, we get that a σ_q-ideal $\mathfrak{J}$ in S is generated by $\mathfrak{J} \cap R'$. This implies that S is σ_q-simple and thus (σ_q, ∂)-simple. Then, it is clear that S is a parametrized Picard-Vessiot ring for $\mathcal{M}_\mathcal{F}$. Finally, for any $c \in S^{\sigma_q}$, the set $\{a \in R' | ac \in R'\}$ is a non-zero σ_q-ideal [3] and by σ_q-simplicity of R', we see that $c \in R'$. We conclude by remarking that $R'^{\sigma_q} = \mathcal{F}^{\sigma_q}$. □

As corollary of the previous proposition, we find

COROLLARY 8.8. *Let $\mathcal{M}_{C(x)}$ be a q-difference module over $C(x)$ and let $\sigma_q(Y) = AY$ be a q-difference system attached to $\mathcal{M}_{C(x)}$. Let R, R_E and $\widetilde{R}$ be the weak parametrized Picard-Vessiot rings attached to $\mathcal{M}_C(x)$, as in §8.2. As above, we write $R = C(x)\{Y, \frac{1}{det(Y)}\}_\partial/\mathfrak{q}$. Then we have two isomorphisms of $\widetilde{C}_E(x)$-(σ_q,∂)-algebras:*

- $\widetilde{S} := \widetilde{C}_E(x)\{Y, \frac{1}{det(Y)}\}_\partial/\mathfrak{q}\widetilde{C}_E(x) \longrightarrow \widetilde{R}$;
- $S \otimes \widetilde{C}_E := C_E(x)\{Y, \frac{1}{det(Y)}\}_\partial/\mathfrak{q}C_E(x) \otimes \widetilde{C}_E \longrightarrow R_E \otimes \widetilde{C_E}$.

PROOF. By Proposition 8.7, applied to $\mathcal{F} = C_E(x)$ and $\mathcal{F} = \widetilde{C}_E(x)$, we find that S (resp. $\widetilde{S}$) is a parametrized Picard-Vessiot ring for $\mathcal{M}_{C_E(x)}$ (resp. $\mathcal{M}_{\widetilde{C}_E(x)}$) such that $S^{\sigma_q} = C_E$ (resp. $\widetilde{S}^{\sigma_q} = \widetilde{C}_E$). Since $\widetilde{C}_E$ is differentially closed, [**HS08**, Proposition 6.16] assures that two parametrized Picard-Vessiot ring for the same q-difference equation over $\widetilde{C}_E(x)$ are isomorphic as $\widetilde{C}_E(x)$-(σ_q,∂)-algebras. The first isomorphism follows from this fact.

The second isomorphism comes from a parametrized version of [**CHS08**, Proposition 2.7]. Its proof follows line by line the proof in the algebraic case, but we give it here for sake of completeness. Let us denote by F_E the fraction field of R_E and let $X = (X_{i,j})$ be a $\nu \times \nu$-matrix of differential indeterminates over F_E. Let $\mathcal{S} := C_E(x)\{X, \frac{1}{det(X)}\}_\partial \subset F_E\{X, \frac{1}{det(X)}\}_\partial$. Define a (σ_q,∂)-structure on $F_E\{X, \frac{1}{det(X)}\}_\partial$ by setting $\sigma_q(X) := AX, \sigma_q(\partial X) := A\partial X + \partial AX, \ldots$. This induces a (σ_q,∂)-structure on $\mathcal{S}$. Since S is a parametrized Picard-Vessiot ring for $\sigma_q(Y) = AY$ view over $C_E(x)$, we can write $S = \mathcal{S}/\mathfrak{p}$, where $\mathfrak{p}$ is a maximal (σ_q,∂)-ideal of $\mathcal{S}$. Now, let $U \in \mathrm{GL}_\nu(R_E)$ be fundamental solution matrix of $\sigma_q(Y) = AY$. Define $Y = (Y_{i,j}) \in \mathrm{GL}(F_E\{X, \frac{1}{det(X)}\}_\partial)$ via $Y := U^{-1}X$ and remark that $\sigma_q(Y) = Y$ and $F_E\{X, \frac{1}{det(X)}\}_\partial = F_E\{Y, \frac{1}{det(Y)}\}_\partial$. Define $\mathcal{S}_1 := C_E\{Y, \frac{1}{det(Y)}\}_\partial$. The ideal $\mathfrak{p} \subset \mathcal{S} \subset F_E\{X, \frac{1}{det(X)}\}_\partial$ generates a (σ_q,∂)-ideal $(\mathfrak{p})$ in $F_E\{X, \frac{1}{det(X)}\}_\partial$, which intersected with $\mathcal{S}_1$ gives a ∂-ideal $\mathfrak{a}$. Since $\widetilde{C}_E$ is differentially closed and $\mathcal{S}_1/\mathfrak{a}$ is differentially finitely generated over C_E, we find a differential homomorphism $\mathcal{S}_1 \otimes \widetilde{C}_E \to \mathcal{S}_1/\mathfrak{a} \to \widetilde{C_E}$. We can extend this homomorphism into a (σ_q,∂)-morphism $F_E\{X, \frac{1}{det(X)}\}_\partial = F_E \otimes \mathcal{S}_1 \to F_E \otimes_{C_E} \widetilde{C_E}$ and restricted to $\mathcal{S}$, we find a (σ_q,∂)-morphism $\mathcal{S} \to F_E \otimes \widetilde{C}_E$, whose Kernel contains $\mathfrak{p}$. By maximality of $\mathfrak{p}$, we have equality and we get an embedding $\iota : S = \mathcal{S}/\mathfrak{p} \to F_E \otimes \widetilde{C}_E$. Now, if we denote by $V \in \mathrm{GL}(S)$ a fundamental solution matrix of $\sigma_q(Y) = AY$, we find, since $(F_E \otimes \widetilde{C}_E)^{\sigma_q} = \widetilde{C}_E$, that $\iota(V) = UC$ with $C \in \mathrm{GL}(\widetilde{C}_E)$. Since S (resp. R_E) R_E) is differentially generated over $C_E(x)$ (resp. $\widetilde{C}_E(x)$) by V (resp U) and the inverse of its determinant, this allows us to conclude that $\iota(S \otimes \widetilde{C}_E) = R_E \otimes \widetilde{C}_E$. □

[3] It is not a (σ_q,∂)-ideal and here the assumption of σ_q-simplicity is crucial.

The comparison between the group of differential tensor automorphisms attached to the fiber functors defined in §8.2 follows from the previous corollary. We refer to Definition 2.10 for the notations. We obtain the following statement:

COROLLARY 8.9. *Let $\mathcal{M}_{C(x)}$, R, R_E and $\widetilde{R}$ be as in Corollary 8.8. Let ω_R (resp. $\omega_{R_E}, \omega_{\widetilde{R}}$) be the differential fiber functor attached to R (resp $R_E, \widetilde{R}$) as in Proposition 2.9. Then,*

$$Aut^{\otimes,\partial}(\mathcal{M}_{C(x)},\omega_R)\otimes_C\widetilde{C}_E \simeq Aut^{\otimes,\partial}(\mathcal{M}_{C_E(x)},\omega_{R_E})\otimes_{C_E}\widetilde{C}_E \simeq Aut^{\otimes,\partial}(\mathcal{M}_{\widetilde{C}_E(x)},\omega_{\widetilde{R}}).$$

PROOF. By Proposition 2.13, we have $Aut^{\otimes,\partial}(\mathcal{M}_{C(x)},\omega_R)\otimes\widetilde{C}_E \simeq G_R^{\partial}\otimes\widetilde{C}_E$, $Aut^{\otimes,\partial}(\mathcal{M}_{C_E(x)},\omega_{R_E})\otimes\widetilde{C}_E \simeq G_{R_E}^{\partial}\otimes\widetilde{C}_E$ and $Aut^{\otimes,\partial}(\mathcal{M}_{\widetilde{C}_E(x)},\widetilde{\omega}_{\widetilde{R}}) \simeq G_{\widetilde{R}}^{\partial}$. We recall that, for instance, G_R^{∂} denotes the differential group scheme of (σ_q,∂)-$C(x)$-automorphism of R. Now, the (σ_q,∂)-isomorphism of Corollary 8.8 translates into functorial isomorphism between $G_R^{\partial}\otimes\widetilde{C}_E, G_{R_E}^{\partial}\otimes\widetilde{C}_E$ and $G_{\widetilde{R}}^{\partial}$ (As in [**CHS08**, Corollary 2.5], it is a consequence of Yoneda lemma). □

We have proved that the group of differential tensor automorphisms of fiber functors attached either to formal solutions, i.e. to ω_R and $\omega_{\widetilde{R}}$, or to meromorphic solutions, i.e. to ω_{R_E}, are forms of the same differential algebraic group scheme defined over C.

8.5. Comparison results for intrinsic Galois groups

We are now concerned with the intrinsic Galois groups, algebraic and parametrized. We first relate them with the Picard-Vessiot groups we have studied previously and then we investigate how they behave through certain type of base field extensions.

8.5.0.1. *Comparison with Picard-Vessiot groups.* Let $\mathcal{M}_{K(x)}$ be a q-difference module defined over $K(x)$. We remind the reader that we have attached to $\mathcal{M}_{K(x)}$ the following groups:

group	fiber functor	field of definition
$Aut^{\otimes}(\mathcal{M}_{C(x)},\omega_R)$	ω_R:$\langle\mathcal{M}_{C(x)}\rangle^{\otimes} \longrightarrow Vect_C$	C
$Aut^{\otimes,\partial}(\mathcal{M}_{C(x)},\omega_R)$	ω_R:$\langle\mathcal{M}_{C(x)}\rangle^{\otimes,\partial} \longrightarrow Vect_C$	C
$Gal(\mathcal{M}_{C(x)},\eta_{C(x)})$	$\eta_{C(x)}$:$\langle\mathcal{M}_{C(x)}\rangle^{\otimes} \longrightarrow Vect_{C(x)}$	$C(x)$
$Gal^{\partial}(\mathcal{M}_{C(x)},\eta_{C(x)})$	$\eta_{C(x)}$:$\langle\mathcal{M}_{C(x)}\rangle^{\otimes,\partial} \longrightarrow Vect_{C(x)}$	$C(x)$
$Aut^{\otimes}(\mathcal{M}_{C_E(x)},\omega_{R_E})$	ω_{R_E}:$\langle\mathcal{M}_{C_E(x)}\rangle^{\otimes} \longrightarrow Vect_{C_E}$	C_E
$Aut^{\otimes,\partial}(\mathcal{M}_{C_E(x)},\omega_{R_E})$	ω_{R_E}:$\langle\mathcal{M}_{C_E(x)}\rangle^{\otimes,\partial} \longrightarrow Vect_{C_E}$	C_E
$Gal(\mathcal{M}_{C_E(x)},\eta_{C_E(x)})$	$\eta_{C_E(x)}$:$\langle\mathcal{M}_{C_E(x)}\rangle^{\otimes} \longrightarrow Vect_{C_E}(x)$	$C_E(x)$
$Gal^{\partial}(\mathcal{M}_{C_E(x)},\eta_{C_E(x)})$	$\eta_{C_E(x)}$:$\langle\mathcal{M}_{C_E(x)}\rangle^{\otimes,\partial} \longrightarrow Vect_{C_E(x)}$	$C_E(x)$

The comparison between the forgetful functors and their corresponding neutral fiber functors is a direct consequence of the more general statement Corollary 7.6. We have:

PROPOSITION 8.10. *Let us denote by $\widetilde{C(x)}$ (resp. $\widetilde{C_E(x)}$) a differential closure of $C(x)$ (resp. $C_E(x)$). We have the following isomorphisms of linear algebraic groups:*

(1) $Aut^{\otimes}(\mathcal{M}_{C(x)}, \omega_R) \otimes_C \widetilde{C(x)} \simeq Gal(\mathcal{M}, \eta_{C(x)}) \otimes_{C(x)} \widetilde{C(x)}$;

(2) $Aut^{\otimes}(\mathcal{M}_{C_E(x)}, \omega_{R_E}) \otimes_{C_E} \widetilde{C_E(x)} \simeq Gal(\mathcal{M}_{C_E(x)}, \eta_{C_E(x)}) \otimes_{C_E(x)} \widetilde{C_E(x)}$;

and the following isomorphisms of linear differential algebraic groups:

(3) $Aut^{\otimes,\partial}(\mathcal{M}_{C(x)}, \omega_R) \otimes_C \widetilde{C(x)} \simeq Gal^{\partial}(\mathcal{M}, \eta_{C(x)}) \otimes_{C(x)} \widetilde{C(x)}$;

(4) $Aut^{\otimes,\partial}(\mathcal{M}_{C_E(x)}, \omega_{R_E}) \otimes_{C_E} \widetilde{C_E(x)} \simeq Gal^{\partial}(\mathcal{M}_{C_E(x)}, \eta_{C_E(x)}) \otimes_{C_E(x)} \widetilde{C_E(x)}$.

Since the dimension of a differential algebraic group as well as the differential transcendence degree (see Definition 2.1) of a field extension is stable up to field extension, one obtains the following corollary

COROLLARY 8.11. *Let $\mathcal{M}_{K(x)}$ be a q-difference module defined over $K(x)$. Let $U \in \mathrm{GL}(\mathcal{M}er(C^*))$ be a fundamental solution matrix attached to $\mathcal{M}_{C_E(x)}$, as in Proposition 8.1.*

Then, the differential transcendence degree of the differential field F_E generated over $C_E(x)$ by the entries of U is equal to the differential dimension of $Gal^{\partial}(\mathcal{M}_{C(x)}, \eta_{C(x)})$.

