Appendices

Appendix 1.

Newton-Ramis polygons and formal Fourier transforms of q-difference equations

In this appendix K is a field of characteristic zero and q is a non zero element of K, which is not a root of unity.

For any positive integer n we set $(n)_q = 1 + \ldots + q^{n-2} + q^{n-1} = \frac{1-q^n}{1-q}$ and $[n]!_q = (n)_q \cdots (2)_q$. If $i \leq n$ is another positive integer we set $\binom{n}{i}_q = \frac{[n]_q!}{[i]!_q [n-i]!_q}$.

1.1. q-analogue of the formal Laplace transform.

In "q-analysis", there are two q-analogues of the formal Laplace transform, namely:

$$\begin{array}{cccc} (\cdot)^+ : & K[\![x]\!] & \longrightarrow & K\left[\left[x^{-1}\right]\right] \\ & F = \sum_{n=0}^{\infty} a_n x^n & \longmapsto & F^+ = \sum_{n=0}^{\infty} [n]!_q a_n x^{-n-1} \end{array}$$

and:

$$\begin{array}{cccc} (\cdot)^\# : & K[\![x]\!] & \longrightarrow & K\left[\left[x^{-1}\right]\right] \\ & F = \sum_{n=0}^{\infty} a_n x^n & \longmapsto & F^\# = \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} a_n x^{-n-1} \end{array}.$$

Since $[n]!_q$ and $q^{n(n-1)/2}$ are polynomial in q of the same degree, if K is a normed field and the norm of q is greater than 1 the two Laplace transforms are equivalent (*cf.* [MZ, §8]), but this is not the case we are interested in. We are interested in the arithmetic point of view, therefore for us $(\cdot)^+$ and $(\cdot)^\#$ have a completely different behavior, linked to the different p-adic behavior of $q^{\frac{n(n-1)}{2}}$ and of $[n]!_q$.

In this appendix we are going to define two q-analogues of the formal Fourier transform, associated to the two Laplace transforms $(\cdot)^+$ and $(\cdot)^\#$ and study their action on the q-analogue of the Newton-Ramis polygon (*cf.* for instance [M, Ch. V, §1] for the differential case).

The Laplace transforms $(\cdot)^+$ and $(\cdot)^\#$ have the following properties:

Lemma 1.1.1. *For all* $F=\sum_{n=0}^{\infty} a_n x^n \in K[\![x]\!]$ *we have*

$$(xF)^+ = -\frac{1}{q} d_{q^{-1}} F^+ \ ,\ (d_qF)^+ = xF^+ - F(0)\ ,$$
$$(xF)^\# = \frac{1}{qx}\varphi_{q^{-1}} F^\# \ ,\ (\varphi_q F)^\# = \frac{1}{q}\varphi_{q^{-1}} F^\# \ .$$

Proof. We deduce the first equality using the relation

$$-\frac{1}{q} d_{q^{-1}} \frac{1}{x^n} = (n)_q \frac{1}{x^{n+1}} \ .$$

All the other formulas easily follow by the definitions. ■

1.2. q-analogue of the formal Fourier transform.

By analogy with the differential case, the previous proposition justifies the definition:

Definition 1.2.1. *We call the maps*

$$\begin{array}{rcll} \mathcal{F}_{q^+} & : K[x,d_q] & \longrightarrow & K\left[x, d_{q^{-1}}\right] \\ & d_q & \longmapsto & x \\ & x & \longmapsto & -\frac{1}{q} d_{q^{-1}} \end{array} \qquad \text{and} \qquad \begin{array}{rcll} \mathcal{F}_{q^\#} & : K[x,\varphi_q] & \longrightarrow & K\left[\frac{1}{x}, \varphi_{q^{-1}}\right] \\ & \varphi_q & \longmapsto & \frac{1}{q}\varphi_{q^{-1}} \\ & x & \longmapsto & \frac{1}{qx}\varphi_{q^{-1}} \end{array}$$

the q^+*-Fourier transform and the* $q^\#$*-Fourier transform respectively.*

Remark 1.2.2.
1) We notice that $\mathcal{F}_{q^+}^{-1} = \lambda \circ \mathcal{F}_{(q^{-1})^+}$, where

$$\begin{array}{rcll} \lambda & : K[x,d_q] & \longrightarrow & K[x,d_q] \\ & d_q & \longmapsto & -\frac{1}{q} d_q \\ & x & \longmapsto & -qx \end{array} \ .$$