PROOF. By [**GGO13**, Proposition 4.28], the functor $Isom^{\otimes,\partial}(\omega_{R_E} \otimes C_E(x), \eta_{C_E(x)})$ is a reduced differential algebraic scheme over $C_E(x)$, represented by R_E. It is also a $Aut^{\otimes,\partial}(\mathcal{M}_{C_E(x)}, \omega_{R_E})$-torsor. It has thus a $\widetilde{C_E(x)}$-point, which gives, by triviality of the torsor, a (σ_q, ∂)-isomorphism between $\widetilde{C_E(x)} \otimes_{C_E(x)} R_E$ and $\widetilde{C_E(x)} \otimes_{C_E} C_E\{Aut^{\otimes,\partial}(\mathcal{M}_{C_E(x)}, \omega_{R_E})\}$. Using the discussion on the differential dimension in §2.1, we get that the differential dimension of $Aut^{\otimes,\partial}(\mathcal{M}_{C_E(x)}, \omega_{R_E})$ equals the differential transcendence degree of F_E over $C_E(x)$. By Proposition 8.10 combined with Corollary 8.9, we find that $Aut^{\otimes,\partial}(\mathcal{M}_{C_E(x)}, \omega_{R_E})$ is isomorphic to $Gal^{\partial}(\mathcal{M}_{C(x)}, \eta_{C(x)})$ over $\widetilde{C}_E(x)$. We conclude by using one more time the fact that the differential dimension of a reduced differential algebraic scheme is invariant by base field extension. □

8.5.0.2. *From $K(x)$ to $C(x)$.* In [**Kat87**, Lemma 1.3.2], it is shown that the group of tensor automorphism of a k-linear neutral fiber functor is invariant up to algebraic field extension of k. For forgetful functors, this is not true. This is essentially due to the fact that, unlike to the case of neutral fiber functors, a vector space stable under the action of the group of tensor automorphism of the forgetful functor is not necessarily an object of the Tannakian category . For q-difference modules defined above $K(x)$, we bypass this difficulties and obtain the following lemma, in which we show that, for any field extension L/K, the parametrized intrinsic Galois group of $\mathcal{M}_{L(x)}$ is equal, up to scalar extension, to the parametrized intrinsic Galois group of $\mathcal{M}_{K'(x)}$, for a convenient finitely generated extension K'/K, with $K' \subset L$.

LEMMA 8.12. *Let L be a field extension of K with $\sigma_q|_L = id$. There exists a finitely generated intermediate field $L/K'/K$ such that*

$$Gal(\mathcal{M}_{L(x)}, \eta_{L(x)}) \cong Gal(\mathcal{M}_{K'(x)}, \eta_{K'(x)}) \otimes_{K'(x)} L(x)$$

and

$$Gal^{\partial}(\mathcal{M}_{L(x)}, \eta_{L(x)}) \cong Gal^{\partial}(\mathcal{M}_{K'(x)}, \eta_{K'(x)}) \otimes_{K'} L(x).$$

These equalities hold when we replace K' by any subfield extension of L containing K'.

PROOF. By definition, $Gal^{\partial}(\mathcal{M}_{L(x)}, \eta_{L(x)})$ is the stabilizer inside $\mathrm{GL}(M_{L(x)})$ of all $L(x)$-vector spaces of the form $W_{L(x)}$ for $\mathcal{W}$ object of $\langle \mathcal{M}_{L(x)} \rangle^{\otimes, \partial}$. [4] Similarly, for any field extension $L/K'/K$, we have

$$Gal^{\partial}(\mathcal{M}_{K'(x)}, \eta_{K'(x)}) = Stab(W_{K'(x)}, \mathcal{W} \text{ object of } \langle \mathcal{M}_{K'(x)} \rangle^{\otimes, \partial}).$$

Then,

$$Gal^{\partial}(\mathcal{M}_{L(x)}, \eta_{L(x)}) \subset Gal^{\partial}(\mathcal{M}_{K'(x)}, \eta_{K'(x)}) \otimes L(x).$$

By noetherianity, the (parametrized) intrinsic Galois group of $\mathcal{M}_{L(x)}$ is defined by a finite family of (differential) polynomial equations, thus we can choose K', which contains the coefficients of the defining equations. $\square$

The corollary below summarizes results of this chapter.

COROLLARY 8.13. *Let $\mathcal{M}_{K(x)}$ be a q-difference module defined over $K(x)$. Let $U \in \mathrm{GL}_\nu(\mathcal{M}er(C^*))$ be a fundamental matrix of meromorphic solutions of $\mathcal{M}_{K(x)}$. Then,*

(1) *the dimension of $Gal(\mathcal{M}_{C(x)}, \eta_{C(x)})$ is equal to the transcendence degree of the field generated by the entries of U over $C_E(x)$, i.e., the algebraic group $Gal(\mathcal{M}_{C(x)}, \eta_{C(x)})$ measures the algebraic relations between the meromorphic solutions of $\mathcal{M}_{C_E(x)}$.*
(2) *the ∂-differential dimension of $Gal^{\partial}(\mathcal{M}_{C(x)}, \eta_{C(x)})$ is equal to the differential transcendence degree of the differential field generated by the entries of U over $\widetilde{C}_E(x)$, i.e., the differential algebraic group $Gal^{\partial}(\mathcal{M}_{C(x)}, \eta_{C(x)})$ encodes the differential algebraic relations between the meromorphic solutions of $\mathcal{M}_{K(x)}$.*
(3) *there exists a finitely generated extension K'/K such that the differential transcendence degree of the differential field generated by the entries of U over $\widetilde{C}_E(x)$ is equal to the differential dimension of $Gal^{\partial}(\mathcal{M}_{K'(x)}, \eta_{K'(x)})$, i.e., it is given by an arithmetic characterization.*

PROOF. The first two statements are proved in Corollary 8.11. The third one is Lemma 8.12. $\square$

[4] One has to understand this equality as a functorial equality for differential scheme defined above $L(x)$.

CHAPTER 9

Specialization of the parameter q

In this chapter we consider the situation in which q is a parameter, that we want to specialize. When we specialize q to q_0 in a q-difference module, we can obtain both a differential module (if $q_0 = 1$) or a q_0-difference module (if $q_0 \neq 1$). Therefore the best framework for studying the reduction of intrinsic Galois groups is André's theory of generalized differential rings (*cf.* [**And01**, 2.1.2.1]). For the reader's convenience, we first recall some definitions and basic facts from [**And01**]). Then we deduce some results on the specialization of intrinsic Galois groups and their differential analogues.

Our purpose is to give a framework where the following result can possibly be analysed more deeply. In [**DV02**, Appendix], the author considers the Heine hypergeometric series. Let a, b, c, q be complex numbers, such that q is non-zero and not a root of unity. The basic q-hypergeometric series:

$$ {}_2\phi_1(a, b, c; q^{-1}, x) = \sum_{n \geq 0} \frac{(a; q^{-1})_n (b; q^{-1})_n}{(c; q^{-1})_n (q^{-1}; q^{-1})_n} x^n, $$

where $(a; q^{-1})_n = (1 - a)(1 - aq^{-1}) \cdots (1 - aq^{-(n-1)})$, is defined if $c \notin q^{\mathbb{Z}_{\leq 0}}$ or if $c \in q^{\mathbb{Z}_{\leq 0}}$ and either $a \in q^{\mathbb{Z}_{\leq 0}}$, $ac^{-1} \in q^{\mathbb{Z}_{\geq 0}}$ or $b \in q^{\mathbb{Z}_{\leq 0}}$, $bc^{-1} \in q^{\mathbb{Z}_{\geq 0}}$. It is a q-analogue of the Gauss hypergeometric series

$$ {}_2F_1(\alpha, \beta, \gamma; x) = \sum_{n \geq 0} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n n!} x^n, $$

where $(\alpha)_n = \alpha(\alpha + 1) \cdots (\alpha + n - 1)$ is the Pochhammer symbol. If γ is a non-positive integer, ${}_2F_1(\alpha, \beta, \gamma; x)$ is defined if and only if either $\alpha \in \mathbb{Z}$, $\gamma \leq \alpha \leq 0$ or $\beta \in \mathbb{Z}$, $\gamma \leq \beta \leq 0$.

The series ${}_2\phi_1(a, b, c; q^{-1}, x)$ is a solution of the basic hypergeometric q-difference equation

$$ (\mathcal{H}_{a,b,c}) \qquad \varphi_q^2 y(x) - \frac{(a + b)x - (1 + cq^{-1})}{abx - cq^{-1}} \varphi_q y(x) + \frac{x - 1}{abx - cq^{-1}} y(x) = 0, $$

which is defined as soon as neither $a = c = 0$ nor $b = c = 0$. Rewriting $(\mathcal{H}_{a,b,c})$ in terms of the operator $d_q := \frac{\sigma_q - 1}{(q-1)x}$, we find

$(\widetilde{\mathcal{H}}_{a,b,c})$

$$ x(c - abqx) d_q^2(y(x)) + \left[\frac{1 - c}{1 - q} + \frac{(1 - a)(1 - b) - (1 - abq)}{1 - q} x \right] d_q(y(x)) - \frac{(1 - a)(1 - b)}{(1 - q)^2} y(x). $$

By replacing a, b, c by $q^\alpha, q^\beta, q^\gamma$ and letting q go to 1, one sees that $(\widetilde{\mathcal{H}}_{a,b,c})$ tends to the hypergeometric differential equation:

$$ (\mathcal{E}_{\alpha,\beta,\gamma}) \qquad y''(x) + \frac{\gamma - (\alpha + \beta + 1)x}{x(1 - x)} y'(x) - \frac{\alpha\beta}{x(1 - x)} y(x) = 0, $$

where α, β, γ are complex parameters. Of course, ${}_2F_1(\alpha, \beta, \gamma; x)$ is a solution of $(\mathcal{E}_{\alpha,\beta,\gamma})$. The following theorem gives necessary and sufficient conditions for the rationality of the solutions of $(\mathcal{H}_{a,b,c})$.

THEOREM 9.1. *Let $\mathcal{Z} = (\mathbb{Z}_{>0} \times \mathbb{Z}_{\leq 0}) \cup (\mathbb{Z}_{\leq 0} \times \mathbb{Z}_{>0})$. Then, the following assertions are equivalent*

(1) *there exists $\alpha, \beta, \gamma \in \mathbb{Z}$, such that $a = q^{\alpha}$, $b = q^{\beta}$, $c = q^{\gamma}$ and*
   - *either $(\alpha, \alpha + 1 - \gamma) \in \mathcal{Z}$ or $(\beta, \beta + 1 - \gamma) \in \mathcal{Z}$,*
   - *either $(\alpha, \beta) \in \mathcal{Z}$ or $(\alpha + 1 - \gamma, \beta + 1 - \gamma) \in \mathcal{Z}$.*

(2) *$(\mathcal{H}_{a,b,c})$ has a basis of solutions in $\mathbb{C}(x)$.*

In the differential setting, we know from [G, Ch. III] that:

THEOREM 9.2. *The following assertions are equivalent:*

(1) *$(\mathcal{E}_{\alpha,\beta,\gamma})$ has a basis of solutions in $\mathbb{C}(x)$;*
(2) *$\alpha, \beta, \gamma \in \mathbb{Z}$ and $|1-\gamma|$, $|\gamma - \alpha - \beta|$ and $|\alpha - \beta|$ are the lengths of the sides of a triangle;*
(3) *the following conditions are satisfied:*
   - *either $(\alpha, \alpha + 1 - \gamma) \in \mathcal{Z}$ or $(\beta, \beta + 1 - \gamma) \in \mathcal{Z}$,*
   - *either $(\alpha, \beta) \in \mathcal{Z}$ or $(\alpha + 1 - \gamma, \beta + 1 - \gamma) \in \mathcal{Z}$.*

The Schwartz list for higher order basic hypergeometric equations has been established by J. Roques (*cf.* [**Roq09**, §8]), and is another example of this phenomenon of confluences of rationality conditions. The framework describe below could give a better insight on the properties of basic hypergeometric series explained above.

9.1. Generalized differential rings

In §9.1, and only in §9.1, we adopt the following more general notation.

DEFINITION 9.3 (*cf.* [**And01**, 2.1.2.1]). Let R be a commutative ring with unit. A generalized differential ring (A, d) over R is an associative R-algebra A endowed with an R-derivation d from A into a left $A \otimes_R A$-module Ω^1, i.e., such that $d(ab) = ad(b) + d(a)b$, where the first product concerns the left A-module structure of Ω^1 and the second product the right A-module structure. The kernel of d, denoted $Const(A)$, is called the set of constants of A.

EXAMPLE 9.4.

(1) Let k be a field and $k(x)$ be the field of rational functions over k. Let $\Omega^1 := dx.k(x)$ with the $k(x)$-$k(x)$-bimodule structure given by $\lambda t = t\lambda$, for all $\lambda \in k(x)$ and $t \in \Omega^1$. The ring $(k(x), \delta)$, with
$$\begin{array}{cccc} \delta: & k(x) & \longrightarrow & \Omega^1 := dx.k(x) \\ & f & \longmapsto & dx.x\dfrac{df}{dx} \end{array},$$
is a generalized differential ring over k, associated to the derivation $x\frac{d}{dx}$.

(2) Let $\mathcal{A}$ be a q-difference ring of the form $\mathcal{O}_K\left[x, \frac{1}{P(x)}, \frac{1}{P(qx)}, \ldots\right]$ with K a σ_q-constant field. Let $\Omega^1 := dx.\mathcal{A}$ with the $\mathcal{A}$-$\mathcal{A}$-bimodule structure given by $\lambda t = t\sigma_q(\lambda)$, for all $\lambda \in \mathcal{A}$ and $t \in \Omega^1$. The ring $(\mathcal{A}, \delta_q)$, with
$$\begin{array}{cccc} \delta_q: & \mathcal{A} & \longrightarrow & \Omega^1 := dx.\mathcal{A} \\ & f & \longmapsto & dx.x\dfrac{\sigma_q(f) - f}{(q-1)x} \end{array},$$
is also a generalized differential ring over $\mathcal{O}_K$, associated to the q-difference algebra $(\mathcal{A}, \sigma_q)$.

(3) Let C denote the ring of constants of a generalized differential ring (A, d) and let $\mathfrak{I}$ be a non-trivial proper prime ideal of C. Then the ring $A_{\mathfrak{I}} := A \otimes C/\mathfrak{I}$ is endowed with a structure of generalized differential ring (*cf.* [**And01**, 3.2.3.7]). In the notations of the example 2) above and of §4.4,

for almost any place $v \in \mathcal{P}_f$ of K, we obtain in this way a generalized differential ring of the form $(\mathcal{A} \otimes_{\mathcal{O}_K} k_v, \delta_{q_v})$.

DEFINITION 9.5 (*cf.* [**And01**, 2.1.2.3]). A morphism of generalized differential rings $(A, d : A \mapsto \Omega^1) \longmapsto (\widetilde{A}, \widetilde{d} : \widetilde{A} \mapsto \widetilde{\Omega}^1)$ is a pair $(u = u^0, u^1)$ where $u^0 : A \mapsto \widetilde{A}$ is a morphism of R-algebras and u^1 is a map from Ω^1 into $\widetilde{\Omega}^1$ satisfying

$$\begin{cases} u^1 \circ d = \widetilde{d} \circ u^0, \\ u^1(a\omega b) = u^0(a)u^1(\omega)u^0(b), \text{ for any } a, b \in A \text{ and any } \omega \in \Omega^1. \end{cases}$$

EXAMPLE 9.6. In the notation of the Example 9.4, the canonical projection $p : A \mapsto A_{\mathfrak{I}}$ induces a morphism u of generalized differential rings from (A, d) into $(A_{\mathfrak{I}}, d)$.

Let B be a generalized differential ring. We denote by $Diff_B$ the category of B-modules with connections (*cf.* [**And01**, 2.2]), i.e., left projective B-modules $\mathcal{M}$ of finite type equipped with a R-linear operator

$$\nabla : \mathcal{M} \longrightarrow \Omega^1 \otimes_A \mathcal{M},$$

such that $\nabla(am) = a\nabla(m) + d(a) \otimes m$. The category $Diff_B$ is abelian, $Const(B)$-linear, monoidal symmetric, *cf.* [**And01**, Theorem 2.4.2.2].

EXAMPLE 9.7. We consider once again the different cases as in Example 9.4:

(1) If $B = (k(x), \delta)$ then $Diff_B$ is the category of differential modules over $k(x)$.
(2) If $B = (\mathcal{A}, \delta_q)$ then $Diff_B$ is the category of q-difference modules over $\mathcal{A}$. In fact, in the notation of the previous sections, it is enough to set $\nabla(m) = dx \otimes \Delta_q(m)$, where $\Delta_q(m) = \frac{\Sigma_q(m) - m}{(q-1)}$ for any $m \in \mathcal{M} = (M, \Sigma_q)$.

Let B be a generalized differential ring. We denote by η_B the forgetful functor from $Diff_B$ into the category of projective B-modules of finite type. For any object $\mathcal{M}$ of $Diff_B$, we consider the forgetful functor η_B induced over the full subcategory $\langle \mathcal{M} \rangle^{\otimes}$ of $Diff_B$ generated by $\mathcal{M}$ and the affine B- group-scheme $Gal(\mathcal{M}, \eta_B)$ defined over B representing the functor $Aut^{\otimes}(\eta_B|_{\langle \mathcal{M} \rangle^{\otimes}})$.

DEFINITION 9.8. The group scheme $Gal(\mathcal{M}, \eta_B)$ over B is called the *intrinsic Galois group* of $\mathcal{M}$.

Let $Constr_B(\mathcal{M})$ be the collection of all constructions of linear algebra of $\mathcal{M}$, i.e., of all the objects of $Diff_B$ deduced from $\mathcal{M}$ by the following B-linear algebraic constructions: direct sums, tensor products, duals, symmetric and antisymmetric products. Then one can show that $Gal(\mathcal{M}, \eta_B)$ is nothing else that the intrinsic Galois group defined in Part 3 in a more restrictive setting (*cf.* [**And01**, 3.2.2.2]):

PROPOSITION 9.9. *Let B be a generalized differential ring and let $\mathcal{M}$ be an object of $Diff_B$. The affine groups scheme $Gal(\mathcal{M}, \eta_B)$ is the stabilizer inside $GL(\mathcal{M})$ of all submodules with connection of some algebraic constructions of $\mathcal{M}$.*

This is not the only Galois group one can define. If we assume the existence of a fiber functor ω from $Diff_B$ into the category of $Const(B)$-module of finite type, we can define the Galois group $Aut^{\otimes}(\omega|_{\langle \mathcal{M} \rangle^{\otimes}})$ of an object $\mathcal{M}$ as the group of tensor automorphism of the fiber functor ω restricted to $\langle \mathcal{M} \rangle^{\otimes}$ (*cf.* [**And01**, 3.2.1.1]). This group characterizes completely the object $\mathcal{M}$. For further reference, we recall the following property (*cf.* [**And01**, Theorem 3.2.2.6]):

PROPOSITION 9.10. *The object $\mathcal{M}$ is trivial if and only if $Aut^{\otimes}(\omega|_{\langle \mathcal{M} \rangle^{\otimes}})$ is a trivial group.*

In certain cases, the category $Diff_B$ can be endowed with a differential structure. Since $Diff_B$ is not necessarily defined over a field, we say that a category $\mathcal{C}$ is a *differential tensor* category, if it satisfies all the axioms of [**Ovc09**, Definition 3] except the assumption that $End(\mathbf{1})$ is a field. We detail below the construction of the prolongation functor associated to $Diff_B$ in some precise cases:

Semi-classic situation: Let us assume that (B, ∂) is a differential subring of the differential field $(L(x), \partial := x\frac{d}{dx})$. Then $Diff_B$ is the category of differential B-modules, equivalently, of left $B[\partial]$-modules M, free and finitely generated over B. We now define a prolongation functor F_∂ for this category as follows. If $\mathcal{M} = (M, \nabla)$ is an object of $Diff_B$ then $F_\partial(\mathcal{M}) = (M^{(1)}, \nabla)$ is the differential module defined by $M^{(1)} = B[\partial]_{\leq 1} \otimes M$, where the tensor product rule is the same one as in §2.2 (i.e., it takes into account the Leibniz rule). If M is an object of $Diff_B$ given by a differential equation $\partial(Y) = AY$, the object $M^{(1)}$ is attached to the differential equation: $\partial(Z) = \begin{pmatrix} A & \partial A \\ 0 & A \end{pmatrix} Z$.

Mixed situation: Let us assume that B is a generalized differential subring of some q(resp. q_v)-difference differential field $(L(x), \delta_q)$ (resp. $(L(x), \delta_{q_v})$). The category $Diff_B$ is the category of q-(resp. q_v-)difference modules. Applying the same constructions than in §2.2, we have that $Diff_B$ is a differential tannakian category and we will denote by F_∂ its prolongation functor.

In both cases, semi-classic and mixed, we can define, as in Chapter 6, the parametrized intrinsic Galois group $Gal^\partial(\mathcal{M}, \eta_B)$ of an object $\mathcal{M}$ of $Diff_B$. If $Constr_B^\partial$ denotes the smallest family of objects deduced from $\mathcal{M}$ by the constructions of linear algebras and the prolongation functor F_∂, then the parametrized analogue of Proposition 9.9 says that the differential group scheme $Gal^\partial(\mathcal{M}, \eta_B)$ is the stabilizer inside $GL(\mathcal{M})$ of all submodules with connection of some constructions of linear differential algebra of $\mathcal{M}$.

Remark 9.11. In the semi-classic situation, the parametrized intrinsic Galois group of a differential module $\mathcal{M}$ is nothing else than the intrinsic Galois group of $\mathcal{M}$. To see this it is enough to notice that there exists a canonical isomorphism:

$$Gal(F_\partial(\mathcal{M}), \eta_{K(x)}) \longrightarrow Gal(\mathcal{M}, \eta_{K(x)}).$$

In fact, such an arrow exists since $\mathcal{M}$ is canonically isomorphic to a differential submodule of $F_\partial(\mathcal{M})$. Since an element $B \in Gal(\mathcal{M}, \eta_{K(x)})$ acts on $F_\partial(\mathcal{M})$ via $\begin{pmatrix} B & \partial B \\ 0 & B \end{pmatrix}$, the arrow is injective. Since an element of $Gal(\mathcal{M}, \eta_{K(x)})$ needs to be sufficiently compatible with the differential structure, it also stabilizes the differential submodules of a construction of $F_\partial(\mathcal{M})$. This last argument proves the surjectivity.

The definition below characterizes the morphisms of generalized differential rings compatible with the differential structure.

Definition 9.12 (*cf.* [**And01**, 2.4.5.1]). Let $u = (u^0, u^1) : (A, d) \mapsto (A', d')$ be a morphism of generalized differential rings. This morphism induces a tensor-compatible functor denoted by u^* from the category $Diff_A$ into the category $Diff_{A'}$. Moreover, let us assume that $Diff_A$ (resp. $Diff_{A'}$) is a differential category and let us denote by F_∂ its prolongation functor. We say that u^* is *differentially compatible* if it commutes with the prolongation functors, i.e., $F_\partial \circ u^* = u^* \circ F_\partial$.

9.2. Specialization of the parameter q and localization of the intrinsic Galois group

We go back to the notation introduced in Chapter 5, in the case where q is transcendent over the base field. So we consider a field K, which is a finite extension of a rational function field $k(q)$. We recall that when speaking of differential algebraic groups, we implicitly require that k is of characteristic zero.

We denote by $\mathcal{P}_f$ the set of places of K such that the associated norms extend, up to equivalence, one of the norms of $k(q)$ attached to an irreducible polynomial $v(q) \in k[q], v(q) \neq q$, by k_v the residue field of K with respect to a place v and by q_v the image of q in k_v.

Let $\mathcal{M} = (M, \Sigma_q)$ be a q-difference module over an algebra $\mathcal{A}$ of the form $\mathcal{O}_K\left[x, \frac{1}{P(x)}, \frac{1}{P(qx)}, \ldots\right]$. For almost all finite place $v \in \mathcal{P}_f$, we can consider the $k_v(x)$-module $M_{k_v(x)} = M \otimes_{\mathcal{A}} k_v(x)$ with the structure induced by Σ_q. In this way, for almost all $v \in \mathcal{P}_f$, we obtain a q_v-difference module $\mathcal{M}_{k_v(x)} = (M_{k_v(x)}, \Sigma_{q_v})$. If we can specialize $\mathcal{M}$ modulo $q-1$, then we get a differential module, whose connection is induced by the action of the operator $\Delta_q = \frac{\Sigma_q - Id}{(q-1)}$ on M. We call the module $\mathcal{M}_{k_v(x)} = (M_{k_v(x)}, \Sigma_{q_v})$ the *specialization* of $\mathcal{M}$ at v. It is naturally equipped with an intrinsic Galois group $Gal(\mathcal{M}_{k_v(x)}, \eta_{k_v(x)})$, associated to the forgetful functor $\eta_{k_v(x)}$. Then, we can ask how the intrinsic Galois group of the specialization $\mathcal{M}_{k_v(x)}$ is related to the specialization at the place v of the equations of the intrinsic Galois group of $\mathcal{M}$. For $v \in \mathcal{C}$, Theorem 7.13 proves that one may recover $Gal(\mathcal{M}_{K(x)}, \eta_{K(x)})$ from the knowledge of almost all of intrinsic Galois groups of its reduction at ϕ_v. In general, for $v \in \mathcal{P}_f$, the specialization of the intrinsic Galois group gives only an upper bound for the intrinsic Galois group of the specialized equation (see Proposition 9.15).

These problems have been studied by Y. André in [**And01**] where he shows, among other things, that the groups of tensor automorphism of neutral fiber functors have a nice behaviour with respect to the specialization.The results of this chapter (see Proposition 9.15 for instance) are nothing more than an adaptation of the results of André to our framework. Moreover, we want to underline the fact that, unlike the neutral fiber functors considered by André, the forgetful functor is automatically compatible with the base change. So that we are, in fact, in a much easier situation than in [**And01**]. However, for sake of completeness, we detail all the statements (since they are not exactly contained in [**And01**]) and proofs. Moreover, we want to emphasize that considering intrinsic Galois group instead of neutral Tannakian groups, allows us to give a description via curvatures of the intrinsic Galois group of a differential equation (see Corollary 9.18).

The following lemma of *localization* relates the intrinsic (parametrized) Galois group of a q-difference module over $K(x)$. This lemma is a version of [**And01**, Lemma 3.2.3.6] for (parametrized) intrinsic Galois groups.

PROPOSITION 9.13. *Let $\mathcal{M}$ be a q-difference module over $K(x)$. Let $\mathcal{A} = \mathcal{O}_K\left[x, \frac{1}{P(x)}, \frac{1}{P(qx)}, \ldots\right]$ be a q-difference sub-algebra of $K(x)$ such that $\mathcal{M}$ is defined over $\mathcal{A}$. Let $v \in \mathcal{P}_f$ and let $\mathcal{A}_v := \mathcal{A} \otimes_{\mathcal{O}_K} k_v$. We have,*

(1) $Gal(\mathcal{M}, \eta_{\mathcal{A}}) \otimes K(x) \simeq Gal(\mathcal{M}_{K(x)}, \eta_{K(x)})$;
(2) $Gal^{\partial}(\mathcal{M}, \eta_{\mathcal{A}}) \otimes K(x) \simeq Gal^{\partial}(\mathcal{M}_{K(x)}, \eta_{K(x)})$
(3) $Gal(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{A}_v, \eta_{\mathcal{A}_v}) \otimes k_v(x) \simeq Gal(\mathcal{M}_{k_v(x)}, \eta_{k_v(x)})$.
(4) $Gal^{\partial}(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{A}_v, \eta_{\mathcal{A}_v}) \otimes k_v(x) \simeq Gal^{\partial}(\mathcal{M}_{k_v(x)}, \eta_{k_v(x)})$.

REMARK 9.14. In the previous section, we have given a description of the intrinsic Galois group $Gal(\mathcal{M}_{K(x)}, \eta_{K(x)})$ *via* the reduction modulo ϕ_v of the operators

$\Sigma_q^{\kappa_v}$. We are unable to give a similar description of $Gal(\mathcal{M},\eta_{\mathcal{A}})$, essentially because Chevalley theorem holds only for algebraic groups over a field.