2) We have

$$\begin{aligned} \mathcal{F}_{q^+}(\varphi_q) &= \mathcal{F}_{q^+}((q-1)x d_q + 1) \\ &= (q-1)\left(-\frac{1}{q} d_{q^{-1}}\right) \circ x + 1 \\ &= \frac{1}{q} + \frac{1-q}{q^2} x d_{q^{-1}} \\ &= \frac{1}{q}\varphi_{q^{-1}} \ . \end{aligned}$$

3) We notice that if $\mathcal{L} = \sum_{i=0}^{\mu} a_i(\frac{1}{x})\varphi_{q^{-1}}^i \in K\left[\frac{1}{x}, \varphi_{q^{-1}}\right]$ is such that $\deg_{\frac{1}{x}} a_i\left(\frac{1}{x}\right) \le i$, there exists a unique $\mathcal{N} \in K[x,\varphi_q]$ such that $\mathcal{F}_{q^\#}(\mathcal{N}) = \mathcal{L}$.

In the following lemma, we verify the expected compatibility between the formal Fourier transform $\mathcal{F}_{q^+}$ (resp. $\mathcal{F}_{q^\#}$) we have just defined and the Laplace transform $(\cdot)^+$ (resp. $(\cdot)^\#$):

Lemma 1.2.3.
1) Let $F \in K[\![x]\!]$ be a series solution of a q-difference linear operator $\mathcal{N} \in K[x, d_q]$ (resp. $K[x, \varphi_q]$) such that $\mu = \deg_{d_q} \mathcal{N}$, then $d^{\mu}_{q^{-1}} \mathcal{F}_{q^+}(\mathcal{N})F^+ = 0$ (resp. $\mathcal{F}_{q^\#}(\mathcal{N})F^\# = 0$).
2) If F^+ is a solution of $\mathcal{L}_1 \in K\left[x, d_{q^{-1}}\right]$ then $\mathcal{F}^{-1}_{q^+}(\mathcal{L}_1)F = 0$.
3) If $F^\#$ is a solution of $\mathcal{L}_2 \in K\left[\frac{1}{x}, \varphi_{q^{-1}}\right]$, then for all $n \in \mathbb{N}$, $n >> 0$, we have $\mathcal{F}^{-1}_{q^\#}(\varphi^n_{q^{-1}} \circ \mathcal{L}_2)F = 0$.

Proof. We prove the statements for $(\cdot)^+$. The proof for $(\cdot)^\#$ is quite similar.

Let

$$\mathcal{N} = \sum_{j=0}^{\mu}\sum_{i=0}^{\nu} a_{i,j} x^i {d_q}^j \in K[x, d_q] \ .$$

By (1.1.1), we deduce that $\mathcal{F}_{q^+}(\mathcal{N})F^+$ is a polynomial of degree at most μ, therefore $d^{\mu}_{q^{-1}} \mathcal{F}_{q^+}(\mathcal{N})F^+ = 0$.