PROOF. We give the proof in the parametrized situation. The algebraic one follows easily. First remark that $\mathcal{A}$ and $\mathcal{A}_v$ are ∂-algebras. Moreover, $(\mathcal{A},\delta_q)$ (resp. $(\mathcal{A}_v,\delta_{q_v})$) is a simple differential ring, i.e., it has no non-trivial δ_q (resp. δ_{q_v})-ideals. It comes from the fact that for a polynomial $Q(x)$ of degree d, the polynomials $\delta_q(Q),\delta_{q_v}(Q)$ have degree strictly inferior to d. However, one has to pay attention that, even if $\mathcal{A}$ is δ_q-simple, it is not a σ_q-simple : the ideal $z\mathcal{A}$ is a σ_q-ideal. Moreover, $\Omega^1(\mathcal{A})$ (resp. $\Omega^1(\mathcal{A}_v)$) is a projective $\mathcal{A}$ (resp. $\mathcal{A}_v$) module of rank 1 (see Example 9.4). In that conditions, the functor of localization $Loc : Diff_{\mathcal{A}} \rightarrow Diff_{K(x)}, \mathcal{N} \mapsto \mathcal{N}\otimes K(x)$ (resp. $Loc_v : Diff_{\mathcal{A}_v} \rightarrow Diff_{k_v(x)}, \mathcal{N} \mapsto \mathcal{N}\otimes k_v(x)$) is full and faithful by [**And01**, 2.5.1.2 and 2.5.2.1]. Moreover, the localization functors commute with the prolongation and forgetful functors so that one can consider their restriction to the differential Tannakian category generated by $\mathcal{M}$ (resp $\mathcal{M}\otimes_{\mathcal{A}}\mathcal{A}_v$). Then, the localization $Loc : \langle\mathcal{M}\rangle^{\otimes,\partial} \rightarrow \langle\mathcal{M}_{K(x)}\rangle^{\otimes,\partial}$ (resp. $Loc_v : \langle\mathcal{M}\otimes_{\mathcal{A}}\mathcal{A}_v\rangle^{\otimes,\partial} \rightarrow \langle\mathcal{M}_{k_v(x)}\rangle^{\otimes,\partial}$) is an equivalence of differential tensor category. The essential surjectivity comes from the fact that if $\mathcal{N}'$ is a sub-object of some $\bigoplus \mathcal{M}_{K(x)}^{\otimes i}\otimes {\mathcal{M}_{K(x)}^{*}}^{\otimes j}\otimes F_\partial^l(\mathcal{M}_{K(x)}^{\otimes r}\otimes {\mathcal{M}_{K(x)}^{*}}^{\otimes s})$ then $\mathcal{N}'=\mathcal{N}\otimes_{\mathcal{A}} K(x)$ where $\mathcal{N} := \mathcal{N}'\cap(\bigoplus \mathcal{M}^{\otimes i}\otimes\mathcal{M}^{*\otimes j}\otimes F_\partial^l(\mathcal{M}^{\otimes r}\otimes\mathcal{M}^{*\otimes s})$ is an object of $Diff_{\mathcal{A}}$. The same reasoning yield modulo v. Finally, we get the isomorphism between the intrinsic Galois groups from these equivalence of differential tensor categories and the fact that they commute with the forgetful functor. □

Finally, we investigate the compatibility of the intrinsic Galois groups with respect to the specialization at the place v. This proposition relies on [**And01**, §3.3]

PROPOSITION 9.15. *Let $(\mathcal{A},\delta_q)$ be the generalized differential ring as in Proposition 9.13. Let v be a finite place of K. For any $\mathcal{M}$ object of $Diff_{\mathcal{A}}$, we have*

$$Gal(\mathcal{M}\otimes_{\mathcal{A}}\mathcal{A}_v,\eta_{\mathcal{A}_v}) \subset Gal(\mathcal{M},\eta_{\mathcal{A}})\otimes\mathcal{A}_v$$

and

$$Gal^{\partial}(\mathcal{M}\otimes_{\mathcal{A}}\mathcal{A}_v,\eta_{\mathcal{A}_v}) \subset Gal^{\partial}(\mathcal{M},\eta_{\mathcal{A}})\otimes\mathcal{A}_v.$$

PROOF. Once again, we do the proof only in the parametrized case. First, we can remark that since k_v is a quotient of $\mathcal{O}_K$, we have $k_v\otimes_{\mathcal{O}_K}k_v$ is isomorphic to k_v. Thus, we are in the situation studied in [**And01**, 3.2.3.4]. Moreover, since $k_v\otimes_{\mathcal{O}_K}k_v=k_v$, the constructions of differential linear algebra of $\mathcal{M}$ commutes with the base change $-\otimes_{\mathcal{O}_K}k_v$:

$$\begin{aligned}&\left(\bigoplus \mathcal{M}^{\otimes i}\otimes\mathcal{M}^{*\otimes j}\otimes F_\partial^l(\mathcal{M}^{\otimes r}\otimes\mathcal{M}^{*\otimes s})\right)\otimes_{\mathcal{A}}\mathcal{A}_v\\ &\qquad=\bigoplus(\mathcal{M}\otimes_{\mathcal{A}}\mathcal{A}_v)^{\otimes i}\otimes(\mathcal{M}\otimes_{\mathcal{A}}\mathcal{A}_v)^{*\otimes j}\otimes F_\partial^l((\mathcal{M}\otimes_{\mathcal{A}}\mathcal{A}_v)^{\otimes r}\otimes(\mathcal{M}\otimes_{\mathcal{A}}\mathcal{A}_v)^{*\otimes s}).\end{aligned}$$

By definition, $Gal^{\partial}(\mathcal{M}\otimes_{\mathcal{A}}\mathcal{A}_v,\eta_{\mathcal{A}_v})=Aut^{\otimes,\partial}(\eta_{\mathcal{A}_v}|_{\langle\mathcal{M}\otimes\mathcal{A}_v\rangle^{\otimes}})$ is the stabilizer inside $GL(\mathcal{M}\otimes_{\mathcal{A}}\mathcal{A}_v)=GL(\mathcal{M})\otimes_{\mathcal{A}}\mathcal{A}_v$ of the sub-objects $\mathcal{W}$ of a construction of differential linear algebra of $\mathcal{M}\otimes_{\mathcal{A}}\mathcal{A}_v=\mathcal{M}\otimes_{\mathcal{O}_K}k_v$. The group $Gal^{\partial}(\mathcal{M},\eta_{\mathcal{A}})$ admits a similar description. Thus, we deduce the inclusion between the intrinsic Galois groups from the compatibility of the construction of differential linear algebra with respect to the base change and from the definition of the parametrized intrinsic Galois group in terms of stabilizer of objects inside the constructions of differential linear algebra . □

REMARK 9.16. Similar results hold for differential equations (*cf.* [**Kat90**, §2.4] and [**And01**, §3.3]). In general one cannot hope for a semicontinuity result. In fact, the differential equation $\frac{y'}{y} = \frac{\lambda}{y}$, with λ complex parameter, has differential Galois group equal to $\mathbb{C}^*$. When one specializes the parameter λ on a rational value λ_0, one gets an equation whose differential Galois group is a cyclic group of order the denominator of λ_0. For all other values of the parameter, the Galois group is $\mathbb{C}^*$.

The situation appears to be more rigid for q-difference equations when q is a parameter. In fact, we can consider the q-difference equation $y(qx) = P(q)y(x)$, with $P(q) \in k(q)$. If we specialize q to a root of unity and we find a finite intrinsic Galois group "too often", we can conclude using Theorem 7.13 that $P(q) \in q^{\mathbb{Z}/r}$, for some positive integer r, and therefore that the intrinsic Galois group of $y(qx) = P(q)y(x)$ over $K(x)$ is finite.

9.3. Upper bounds for the intrinsic Galois group of a differential equation

Let us consider a q-difference module $\mathcal{M} = (M, \Sigma_q)$ over $\mathcal{A}$ that admits a reduction modulo the $(q-1)$-adic place of K, i.e., such that we can specialize the parameter q to 1. To simplify notation, let us denote by k_1 the residue field of K modulo $q-1$.

In this case the specialized module $\mathcal{M}_{k_1(x)} = (M_{k_1(x)}, \Delta_1)$ is a differential module. We can deduce from the results above that:

Corollary 9.17.

$$Gal(\mathcal{M}_{k_1(x)}, \eta_{k_1(x)}) \subset Gal(\mathcal{M}, \eta_{\mathcal{A}}) \otimes k_1(x).$$

and

$$Gal^{\partial}(\mathcal{M}_{k_1(x)}, \eta_{k_1(x)}) \subset Gal^{\partial}(\mathcal{M}, \eta_{\mathcal{A}}) \otimes k_1(x).$$

Proof. Proposition 9.15 says that:

$$Gal(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{A}/(q-1), \eta_{\mathcal{A}/(q-1)}) \subset Gal(\mathcal{M}, \eta_{\mathcal{A}}) \otimes \mathcal{A}/(q-1),$$

and

$$Gal^{\partial}(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{A}/(q-1), \eta_{\mathcal{A}/(q-1)}) \subset Gal^{\partial}(\mathcal{M}, \eta_{\mathcal{A}}) \otimes \mathcal{A}/(q-1),$$

We conclude applying Proposition 9.13:

$$Gal(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{A}/(q-1), \eta_{\mathcal{A}/(q-1)}) \otimes_{\mathcal{A}/(q-1)} k_1(x) \cong Gal(\mathcal{M}_{k_1(x)}, \eta_{k_1(x)}),$$

and

$$Gal^{\partial}(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{A}/(q-1), \eta_{\mathcal{A}/(q-1)}) \otimes_{\mathcal{A}/(q-1)} k_1(x) \cong Gal^{\partial}(\mathcal{M}_{k_1(x)}, \eta_{k_1(x)}).$$

□

On the other hand, given a $k(x)/k$-differential module $\mathcal{M} = (M, \nabla)$, we can fix a basis $\underline{e}$ of M such that

$$\nabla(\underline{e}) = \underline{e}G(x),$$

where we have identified ∇ with $\nabla\left(\frac{d}{dx}\right)$. The horizontal vectors for ∇ are solutions of the system $Y'(x) = -G(x)Y(x)$. Then, if $K/k(q)$ is a finite extension, we can define a natural q-difference module structure over $M_{K(x)} = M \otimes_{k(x)} K(x)$ setting

$$\Sigma_q(\underline{e}) = \underline{e}\left(1 + (q-1)xG(x)\right),$$

and extending the action of Σ_q to $M_{K(x)}$ by semi-linearity. The definition of Σ_q depends on the choice of $\underline{e}$, so that we should rather write $\Sigma_q^{(\underline{e})}$, which we avoid to not complicate the notation. Thus, starting from a differential module $\mathcal{M}$ we can find a q-difference module $\mathcal{M}_{K(x)}$ such that $\mathcal{M}$ is the specialization of $\mathcal{M}_{K(x)}$ at the place of K defined by $q = 1$. The q-deformation we have considered here is somehow trivial and does not correspond, for instance, to the process used to deform a hypergeometric differential equation into a q-hypergeometric equation. Anyway, we just want to show that a q-deformation combined with our results gives an arithmetic description of the intrinsic Galois group of a differential equation. This description depends obviously of the process of q-deformation and its refinement is strongly related to the sharpness of the q-deformation used.

Using the "trivial" q-deformation, we have the following description

COROLLARY 9.18. *The intrinsic Galois group of $\mathcal{M} = (M, \nabla)$ is contained in the "specialization at $q = 1$" of the smallest algebraic subgroup of* $\mathrm{GL}(M_{K(x)})$ *containing the reduction modulo ϕ_v of $\Sigma_q^{\kappa_v}$:*

$$\Sigma_q^{\kappa_v} \underline{e} = \underline{e} \prod_{i=0}^{\kappa_v - 1} \left(1 + (q-1)q^i x G(q^i x)\right),$$

for almost all $v \in \mathcal{C}_K$.

COROLLARY 9.19. *Suppose that k is algebraically closed. Then a differential module (M, ∇) is trivial over $k(x)$ if and only if there exists a basis $\underline{e}$ such that $\nabla(\underline{e}) = \underline{e}G(x)$ and for almost all primitive roots of unity ζ in a fixed algebraic closure $\overline{k}$ of k we have:*

$$\left[\prod_{i=0}^{n-1} \left(1 + (q-1)q^i x G(q^i x)\right)\right]_{q=\zeta} = \textit{identity matrix},$$

where n is the order of ζ.

PROOF. If the identity above is verified, then the Galois group of (M, ∇) is trivial, which implies that (M, ∇) is trivial over $k(x)$. On the other hand, if (M, ∇) is trivial over $k(x)$, there exists a basis $\underline{e}$ of M over $k(x)$ such that $\nabla(\underline{e}) = 0$. This ends the proof. $\square$

Part 5

Comparison with the non-linear theory

CHAPTER 10

Preface to Part 5. The Galois D-groupoid of a q-difference system, by Anne Granier

We recall here the definition of the Galois D-groupoid of a q-difference system, and how to recover groups from it in the case of a linear q-difference system. This appendix thus consists in a summary of Chapter 3 of [**Gra09**].

10.1. Definitions

We need to recall first Malgrange's definition of D-groupoids, following [**Mal01**] but specializing it to the base space $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{C}^\nu$ as in [**Gra09**] and [**Gra**], and to explain how it allows to define a Galois D-groupoid for q-difference systems.

Fix $\nu \in \mathbb{N}^*$, and denote by M the analytic complex variety $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{C}^\nu$. We call *local diffeomorphism of* M any biholomorphism between two open sets of M, and we denote by $Aut(M)$ the set of germs of local diffeomorphisms of M. Essentially, a D-groupoid is a subgroupoid of $Aut(M)$ defined by a system of partial differential equations.

Let us precise what is the object which represents the system of partial differential equations in this rough definition.

A germ of a local diffeomorphism of M is determined by the coordinates denoted by $(x, X) = (x, X_1, \ldots, X_\nu)$ of its source point, the coordinates denoted by $(\bar{x}, \bar{X}) = (\bar{x}, \bar{X}_1, \ldots, \bar{X}_\nu)$ of its target point, and the coordinates denoted by $\frac{\partial \bar{x}}{\partial x}, \frac{\partial \bar{x}}{\partial X_1}, \ldots, \frac{\partial \bar{X}_1}{\partial x}, \ldots, \frac{\partial^2 \bar{x}}{\partial x^2}, \ldots$ which represent its partial derivatives evaluated at the source point. We also denote by δ the polynomial in the coordinates above, which represents the Jacobian of a germ evaluated at the source point. We will allow us abbreviations for some sets of these coordinates, as for example $\frac{\partial \bar{X}}{\partial X}$ to represent all the coordinates $\frac{\partial \bar{X}_i}{\partial X_j}$ and $\partial \bar{X}$ for all the coordinates $\frac{\partial \bar{X}_i}{\partial x_j}$, $\frac{\partial \bar{X}_i}{\partial \bar{x}_j}$, $\frac{\partial \bar{X}_i}{\partial X_j}$ and $\frac{\partial \bar{X}_i}{\partial \bar{X}_j}$.

We denote by r any positive integer. We call *partial differential equation*, or only *equation*, of order $\leq r$ any fonction $E(x, X, \bar{x}, \bar{X}, \partial\bar{x}, \partial\bar{X}, \ldots, \partial^r\bar{x}, \partial^r\bar{X})$ which locally and holomorphically depends on the source and target coordinates, and polynomially on δ^{-1} and on the partial derivative coordinates of order $\leq r$. These equations are endowed with a sheaf structure on $M \times M$ which we denote by $\mathcal{O}_{J_r^*(M,M)}$. We then denote by $\mathcal{O}_{J^*(M,M)}$ the sheaf of all the equations, that is the direct limit of the sheaves $\mathcal{O}_{J_r^*(M,M)}$. It is endowed with natural derivations of the equations with respect to the source coordinates. For example, one has: $D_x \cdot \frac{\partial \bar{X}_i}{\partial X_j} = \frac{\partial^2 \bar{X}_i}{\partial x \partial X_j}$.

We will consider the pseudo-coherent (in the sense of [**Mal01**]) and differential ideal [1] $\mathcal{I}$ of $\mathcal{O}_{J^*(M,M)}$ as the systems of partial differential equations in the

[1] We will say everywhere differential ideal for sheaf of differential ideal.

definition of D-groupoid. A *solution* of such an ideal $\mathcal{I}$ is a germ of a local diffeomorphism $g : (M,a) \to (M,g(a))$ such that, for any equation E of the fiber $\mathcal{I}_{(a,g(a))}$, the function defined by $(x,X) \mapsto E((x,X), g(x,X), \partial g(x,X), \ldots)$ is null in a neighbourhood of a in M. The solutions of $\mathcal{I}$ is denoted by $sol(\mathcal{I})$ and forms a set groupoid.

The set $Aut(M)$ is endowed with a groupoid structure for the composition c and the inversion i of the germs of local diffeomorphisms of M. We thus have to characterize, with the comorphisms c^* and i^* defined on $\mathcal{O}_{J^*(M,M)}$, the systems of partial differential equations $\mathcal{I} \subset \mathcal{O}_{J^*(M,M)}$ whose set of solutions $sol(\mathcal{I})$ is a subgroupoid of $Aut(M)$.

We call *groupoid of order* r on M the subvariety of the space of invertible jets of order r defined by a coherent ideal $\mathcal{I}_r \subset \mathcal{O}_{J_r^*(M,M)}$ such that *(i)*: all the germs of the identity map of M are solutions of $\mathcal{I}_r$, such that *(ii)*: $c^*(\mathcal{I}_r) \subset \mathcal{I}_r \otimes \mathcal{O}_{J_r^*(M,M)} + \mathcal{O}_{J_r^*(M,M)} \otimes \mathcal{I}_r$, and such that *(iii)*: $\iota^*(\mathcal{I}_r) \subset \mathcal{I}_r$. The solutions of such an ideal $\mathcal{I}_r$ form a subgroupoid of $Aut(M)$.

DEFINITION 10.1. According to [**Mal01**], a *D-groupoid* $\mathcal{G}$ on M is a subvariety of the space $(M^2, \mathcal{O}_{J^*(M,M)})$ of invertible jets defined by a reduced, pseudo-coherent and differential ideal $\mathcal{I}_{\mathcal{G}} \subset \mathcal{O}_{J^*(M,M)}$ such that

- *(i')* all the germs of the identity map of M are solutions of $\mathcal{I}_{\mathcal{G}}$,
- *(ii')* for any relatively compact open set U of M, there exists a closed complex analytic subvariety Z of U of codimension ≥ 1, and a positive integer $r_0 \in \mathbb{N}$ such that, for all $r \geq r_0$ and denoting by $\mathcal{I}_{\mathcal{G},r} = \mathcal{I}_{\mathcal{G}} \cap \mathcal{O}_{J_r^*(M,M)}$, one has, above $(U \setminus Z)^2$: $c^*(\mathcal{I}_{\mathcal{G},r}) \subset \mathcal{I}_{\mathcal{G},r} \otimes \mathcal{O}_{J_r^*(M,M)} + \mathcal{O}_{J_r^*(M,M)} \otimes \mathcal{I}_{\mathcal{G},r}$,
- *(iii')* $\iota^*(\mathcal{I}_{\mathcal{G}}) \subset \mathcal{I}_{\mathcal{G}}$.

The ideal $\mathcal{I}_{\mathcal{G}}$ totally determines the D-groupoid $\mathcal{G}$, so we will rather focus on the ideal $\mathcal{I}_{\mathcal{G}}$ than its solution $sol(\mathcal{I}_{\mathcal{G}})$ in $Aut(M)$. Thanks to the analytic continuation theorem, $sol(\mathcal{I}_{\mathcal{G}})$ is a subgroupoid of $Aut(M)$.

The flexibility introduced by Malgrange in his definition of D-groupoid allows him to obtain two main results. Theorem 4.4.1 of [**Mal01**] states that the reduced differential ideal of $\mathcal{O}_{J^*(M,M)}$ generated by a coherent ideal $\mathcal{I}_r \subset \mathcal{O}_{J_r^*(M,M)}$ which satisfies the previous conditions *(i)*,*(ii)*, and *(iii)* defines a D-groupoid on M. Theorem 4.5.1 of [**Mal01**] states that for any family of D-groupoids on M defined by a family of ideals $\{\mathcal{G}^i\}_{i\in I}$, the ideal $\sqrt{\sum \mathcal{G}^i}$ defines a D-groupoid on M called *intersection*. The terminology is legitimated by the equality: $sol(\sqrt{\sum \mathcal{G}^i}) = \cap_{i\in I} sol(\mathcal{G}^i)$. This last result allows to define the notion of D-envelope of any subgroupoid of $Aut(M)$.

Fix $q \in \mathbb{C}^*$, and let $Y(qx) = F(x, Y(x))$ be a (non-linear) q-difference system, with $F(x,X) \in \mathbb{C}(x,X)^\nu$. Consider the set subgroupoid of $Aut(M)$ generated by the germs of the application $(x,X) \mapsto (qx, F(x,X))$ at any point of M where it is well defined and invertible, and denote it by $Dyn(F)$. The Galois D-groupoid of the q-difference system $Y(qx) = F(x,Y(x))$ is the D-enveloppe of $Dyn(F)$, that is the *intersection* of the D-groupoids on M whose set of solutions contains $Dyn(F)$.

10.2. A bound for the Galois D-groupoid of a linear q-difference system

For all the following, consider a rational linear q-difference system $Y(qx) = A(x)Y(x)$, with $A(x) \in GL_\nu(\mathbb{C}(x))$. We denote by $\mathcal{G}al(A(x))$ the Galois D-groupoid of this system as defined at the end of the previous section 10.1, we denote by

$\mathcal{I}_{\mathcal{G}al(A(x))}$ its defining ideal of equations, and by $sol(\mathcal{G}al(A(x)))$ its groupoid of solutions.

The elements of the dynamics $Dyn(A(x))$ of $Y(qx) = A(x)Y(x)$ are the germs of the local diffeomorphisms of M of the form $(x, X) \mapsto (q^k x, A_k(x)X)$, with:

$$A_k(x) = \begin{cases} Id_n & \text{if } k = 0, \\ \prod_{i=0}^{k-1} A(q^i x) & \text{if } k \in \mathbb{N}^*, \\ \prod_{i=k}^{-1} A(q^i x)^{-1} & \text{if } k \in -\mathbb{N}^*. \end{cases}$$

The first component of these diffeomorphisms is independent on the variables X and depends linearly on the variable x, and the second component depends linearly on the variables X. These properties can be expressed in terms of partial differential equations. This gives an *upper bound* for the Galois D-groupoid $\mathcal{G}al(A(x))$ which is defined in the following proposition.

PROPOSITION 10.2. *The coherent ideal:*

$$\left\langle \frac{\partial \bar{x}}{\partial X}, \frac{\partial \bar{x}}{\partial x} x - \bar{x}, \partial^2 \bar{x}, \frac{\partial \bar{X}}{\partial X} X - \bar{X}, \frac{\partial^2 \bar{X}}{\partial X^2} \right\rangle \subset \mathcal{O}_{J_2^*(M,M)}$$

satisfies the conditions (i),(ii), and (iii) of 10.1. Hence, thanks to Theorem 4.4.1 of [**Mal01**], *the reduced differential ideal $\mathcal{I}_{\mathcal{L}in}$ it generates defines a D-groupoid $\mathcal{L}in$. Its solutions $sol(\mathcal{L}in)$ are the germs of the local diffeomorphisms of M of the form:*

$$(x, X) \mapsto (\alpha x, \beta(x) X),$$

with $\alpha \in \mathbb{C}^*$ and locally, $\beta(x) \in GL_\nu(\mathbb{C})$ for all x.
They contain $Dyn(A(x))$, and therefore, given the definition of $\mathcal{G}al(A(x))$, one has the inclusion

$$\mathcal{G}al(A(x)) \subset \mathcal{L}in,$$

which means that:

$$\mathcal{I}_{\mathcal{L}in} \subset \mathcal{I}_{\mathcal{G}al(A(x))} \text{ and } sol(\mathcal{G}al(A(x))) \subset sol(\mathcal{L}in).$$

PROOF. *cf* proof of Proposition 3.2.1 of [**Gra09**] for more details. □

REMARK 10.3. Given their shape, the solutions of $\mathcal{L}in$ are naturally defined in neighborhoods of transversals $\{x_a\} \times \mathbb{C}^\nu$ of M. Actually, consider a particular element of $sol(\mathcal{L}in)$, that is precisely a germ at a point $(x_a, X_a) \in M$ of a local diffeomorphism g of M of the form $(x, X) \mapsto (\alpha x, \beta(x)X)$. Consider then a neighborhood Δ of x_a in $P^1\mathbb{C}$ where the matrix $\beta(x)$ is well defined and invertible, consider the "cylinders" $T_s = \Delta \times \mathbb{C}^\nu$ and $T_t = \alpha\Delta \times \mathbb{C}^\nu$ of M, and the diffeomorphism $\tilde{g} : T_s \to T_t$ well defined by $(x, X) \to (\alpha x, \beta(x)X)$. Therefore, according to the previous Proposition 10.2, all the germs of $\tilde{g}$ at the points of T_s are in $sol(\mathcal{L}in)$ too.

The defining ideal $\mathcal{I}_{\mathcal{L}in}$ of the bound $\mathcal{L}in$ is generated by very simple equations. This allows to reduce modulo $\mathcal{I}_{\mathcal{L}in}$ the equations of $\mathcal{I}_{\mathcal{G}al(A(x))}$ and obtain some simpler representative equations, in the sense that they only depend on some variables.

PROPOSITION 10.4. *Let $r \geq 2$. For any equation $E \in \mathcal{I}_{\mathcal{G}al(A(x))}$ of order r, there exists an invertible element $u \in \mathcal{O}_{J_r^*(M,M)}$, an equation $L \in \mathcal{I}_{\mathcal{L}in}$ of order r, and an equation $E_1 \in \mathcal{I}_{\mathcal{G}al(A(x))}$ of order r only depending on the variables written below, such that:*

$$uE = L + E_1 \left(x, X, \frac{\partial \bar{x}}{\partial x}, \frac{\partial \bar{X}}{\partial X}, \frac{\partial^2 \bar{X}}{\partial x \partial X}, \dots \frac{\partial^r \bar{X}}{\partial x^{r-1} \partial X} \right).$$

PROOF. The invertible element u is a convenient power of δ. The proof consists then in performing the divisions of the equation uE, and then its succesive remainders, by the generators of $\mathcal{I}_{\mathcal{L}in}$. More details are given in the proof of Proposition 3.2.3 of [**Gra09**]. □

10.3. Groups from the Galois D-groupoid of a linear q-difference system

We are going to prove that the solutions of the Galois D-groupoid $\mathcal{G}al(A(x))$ are, like the solutions of the bound $\mathcal{L}in$, naturally defined in neighbourhoods of transversals of M. This property, together with the groupoid structure of $sol(\mathcal{G}al(A(x)))$, allows to exhibit groups from the solutions of $\mathcal{G}al(A(x))$ which fix the transversals.

According to Proposition 10.2, an element of $sol(\mathcal{G}al(A(x)))$ is also an element of $sol(\mathcal{L}in)$. Therefore, it is a germ at a point $a = (x_a, X_a) \in M$ of a local diffeomorphism $g : (M, a) \to (M, g(a))$ of the form $(x, X) \mapsto (\alpha x, \beta(x)X)$, such that, for any equation $E \in \mathcal{I}_{\mathcal{G}al(A(x))}$, one has $E((x, X), g(x, X), \partial g(x, X), \ldots) = 0$ in a neighbourhood of a in M.

Consider an open connected neighbourhood Δ of x_a in $\mathbb{P}^1_{\mathbb{C}}$ on which the matrix β is well-defined and invertible, that is where β can be prolongated in a matrix $\beta \in GL_\nu(\mathcal{O}(\Delta))$. Consider the "cylinders" $T_s = \Delta \times \mathbb{C}^\nu$ and $T_t = \alpha\Delta \times \mathbb{C}^\nu$ of M, and the diffeomorphism $\tilde{g} : T_s \to T_t$ defined by $(x, X) \to (\alpha x, \beta(x)X)$.

PROPOSITION 10.5. *The germs at all points of T_s of the diffeomorphism $\tilde{g}$ are elements of $sol(\mathcal{G}al(A(x)))$.*

PROOF. For all $r \in \mathbb{N}$, the ideal $(\mathcal{I}_{\mathcal{G}al(A(x))})_r = \mathcal{I}_{\mathcal{G}al(A(x))} \cap \mathcal{O}_{J_r^*(M,M)}$ is coherent. Thus, for any point $(y_0, \bar{y}_0) \in M^2$, there exists an open neighbourhood Ω of $(y_0, \bar{y}_0)$ in M^2, and equations $E_1^\Omega, \ldots, E_l^\Omega$ of $(\mathcal{I}_{\mathcal{G}al(A(x))})_r$ defined on the open set Ω such that:

$$\left((\mathcal{I}_{\mathcal{G}al(A(x))})_r\right)_{|\Omega} = \left(\mathcal{O}_{J_r^*(M,M)}\right)_{|\Omega} E_1^\Omega + \cdots + \left(\mathcal{O}_{J_r^*(M,M)}\right)_{|\Omega} E_l^\Omega.$$

Let $a_1 \in T_s = \Delta \times \mathbb{C}^\nu$. Let $\gamma : [0, 1] \to T_s$ be a path in T_s such that $\gamma(0) = a$ and $\gamma(1) = a_1$. Let $\{\Omega_0, \ldots, \Omega_N\}$ be a finite covering of the path $\gamma([0, 1]) \times \tilde{g}(\gamma([0, 1]))$ in $T_s \times T_t$ by connected open sets $\Omega \subset (T_s \times T_t)$ like above, and such that the origin $(\gamma(0), g(\gamma(0))) = (a, g(a))$ belongs to Ω_0.
The germ of g at the point a is an element of $sol(\mathcal{G}al(A(x)))$. Therefore, one has $E_k^{\Omega_0}((x, X), g(x, X), \partial g(x, X), \ldots) \equiv 0$ in a neighbourhood of a, for all $1 \leq l \leq k$. Moreover, by analytic continuation, one has also $E_k^{\Omega_0}(x, X, \tilde{g}(x, X), \partial\tilde{g}(x, X), \ldots) \equiv 0$ on the source projection of Ω_0 in M. It means that the germs of $\tilde{g}$ at any point of the source projection of Ω_0 are solutions of $(\mathcal{I}_{\mathcal{G}al(A(x))})_r$.
Then, step by step, one gets that the germs of $\tilde{g}$ at any point of the source projection of Ω_k are solutions of $(\mathcal{I}_{\mathcal{G}al(A(x))})_r$ and, in particular, the germ of $\tilde{g}$ at the point a_1 is also a solution of $(\mathcal{I}_{\mathcal{G}al(A(x))})_r$. □

This Proposition 10.5 means that any solution of the Galois D-groupoid $\mathcal{G}al(A(x))$ is naturally defined in a neighbourhood of a transversal of M, above.

REMARK 10.6. In some sense, the "equations" counterpart of this proposition is Lemma 11.12.

The solutions of $\mathcal{G}al(A(x))$ which fix the transversals of M can be interpreted as solutions of a sub-D-groupoid of $\mathcal{G}al(A(x))$, partly because this property can be interpreted in terms of partial differential equations. Actually, a germ of a diffeomorphism of M fix the transversals of M if and only if it is a solution of the equation $\bar{x} - x$.

The ideal of $\mathcal{O}_{J_0^*(M,M)}$ generated by the equation $\bar{x} - x$ satisfies the conditions *(i)*,*(ii)*, and *(iii)* of 10.1. Hence, thanks to Theorem 4.4.1 of [**Mal01**], the reduced differential ideal it generates defines a D-groupoid:

DEFINITION 10.7. We call $\mathcal{T}rv$ the D-groupoid generated by the equation $\bar{x} - x$.

Its solutions, $sol(\mathcal{T}rv)$, are the germs of the local diffeomorphisms of M of the form: $(x, X) \mapsto (x, \bar{X}(x, X))$.

DEFINITION 10.8. We call $\widetilde{\mathcal{G}al(A(x))}$ the *intersection* D-groupoid $\mathcal{G}al(A(x)) \cap \mathcal{T}rv$, in the sense of Theorem 4.5.1 of [**Mal01**], whose defining ideal of equations $\mathcal{I}_{\widetilde{\mathcal{G}al(A(x))}}$ is generated by $\mathcal{I}_{\mathcal{G}al(A(x))}$ and $\mathcal{I}_{\mathcal{T}rv}$.

The solutions of $sol(\widetilde{\mathcal{G}al(A(x))})$ coincide with $sol(\mathcal{G}al(A(x))) \cap sol(\mathcal{T}rv)$, that are exactly the solutions of $\mathcal{G}al(A(x))$ of the form $(x, X) \mapsto (x, \beta(x)X)$. They are also naturally defined in neighbourhoods of transversals of M.

PROPOSITION 10.9. *Let $x_0 \in \mathbb{P}^1_{\mathbb{C}}$. The set of solutions of $\widetilde{\mathcal{G}al(A(x))}$ defined in a neighbourhood of the transversal $\{x_0\} \times \mathbb{C}^\nu$ of M can be identified with a subgroup of $GL_\nu(\mathbb{C}\{x - x_0\})$.*

PROOF. The solutions of the D-groupoid $\widetilde{\mathcal{G}al(A(x))}$ defined in a neighbourhood of the transversal $\{x_0\} \times \mathbb{C}^\nu$ can be considered, without loosing any information, only in a neighbourhood of the stable point $(x_0, 0) \in M$. At this point, the groupoid structure of $sol(\widetilde{\mathcal{G}al(A(x))})$ is in fact a group structure because the source and target points are always $(x_0, 0)$. Thus, considering the matrices $\beta(x)$ in the solutions $(x, X) \mapsto (x, \beta(x)X)$ of $\widetilde{\mathcal{G}al(A(x))})$ defined in a neighbourhood of $\{x_0\} \times \mathbb{C}^\nu$, one gets a subgroup of $GL_\nu(\mathbb{C}\{x - x_0\})$. More details are given in the proof of Proposition 3.3.2 of [**Gra09**]. □

In the particular case of a constant linear q-difference system, that is with $A(x) = A \in GL_\nu(\mathbb{C})$, the solutions of the Galois D-groupoid $\mathcal{G}al(A)$ are in fact global diffeomorphisms of M, and the set of those that fix the transversals of M can be identified with an algebraic subgroup of $GL_\nu(\mathbb{C})$. This can be shown using a better bound than $\mathcal{L}in$ for the Galois D-groupoid of a constant linear q-difference system (*cf* Proposition 3.4.2 of [**Gra09**]), or computing the D-groupoid $\mathcal{G}al(A)$ directly (*cf* Theorem 2.1 of [**Gra**] or Theorem 4.2.7 of [**Gra09**]). Moreover, the explicit computation allows to observe that this subgroup corresponds to the usual q-difference Galois group as described in [**Sau04b**] of the constant linear q-difference system $X(qx) = AX(x)$ (*cf.* Theorem 4.4.2 of [**Gra09**] or Theorem 2.4 of [**Gra**]).

CHAPTER 11

Comparison of the parametrized intrinsic Galois group with the Galois D-groupoid

A. Granier has defined a D-groupoid for non-linear q-difference equations, in analogy with Malgrange D-groupoid for non-linear differential equations (see the previous chapter). Roughly, this D-groupoid corresponds to the largest sheaf of analytic differential equations that kill the dynamics of the non-linear q-difference equation.

In this section we prove that the Malgrange-Granier D-groupoid, in the special case of a linear q-difference equation, essentially "coincides" with the parametrized intrinsic Galois group of the equation. This result, which is Corollary 11.10, is not a priori straightforward because one has to compare a D-groupoid defined as a sheaf of differential ideal over an analytic variety and a differential algebraic group *à la Kolchin*. This answers a question of Malgrange ([**Mal09**, page 2]).

Our proof is divided in three main steps. The first one relies on Theorem 7.13 and allows us to compare the parametrized intrinsic Galois group with the smallest differential algebraic variety that contains the dynamic, namely its Kolchin closure. Then, we sheafify the defining equations of the Kolchin closure in order to get an algebraic D-groupoid, which is defined by the largest set of algebraic differential equations that kill the dynamic. Finally thanks to GAGA arguments, we show that the defining equations of the Malgrange-Granier D-groupoid are global and algebraic and thus coincide with the ones of our algebraic D-groupoid. In the differential case, the problem of the algebraicity of the D-groupoid has been tackled in more recent works by B. Malgrange himself.

In the special case of a linear differential equation, Malgrange proves that his D-groupoid, allows to recover the Picard-Vessiot Galois group (see [**Mal01**]). The foliation associated to the solutions of the non-linear differential equation, which exists due to the Cauchy theorem, plays a central role in his proof, and actually in the whole theory. There is a true hindrance to prove a Cauchy theorem and define a foliation over $\mathbb{C}$ attached to a q-difference system. First of all, the solutions of a q-difference equation must be defined over a q-invariant domain and they usually have an essential singularity at 0 and at ∞. This fact prevents the existence of a local solution on a compact domain and therefore a transposition of the Cauchy theorem. To overcome the lack of local solutions, we use Theorem 7.13 as a crucial ingredient of our proof. However, some steps of our proof are similar to Malgrange theorem (*cf.* [**Mal01**]) and Granier's proof in the case of q-difference system with constant coefficients (see [**Gra**, §2.1]). In §11.4 below, we show how in Malgrange or Granier's former comparison results, a parametrized intrinsic Galois group is hidden and why the parametrized structure is inherent to Malgrange's D-groupoid constructions.

Our results shall give some hints to compare the algebraic definitions of Morikawa of the Galois group of a non-linear q-difference equation and the analytic definitions of A.Granier (*cf.* [**Mor09**], [**MU09**], [**Ume10**]).

11.1. The Kolchin closure of the Dynamics and the Malgrange-Granier groupoid

Let $q \in \mathbb{C}^*$ be not a root of unity and let $A(x) \in \mathrm{GL}_\nu(\mathbb{C}(x))$. We consider the linear q-difference system

$$Y(qx) = A(x)Y(x). \tag{11.1}$$

We set:

$$\begin{aligned}
&A_k(x) := A(q^{k-1}x)\dots A(qx)A(x) \text{ for all } k \in \mathbb{Z},\ k > 0;\\
&A_0(x) = Id_\nu\\
&A_k(x) := A(q^k x)^{-1}A(q^{k+1}x)^{-1}\dots A(q^{-1}x)^{-1} \text{ for all } k \in \mathbb{Z},\ k < 0,
\end{aligned}$$

so that $Y(q^k x) = A_k(x)Y(x)$, for any $k \in \mathbb{Z}$. Following Chapter 10, we denote by M the analytic complex variety $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{C}^\nu$, by $\mathcal{G}al(A(x))$ the Galois D-groupoid of the system (11.1), i.e., the D-envelop of the dynamics

$$Dyn(A(x)) = \left\{(x,X) \longmapsto (q^k x, A_k(x)X) \,:\, k \in \mathbb{Z}\right\} \tag{11.2}$$

in the space of jets $J^*(M,M)$. We keep the notation of Chapter 10, which is preliminary to the content of this section.

Warning. Following Malgrange and the convention in Chapter 10, we say that a D-groupoid $\mathcal{H}$ is contained in a D-groupoid $\mathcal{G}$ if the groupoid of solutions of $\mathcal{H}$ is contained in the groupoid of solutions of $\mathcal{G}$. We will write $sol(\mathcal{H}) \subset sol(\mathcal{G})$ or equivalently $\mathcal{I}_{\mathcal{G}} \subset \mathcal{I}_{\mathcal{H}}$, where $\mathcal{I}_{\mathcal{G}}$ and $\mathcal{I}_{\mathcal{H}}$ are the (sheaves of) ideals of definition of $\mathcal{G}$ and $\mathcal{H}$, respectively.

Notation. In this section we introduce many tools that we use to get the proof of our main result Corollary 11.10. For the reader convenience we make a list of them here, with the reference for their definitions:

$Dyn(A(x))$,	(11.2);		
$\mathcal{G}al(A(x))$,	§10.2;	$\widetilde{\mathcal{G}al(A(x))}$,	Definition 10.8;
$\mathcal{G}al^{alg}(A(x))$,	Definition 11.1;	$\widetilde{\mathcal{G}al^{alg}(A(x))}$,	Definition 11.6;
$\mathcal{K}ol(A(x))$,	Definition 11.1;	$\widetilde{\mathcal{K}ol(A(x))}$,	Definition 11.3;
$\mathcal{L}in$,	Proposition 10.2;	$\mathcal{T}rv$,	Definition 10.7.

11.2. The groupoid $\mathcal{G}al^{alg}(A(x))$

Let $\mathbb{C}(x)\left\{T, \frac{1}{\det T}\right\}_\partial$, with $T = (T_{i,j} : i,j = 0,1,\dots,\nu)$, be the algebra of differential rational functions over $\mathrm{GL}_{\nu+1}(\mathbb{C}(x))$. We consider the following morphism of ∂-$\mathbb{C}[x]$-algebras

$$\tau : \qquad \mathbb{C}[x]\left\{T, \tfrac{1}{\det T}\right\}_\partial \longrightarrow H^0(M \times_{\mathbb{C}} M, \mathcal{O}_{J^*(M,M)})$$

$$\begin{pmatrix} T_{0,0} & T_{0,1} & \dots & T_{0,\nu} \\ T_{1,0} & & & \\ \vdots & & (T_{i,j})_{i,j} & \\ T_{\nu,0} & & & \end{pmatrix} \longmapsto \begin{pmatrix} \frac{\partial \overline{x}}{\partial x} & \frac{\partial \overline{x}}{\partial X_1} & \dots & \frac{\partial \overline{x}}{\partial X_\nu} \\ \frac{\partial \overline{X_1}}{\partial x} & & & \\ \vdots & & \left(\frac{\partial \overline{X_i}}{\partial X_j}\right)_{i,j} & \\ \frac{\partial \overline{X_\nu}}{\partial x} & & & \end{pmatrix}$$

from $\mathbb{C}[x]\left\{T, \frac{1}{\det T}\right\}_\partial$ to the global sections $H^0(M \times_{\mathbb{C}} M, \mathcal{O}_{J^*(M,M)})$ of $\mathcal{O}_{J^*(M,M)}$, that can be thought as the algebra of global partial differential equations over $M \times M$. The image by τ of the differential ideal

$$\mathcal{I} = \left(T_{0,1}, \dots, T_{0,\nu}, T_{1,0}, \dots, T_{\nu,0}, \partial(T_{0,0})\right),$$

that defines the differential algebraic group

$$\left\{ diag(\alpha, \beta(x)) := \begin{pmatrix} \alpha & 0 \\ 0 & \beta(x) \end{pmatrix} \ : \ \text{where } \alpha \in \mathbb{C}^* \text{ and } \beta(x) \in \mathrm{GL}_\nu(\mathbb{C}(x)) \right\},$$

is contained in the ideal $\mathcal{I}_{\mathcal{L}in}$ defining the D-groupoid $\mathcal{L}in$ (*cf.* Proposition 10.2).

DEFINITION 11.1. We call $\mathcal{K}ol(A(x))$ the smallest differential subvariety of $\mathrm{GL}_{\nu+1}(\mathbb{C}(x))$, defined over $\mathbb{C}(x)$, which contains

$$\left\{ diag(q^k, A_k(x)) := \begin{pmatrix} q^k & 0 \\ 0 & A_k(x) \end{pmatrix} \ : \ k \in \mathbb{Z} \right\},$$

and has the following property: if we call $I_{\mathcal{K}ol(A(x))}$ the differential ideal defining $\mathcal{K}ol(A(x))$ and $I'_{\mathcal{K}ol(A(x))} = I_{\mathcal{K}ol(A(x))} \cap \mathbb{C}[x]\left\{T, \frac{1}{\det T}\right\}_\partial$, then the (sheaf of) differential ideal $\langle \mathcal{I}_{\mathcal{L}in}, \tau(I'_{\mathcal{K}ol(A(x))})\rangle$ generates a D-groupoid, that we will call $\mathcal{G}al^{alg}(A(x))$, in the space of jets $J^*(M, M)$.

REMARK 11.2. The definition above requires some explanations:

- The phrase "smallest differential subvariety of $\mathrm{GL}_{\nu+1}(\mathbb{C}(x))$" must be understood in the following way. The ideal of definition of $\mathcal{K}ol(A(x))$ is the largest differential ideal of $\mathbb{C}(x)\left\{T, \frac{1}{\det T}\right\}_\partial$ which admits the matrices $diag(q^k, A_k(x))$ as solutions for any $k \in \mathbb{Z}$ and verifies the second requirement of the definition. Then $I_{\mathcal{K}ol(A(x))}$ is radical and the Ritt-Raudenbush theorem (*cf.* Theorem 7.7 above) implies that $I_{\mathcal{K}ol(A(x))}$ is finitely ∂-generated. Of course, the $\mathbb{C}(x)$-rational points of $\mathcal{K}ol(A(x))$ may give very poor information on its structure, so we would rather speak of solutions in a differential closure of $\mathbb{C}(x)$.
- The structure of D-groupoid has the following consequence on the points of $\mathcal{K}ol(A(x))$: if $diag(\alpha, \beta(x))$ and $diag(\gamma, \delta(x))$ are two matrices with entries in a differential extension of $\mathbb{C}(x)$ that belong to $\mathcal{K}ol(A(x))$ then the matrix $diag(\alpha\gamma, \beta(\gamma x)\delta(x))$ belongs to $\mathcal{K}ol(A(x))$. In other words, the set of local diffeomorphisms $(x, X) \mapsto (\alpha x, \beta(x)X)$ of $M \times M$ such that $diag(\alpha, \beta(x))$ belongs to $\mathcal{K}ol(A(x))$ forms a set theoretic groupoid. We could have supposed only that $\mathcal{K}ol(A(x))$ is a differential variety and the solutions of $\mathcal{K}ol(A(x))$ form a groupoid in the sense above, but this wouldn't have been enough. In fact, it is not known if a sheaf of differential ideals of $J^*(M, M)$ whose solutions forms a groupoid is actually a D-groupoid (*cf.* Definition 10.1, and in particular conditions (ii') and (iii')). B. Malgrange told us that he can only prove this statement for a Lie algebra.

The differential variety $\mathcal{K}ol(A(x))$ is going to be a bridge between the parametrized intrinsic Galois group and the Galois D-groupoid $\mathcal{G}al(A(x))$ defined in the previous chapter, *via* the following theorem.

DEFINITION 11.3. Let $\mathcal{M}^{(A)}_{\mathbb{C}(x)} := (\mathbb{C}(x)^\nu, \Sigma_q : X \mapsto A^{-1}\sigma_q(X))$ be the q-difference module over $\mathbb{C}(x)$ associated to the system $Y(qx) = A(x)Y(x)$, where $\sigma_q(X)$ is defined componentwise. We call $\widetilde{\mathcal{K}ol(A(x))}$ the differential group over $\mathbb{C}(x)$ defined by the differential ideal $\langle I_{\mathcal{K}ol(A(x))}, T_{0,0} - 1\rangle$ in $\mathbb{C}(x)\left\{T, \frac{1}{\det T}\right\}_\partial$.

Notice that, as for the Zariski closure, the Kolchin closure does not commute with the intersection, therefore $\widetilde{\mathcal{K}ol(A(x))}$ is not the Kolchin closure of $\{A_k(x)\}_k$. Then we have:

THEOREM 11.4. *$Gal^\partial(\mathcal{M}^{(A)}_{\mathbb{C}(x)}, \eta_{\mathbb{C}(x)}) \cong \widetilde{\mathcal{K}ol(A(x))}$.*

REMARK 11.5. One can define in exactly the same way an algebraic subvariety $\mathcal{Z}ar(A)$ of $\mathrm{GL}_{\nu+1}(\mathbb{C}(x))$ containing the dynamics of the system and such that

$$\{(x,X)\mapsto(\alpha x,\beta(x)X) : diag(\alpha,\beta(x))\in\mathcal{Z}ar(A)\}$$

is a subgroupoid of the groupoid of diffeomorphisms of $M\times M$. Then one proves in the same way that $\widetilde{\mathcal{Z}ar(A)}$ coincide with the intrinsic Galois group, introduced in Chapter 6.

PROOF OF THEOREM 11.4. Let $\mathcal{N}=constr^{\partial}(\mathcal{M})$ be a construction of differential algebra of $\mathcal{M}$. We can consider:

- The basis denoted by $constr^{\partial}(\underline{e})$ of $\mathcal{N}$ built from the canonical basis $\underline{e}$ of $\mathbb{C}(x)^{\nu}$, applying the same constructions of linear differential algebra.
- For any $\beta\in\mathrm{GL}_{\nu}(\mathbb{C}(x))$, the matrix $constr^{\partial}(\beta)$ acting on $\mathcal{N}$ with respect to the basis $constr^{\partial}(\underline{e})$, obtained from β by functoriality. Its coefficients lies in $\mathbb{C}(x)[\beta,\partial(\beta),...]$
- Any $\psi=(\alpha,\beta)\in\mathbb{C}^{*}\times\mathrm{GL}_{\nu}(\mathbb{C}(x))$ acts semilinearly on $\mathcal{N}$ in the following way: $\psi\underline{e}=(constr^{\partial}(\beta))^{-1}\underline{e}$ and $\phi(f(x)n)=f(\alpha x)n$, for any $f(x)\in\mathbb{C}(x)$ and $n\in\mathcal{N}$. In particular, $(q^{k},A_{k}(x))\in\mathbb{C}^{*}\times\mathrm{GL}_{\nu}(\mathbb{C}(x))$ acts as Σ_{q}^{k} on $\mathcal{N}$.

A sub-q-difference module $\mathcal{E}$ of $\mathcal{N}$ correspond to an invertible matrix $F\in\mathrm{GL}_{\nu}(\mathbb{C}(x))$ such that

$$F(q^{k}x)^{-1}constr^{\partial}(A_{k})F(x)=\begin{pmatrix}*&*\\0&*\end{pmatrix},\ \text{for any } k\in\mathbb{Z}. \tag{11.3}$$

Now, $(1,\beta)\in\mathbb{C}^{*}\times\mathrm{GL}_{\nu}(\mathbb{C}(x))$ stabilizes $\mathcal{E}$ if and only if

$$F(x)^{-1}constr^{\partial}(\beta)F(x)=\begin{pmatrix}*&*\\0&*\end{pmatrix}. \tag{11.4}$$

Equation (11.3) corresponds to a differential polynomial $L(T_{0,0},(T_{i,j})_{i,j\geq1})$ belonging to $\mathbb{C}(x)\left\{T,\frac{1}{\det T}\right\}_{\partial}$ and having the property that $L(q^{k},(A_{k}))=0$, for all $k\in\mathbb{Z}$. On the other hand (11.4) corresponds to $L(1,(T_{i,j})_{i,j\geq1}))$. It means that the solutions of the differential ideal $\langle I_{\mathcal{K}ol(A(x))},T_{0,0}-1\rangle\subset\mathbb{C}(x)\left\{T,\frac{1}{\det T}\right\}_{\partial}$ stabilize all the sub-q-difference modules of all the constructions of differential algebra, and hence that

$$\widetilde{\mathcal{K}ol(A(x))}\subset Gal^{\partial}(\mathcal{M}_{\mathbb{C}(x)},\eta_{\mathbb{C}(x)}).$$

Let us prove the inverse inclusion. In the notation of Theorem 7.19, there exists a finitely generated extension K of $\mathbb{Q}$ and a σ_q-stable subalgebra $\mathcal{A}$ of $K(x)$ of the forms considered in §7.2 such that:

1. $A(x)\in\mathrm{GL}_{\nu}(\mathcal{A})$, so that it defines a q-difference module $\mathcal{M}_{K(x)}^{(A)}$ over $K(x)$;
2. $Gal^{\partial}(\mathcal{M}_{K(x)}^{(A)},\eta_{K(x)})\otimes_{K(x)}\mathbb{C}(x)\cong Gal^{\partial}(\mathcal{M}_{\mathbb{C}(x)}^{(A)},\eta_{\mathbb{C}(x)})$;
3. $\mathcal{K}ol(A(x))$ is defined over $\mathcal{A}$, i.e., there exists a differential ideal I in the differential ring $\mathcal{A}\{T,\frac{1}{\det(T)}\}_{\partial}$ such that I generates $I_{\mathcal{K}ol(A(x))}$ in $\mathbb{C}(x)\left\{T,\frac{1}{\det T}\right\}_{\partial}$.

For any element $\widetilde{L}$ of the defining ideal of $\widetilde{\mathcal{K}ol(A(x))}$ over $\mathcal{A}$, there exists

$$L(T_{0,0};T_{i,j},i,j=1,\dots,\nu)\in I\subset\mathcal{A}\left\{T,\frac{1}{\det(T)}\right\}_{\partial},$$

such that $L\in\mathcal{I}_{\mathcal{K}ol(A(x))}$ and $\widetilde{L}=L(1;T_{i,j},i,j=1,\dots,\nu)$. If q is an algebraic number, other than a root of unity, or if q is transcendental, then, for almost all places $v\in\mathcal{C}$, we have

$$\widetilde{L}(A_{\kappa_v})\equiv L(1,A_{\kappa_v})\equiv L(q^{\kappa_v},A_{\kappa_v})\equiv 0 \text{ modulo } \phi_v.$$

This shows that $\widetilde{\mathcal{K}ol(A(x))}$ is a differential subgroup of $\mathrm{GL}_\nu(\mathbb{C}(x))$ which contains a non-empty cofinite set of v-curvatures, in the sense of Theorem 7.19. Therefore, $\widetilde{\mathcal{K}ol(A(x))}$ contains the parametrized intrinsic Galois group of $\mathcal{M}_{\mathbb{C}(x)}^{(A)}$. $\square$

Definition 11.6. We call $\widetilde{\mathcal{G}al^{alg}(A(x))}$ the intersection of $\mathcal{G}al^{alg}(A(x))$ and $\mathcal{T}rv$.

It follows from the definition that the D-groupoid $\widetilde{\mathcal{G}al^{alg}(A(x))}$ is generated by its global equations, i.e., by $\mathcal{L}in$ and the image of the equations of $\widetilde{\mathcal{K}ol(A(x))}$ by the morphism τ. Therefore we deduce from Theorem 11.4 the following statement:

Corollary 11.7. *As a D-groupoid, $\widetilde{\mathcal{G}al^{alg}(A(x))}$ is generated by its global sections, namely the D-groupoid $\mathcal{L}in$ and the image of the equations of $Gal^{\partial}(\mathcal{M}_{\mathbb{C}(x)}^{(A)}, \eta_{\mathbb{C}(x)})$* via *the morphism τ.*

Remark 11.8. The corollary above says not only that a germ of diffeomorphism $(x, X) \mapsto (x, \beta(x)X)$ of M is solution of $\widetilde{\mathcal{G}al^{alg}(A(x))}$ if and only if $\beta(x)$ is solution of the differential equations defining the parametrized intrinsic Galois group of $\mathcal{M}_{\mathbb{C}(x)}^{(A)} = (\mathbb{C}(x)^\nu, X \mapsto A(x)^{-1}\sigma_q(X))$, but also that the two differential defining ideals "coincide".

The D-groupoid $\widetilde{\mathcal{G}al^{alg}(A(x))}$ is a differential analog of the D-groupoid generated by an algebraic group introduced in [**Mal01**, Proposition 5.3.2] by B. Malgrange.

11.3. The Galois D-groupoid $\mathcal{G}al(A(x))$ vs the intrinsic parametrized Galois group

Since $Dyn(A(x))$ is contained in the solutions of $\mathcal{G}al^{alg}(A(x))$, we have

$$sol(\mathcal{G}al(A(x))) \subset sol(\mathcal{G}al^{alg}(A(x)))$$

and

$$sol(\widetilde{\mathcal{G}al(A(x))}) \subset sol(\widetilde{\mathcal{G}al^{alg}(A(x))}).$$

as already mentioned, the solution are to be found in some differential closure of $(\mathbb{C}(x), \partial)$.

Theorem 11.9. *The solutions of the D-groupoid $\widetilde{\mathcal{G}al(A(x))}$ (resp. $\mathcal{G}al(A(x))$) coincide with the solutions of $\widetilde{\mathcal{G}al^{alg}(A(x))}$ (resp. $\mathcal{G}al^{alg}(A(x))$).*

Combining the theorem above with Corollary 11.7, we immediately obtain:

Corollary 11.10. *The solutions of the D-groupoid $\widetilde{\mathcal{G}al(A(x))}$ are germs of diffeomorphisms of the form $(x, X) \longmapsto (x, \beta(x)X)$, such that $\beta(x)$ is a solution of the differential equations defining $Gal^{\partial}(\mathcal{M}_{\mathbb{C}(x)}^{(A)}, \eta_{\mathbb{C}(x)})$, and* vice versa.

Remark 11.11. The corollary above says that the solutions of $\widetilde{\mathcal{G}al(A(x))}$ in a neighborhood of a transversal $\{x_0\} \times \mathbb{C}^\nu$ (*cf.* Proposition 10.9 below), rational over a differential extension $\mathcal{F}$ of $\mathbb{C}(x)$, correspond one-to-one with the solutions $\beta(x) \in \mathrm{GL}_\nu(\mathcal{F})$ of the differential equations defining the parametrized intrinsic Galois group.

It does not say that the two defining differential ideals can be compared. We actually don't prove that $\mathcal{G}al(A(x))$ is an "algebraic D-groupoid" and therefore that $\mathcal{G}al^{alg}(A(x))$ and $\mathcal{G}al(A(x))$ coincide as D-groupoids.

Proof of Theorem 11.9. Let $\mathcal{I}$ be the differential ideal of $\mathcal{G}al(A(x))$ in $\mathcal{O}_{J^*(M,M)}$ and let $\mathcal{I}_r$ be the sub-ideal of $\mathcal{I}$ of order $\leq r$. We consider the morphism of analytic varieties given by

$$\begin{array}{rccc} \iota: & \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} & \longrightarrow & M \times_{\mathbb{C}} M \\ & & & \\ & (x, \overline{x}) & \longmapsto & (x, 0, \overline{x}, 0) \end{array}$$

and the inverse image $\mathcal{J}_r := \iota^{-1}\mathcal{I}_r$ (resp. $\mathcal{J} := \iota^{-1}\mathcal{I}$) of the sheaf $\mathcal{I}_r$ (resp. $\mathcal{I}$) over $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$. We consider similarly to [**Mal01**, Lemma 5.3.3], the evaluation $ev(\iota^{-1}\mathcal{I})$ at $X = \overline{X} = \frac{\partial^i \overline{X}}{\partial x^i} = 0$ of the equations of $\iota^{-1}\mathcal{I}$ and we denote by $ev(\mathcal{I})$ the direct image by ι of the sheaf $ev(\iota^{-1}\mathcal{I})$.

The following lemma is crucial in the proof of the Theorem 11.9:

Lemma 11.12. *A germ of local diffeomorphism $(x, X) \mapsto (\alpha x, \beta(x)X)$ of M is solution of $\mathcal{I}$ if and only if it is solution of $ev(\mathcal{I})$.*

Proof. First of all, we notice that $\mathcal{I}$ is contained in $\mathcal{L}in$. Moreover the solutions of $\mathcal{I}$, that are diffeomorphisms mapping a neighborhood of $(x_0, X_0) \in M$ to a neighborhood of $(\overline{x}_0, \overline{X}_0)$, can be naturally continued to diffeomorphisms of a neighborhood of $x_0 \times \mathbb{C}^\nu$ to a neighborhood of $\overline{x}_0 \times \mathbb{C}^\nu$. Therefore it follows from the particular structure of the solutions of $\mathcal{L}in$, that they are also solutions of $ev\,(\mathcal{I})$ (*cf.* Proposition 10.2).

Conversely, let the germ of diffeomorphism $(x, X) \mapsto (\alpha x, \beta\,(x)\,X)$ be a solution of $ev\,(\mathcal{I})$ and $E \in \mathcal{I}_r$. It follows from Proposition 10.4 that there exists $E_1 \in \mathcal{I}$ of order r, only depending on the variables x,X,$\frac{\partial \bar{x}}{\partial x}$, $\frac{\partial \bar{X}}{\partial X}$,$\frac{\partial^2 \bar{X}}{\partial x \partial X}$, $\dots$, $\frac{\partial^r \bar{X}}{\partial x^{r-1} \partial X}$, such that $(x, X) \mapsto (\alpha x, \beta\,(x)\,X)$ is solution of E if and only if it is solution of E_1. So we will focus on equations on the form E_1 and, to simplify notation, we will write E for E_1.

By assumption $(x, X) \mapsto (\alpha x, \beta\,(x)\,X)$ is solution of

$$E\left(x, 0, \frac{\partial \bar{x}}{\partial x}, \frac{\partial \bar{X}}{\partial X}, \frac{\partial^2 \bar{X}}{\partial x \partial X}, \dots \frac{\partial^r \bar{X}}{\partial x^{r-1} \partial X}\right)$$

and we have to show that $(x, X) \mapsto (\alpha x, \beta\,(x)\,X)$ is a solution of E. We consider the Taylor expansion of E:

$$E\left(x, X, \frac{\partial \bar{x}}{\partial x}, \frac{\partial \bar{X}}{\partial X}, \frac{\partial^2 \bar{X}}{\partial x \partial X}, \dots \frac{\partial^r \bar{X}}{\partial x^{r-1} \partial X}\right) = \sum_\alpha E_\alpha\,(x, X)\,\partial^\alpha,$$

where ∂^α is a monomial in the coordinates $\frac{\partial \bar{x}}{\partial x}, \frac{\partial \bar{X}}{\partial X}, \frac{\partial^2 \bar{X}}{\partial x \partial X}, \dots \frac{\partial^r \bar{X}}{\partial x^{r-1} \partial X}$. Developing the $E_\alpha\,(x, X)$ with respect to $X = (X_1, \dots, X_\nu)$ we obtain:

$$E = \sum \left(\sum_\alpha \left(\frac{\partial^{\underline{k}} E_\alpha}{\partial X^{\underline{k}}} \right) (x, 0)\,\partial^\alpha \right) X^{\underline{k}},$$

with $\underline{k} \in (\mathbb{Z}_{\geq 0})^\nu$. If we show that for any $\underline{k}$ the germ $(x, X) \mapsto (\alpha x, \beta\,(x)\,X)$ verifies the equation

$$B_{\underline{k}} := \sum_\alpha \left(\frac{\partial^{\underline{k}} E_\alpha}{\partial X^{\underline{k}}} \right) (x, 0)\,\partial^\alpha$$

we can conclude. For $\underline{k} = (0, \dots, 0)$, there is nothing to prove since $B_{\underline{0}} = ev\,(E)$.

Let D_{X_i} be the derivation of $\mathcal{I}$ corresponding to $\frac{\partial}{\partial X_i}$, The differential equation

$$D_{X_i}\,(E) = \sum_\alpha \left(\frac{\partial E_\alpha}{\partial X_i} \right) (x, X)\,\partial^\alpha + \sum_\alpha E_\alpha\,(x, X)\,D_{X_i}\,(\partial^\alpha)$$

is still in $\mathcal{I}$, since $\mathcal{I}$ is a differential ideal. Therefore by assumption $(x,X) \mapsto (\alpha x, \beta(x) X)$ is a solution of

$$ev\left(D_{X_i}E\right) = \sum_{\alpha} \left(\frac{\partial E_\alpha}{\partial X_i}\right)(x,0)\,\partial^\alpha + \sum_{\alpha} E_\alpha(x,0)\, D_{X_i}\left(\partial^\alpha\right).$$

Since $D_{X_i}(\partial^\alpha) \in \mathcal{L}in$ and $(x,X) \mapsto (\alpha x, \beta(x) X)$ is a solution of $\mathcal{L}in$, we conclude that $(x,X) \mapsto (\alpha x, \beta(x) X)$ is a solution of

$$\sum_{\alpha} \left(\frac{\partial E_\alpha}{\partial X}\right)(x,0)\,\partial^\alpha$$

and therefore of B_1. Iterating the argument, one deduce that $(x,X) \mapsto (\alpha x, \beta(x) X)$ is solution of $B_{\underline{k}}$ for any $\underline{k} \in (\mathbb{Z}_{\geq 0})^\nu$, which ends the proof of the lemma. □

We go back to the proof of Theorem 11.9. Lemma 11.12 proves that the solutions of $\mathcal{G}al(A(x))$ coincide with those of the D-groupoid Γ generated by $\mathcal{L}in$ and $ev(\mathcal{I})$, defined on the open neighborhoods of any $x_0 \times \mathbb{C}^\nu \in M$. By intersection with the equation $\mathcal{T}rv$, the same holds for the transversal groupoids $\widetilde{\mathcal{G}al(A(x))}$ and $\widetilde{\Gamma}$.

Since $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ and $M \times_{\mathbb{C}} M$ are locally compact and $\mathcal{I}_r$ is a coherent sheaf over $M \times_{\mathbb{C}} M$, the sheaf $\mathcal{J}_r$ is an analytic coherent sheaf over $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ and so is its quotient $ev(\iota^{-1}(\mathcal{I}_r))$. By [**Ser56**, Theorem 3], there exists an algebraic coherent sheaf $\mathbb{J}_r$ over the projective variety $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ such that the analyzation of $\mathbb{J}_r$ coincides with $ev(\iota^{-1}(\mathcal{I}_r))$. This implies that $ev(\mathcal{I})$ is generated by algebraic differential equations which by definition have the dynamics for solutions.

We thus have that the $sol(\Gamma) = sol(\mathcal{G}al(A(x))) \subset sol(\mathcal{G}al^{alg}(A(x)))$. Since both Γ and $\mathcal{G}al^{alg}(A(x))$ are algebraic, the minimality of the variety $\mathcal{K}ol(A(x))$ implies that $sol(\mathcal{G}al^{alg}(A(x))) \subset sol(\Gamma)$. We conclude that the solutions of $\mathcal{G}al(A(x))$ coincide with those $\mathcal{G}al^{alg}(A(x))$. The same hold for $\widetilde{\mathcal{G}al(A(x))}$, $\widetilde{\Gamma}$ and $\widetilde{\mathcal{G}al^{alg}(A(x))}$). This concludes the proof. □

11.4. Comparison with known results

In [**Mal01**], B. Malgrange proves that the Galois-D-groupoid of a linear differential equation allows to recover, in the special case of a linear differential equation, the Picard-Vessiot Galois group over $\mathbb{C}$. This is not in contradiction with the result above, since:

- due to the fact that local solutions of a linear differential equation form a $\mathbb{C}$-vector space (rather than a vector space on the field of elliptic functions!), [**Kat82**, Proposition 4.1] shows that the intrinsic Galois group and the Picard-Vessiot Galois group in the differential setting become isomorphic above a certain extension of the local ring. For more details on the relation between the intrinsic Galois group and the usual Galois group see [**Pil02**, Corollary 3.3].
- it is not difficult to prove that, in the differential setting, the Picard-Vessiot Galois group and parametrized Galois group with respect to $\frac{d}{dx}$ coincide. See Remark 9.11.

Therefore B. Malgrange actually finds a parametrized intrinsic Galois group, which is hidden in his construction. The steps of the proof above are the same as in his proof, apart that, to compensate the lack of good local solutions, we are obliged to use Theorem 7.13. Anyway, the application of Theorem 7.13 appears to be very natural, if one considers how close the definition of the dynamics of a linear q-difference system and the definition of the curvatures are.

In [**Gra**], A. Granier shows that in the case of a q-difference system with constant coefficients the groupoid that fixes the transversals in $\mathcal{G}al(A(x))$ is the Picard-Vessiot Galois group, i.e., an algebraic group defined over $\mathbb{C}$. Once again, this is not in contradiction with our results. In fact, under this assumption, it is not difficult to show that the parametrized intrinsic Galois group is defined over $\mathbb{C}$. Moreover the parametrized intrinsic Galois groups and the intrinsic Galois group coincide, in fact if $\mathcal{M}$ is a q-difference module over $\mathbb{C}(x)$ associated with a constant q-difference system, it is easy to prove that the prolongation functor F_∂ acts trivially on $\mathcal{M}$, namely $F_\partial(\mathcal{M}) \cong \mathcal{M} \oplus \mathcal{M}$. Finally, to conclude that the intrinsic Galois group coincide with the usual one, it is enough to notice that they are associated with the same fiber functor, or equivalently that they stabilize exactly the same objects.

Because of these results, G. Casale and J. Roques have conjectured that "for linear (q-)difference systems, the action of Malgrange groupoid on the fibers gives the classical Galois groups" (*cf.* [**CR08**]). In *loc. cit.*, they give two proofs of their main integrability result: one of them relies on their conjecture. Here we have proved that the Galois-D-groupoid allows to recover exactly the parametrized intrinsic Galois group. By taking the Zariski closure one can also recover the algebraic intrinsic Galois group. The comparison theorems in Part 4 imply that we can also recover the Picard-Vessiot Galois group (*cf.* [**vdPS97**], [**Sau04b**]), performing a Zariski closure and a convenient field extension, and the parametrized Galois group (*cf.* [**HS08**]), performing a field extension.

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