Let

$$\mathcal{L}_1 = \sum_{j=0}^{\mu}\sum_{i=0}^{\nu} a_{i,j} x^i d^j_{q^{-1}} \in K\left[x, {d_q}^{-1}\right] \ ,$$

then $\left(\mathcal{F}^{-1}_{q^+}(\mathcal{L}_1)F\right)^+$ is a polynomial of degree at most μ. We obtain

$$d^{\mu}_{q^{-1}} \left(\mathcal{F}^{-1}_{q^+}(\mathcal{L}_1)F\right)^+ = \left((-qx)^{\mu} \mathcal{F}^{-1}_{q^+}(\mathcal{L}_1)F\right)^+ = 0$$

and finally $(-qx)^{\mu}\mathcal{F}^{-1}_{q^+}(\mathcal{L}_1)F = 0$. ■

1.3. Newton-Ramis polygons.

Let us consider a linear q-difference operator

$$\mathcal{N} = \sum_{i=0}^{\mu} a_i(x) x^i {d_q}^i = \sum_{i=0}^{\mu} b_i(x) \varphi^i_q \ , \tag{1.3.0.1}$$

with $b_j(x), a_j(x) \in K[x]$. By [DVIII, (2.1.10)], we obtain

$$\begin{aligned} \mathcal{N} &= \sum_{j=0}^{\mu} b_j(x) \sum_{i=0}^{j} \binom{j}{i}_q (1-q)^i q^{i(i-1)/2} x^i {d_q}^i \\ &= \sum_{i=0}^{\mu} (1-q)^i q^{i(i-1)/2} \left(\sum_{j=i}^{\mu} \binom{j}{i}_q b_j(x)\right) x^i {d_q}^i \ , \end{aligned} \tag{1.3.0.2}$$

therefore $a_i(x) = (1-q)^i q^{i(i-1)/2} \sum_{j=i}^{\mu} \binom{j}{i}_q b_j(x)$.

Definition 1.3.1. *We define the Newton polygon $NP_0(\mathcal{N})$ of $\mathcal{N}$ at zero to be the convex hull of the following subset of $\mathbb{R}^2$*

$$\bigcup_{b_i(x)\neq 0}\{(u,v)\in\mathbb{R}^2 : u=i, v\geq \mathrm{ord}_x b_i(x)\}\ .$$

Remark 1.3.2. The set $NP_0(\mathcal{N})$ is a convex polygon with a finite number of finite sides.

Definition 1.3.3. *We say that $\mathcal{N}$ is regular singular at zero (or that $\mathcal{N}$ has a regular singularity at zero) if $NP_0(\mathcal{N})$ has only one finite side whose slope is equal to zero.*

Definition 1.3.4. *We say that $\mathcal{N}$ is irregular singular at zero (or that $\mathcal{N}$ has an irregular singularity at zero) if $NP_0(\mathcal{N})$ does not have a regular singularity at zero.*

Definition 1.3.5. *We say that $\mathcal{N}$ has a regular singularity at ∞ (resp. an irregular singularity at ∞) if it has a regular singularity at zero (resp. an irregular singularity at zero) with respect to $t=\frac{1}{x}$.*

In the ordinary differential equations theory, one usually defines a more general polygon, called the Newton-Ramis polygon, which gives at the same time informations about the slopes at zero and at ∞ (*cf.* [M, V, §1]). Inspired by this construction, we introduce a modification of the above definition of the Newton polygon.

First of all we notice that by setting $t=\frac{q^\mu}{x}$ in (1.3.0.1) we obtain the q-difference operator

$$\varphi_q^\mu\circ\mathcal{N}=\sum_{i=0}^{\mu} b_i\left(\frac{1}{t}\right)\varphi_q^{\mu-i}\ ,$$

hence $NP_\infty(\mathcal{N})$ is obtain by shifting the polygon

$$\text{convex hull of }\bigcup_{b_i(x)\neq 0}\left\{(u,v)\in\mathbb{R}^2 : u=\mu-i, v\geq \mathrm{ord}_t b_i\left(\frac{1}{t}\right)\right\}\ .$$

Since $\mathrm{ord}_t b_i\left(\frac{1}{t}\right)=-\deg_x b_i(x)$, the following definition becomes natural:

Definition 1.3.6. *Let $\mathcal{N}=\sum_{i=0}^{\mu} a_i(x)x^i {d_q}^i=\sum_{i=0}^{\mu} b_i(x)\varphi_q^i$, such that $b_j(x),a_j(x)\in K[x]$. Then we define the Newton-Ramis polygon of $\mathcal{N}$ with respect to φ_q (and we write $NRP_{\varphi_q}(\mathcal{N})$) (resp. with respect to d_q (and we write $NRP_{d_q}(\mathcal{N})$)) to be the convex hull of the following set*

$$\bigcup_{b_i(x)\neq 0}\{(u,v)\in\mathbb{R}^2 : u=i, \deg_x b_i(x)\geq v\geq \mathrm{ord}_x b_i(x)\}\subset\mathbb{R}^2\ .$$

$$\left(\textit{resp.}\quad\bigcup_{a_i(x)\neq 0}\{(u,v)\in\mathbb{R}^2 : 0u\leq i, \deg_x a_i(x)\geq v\geq \mathrm{ord}_x a_i(x)\}\subset\mathbb{R}^2\right)\ .$$

We say that the vertical sides of $NRP_{\varphi_q}(\mathcal{N})$) (resp. $NRP_{d_q}(\mathcal{N})$) have slope ∞.

Remark 1.3.7. The reason why we define two different Newton-Ramis polygons is that $NRP_{\varphi_q}(\mathcal{N})$ is more adapted to describe the behavor of $\mathcal{F}_{q\#}$ and $NRP_{d_q}(\mathcal{N})$ is more adapted to $\mathcal{F}_{q^+}$.

As expected we obtain:

Proposition 1.3.8. *We have*

$$NRP_{\varphi_q}(\mathcal{N}) = NP_0(\mathcal{N}) \cap \{(\mu - u, -v) \in \mathbb{R}^2 : (u,v) \in NP_\infty(\mathcal{N})\} ,$$

therefore the slopes of the upper sides of $NRP_{\varphi_q}(\mathcal{N})$ are the finite slopes of $NP_\infty(\mathcal{N})$ and the slopes of the lower sides are the finite slopes of $NP_0(\mathcal{N})$.

Moreover we have

$$NRP_{d_q}(\mathcal{N}) = \bigcup_{(u_0,v_0)\in NRP_{\varphi_q}(\mathcal{N})} \{(u, v_0) \in \mathbb{R}^2 : u \leq u_0\} .$$

Proof. The first part of the proposition follows by the definition of $NRP_{\varphi_q}(\mathcal{N})$. The second part follows by (1.3.0.2). ■

1.4. Behavior of the Newton-Ramis polygon under the q-analogue of the formal Fourier transform.

The following proposition describes the behavior of the Newton-Ramis polygon with respect to $\mathcal{F}_{q^+}$ and $\mathcal{F}_{q\#}$.

Proposition 1.4.1. *We denote $NRP_{\varphi_{q^{-1}}}\left(\mathcal{F}_{q\#}(\mathcal{N})\right)$ the Newton-Ramis polygon of $\mathcal{F}_{q\#}(\mathcal{N})$ defined with respect to x and $\varphi_{q^{-1}}$ and $NRP_{d_{q^{-1}}}\left(\mathcal{F}_{q^+}(\mathcal{N})\right)$ the Newton-Ramis polygon of $\mathcal{F}_{q^+}(\mathcal{N})$ defined with respect to x and $d_{q^{-1}}$. The map*

$$\begin{array}{ccc} NRP_{\varphi_q}(\mathcal{N}) & \longrightarrow & NRP_{\varphi_{q^{-1}}}\left(\mathcal{F}_{q\#}(\mathcal{N})\right) \\ (u,v) & \longmapsto & (u+v,-v) \end{array}$$

$$\left(resp. \begin{array}{ccc} NRP_{d_q}(\mathcal{N}) & \longrightarrow & NRP_{d_{q^{-1}}}\left(\mathcal{F}_{q^+}(\mathcal{N})\right) \\ (u,v) & \longmapsto & (u+v,-v) \end{array} \right)$$

is a bijection between $NRP_{\varphi_q}(\mathcal{N})$ and $NRP_{\varphi_{q^{-1}}}\left(\mathcal{F}_{q\#}(\mathcal{N})\right)$ (resp. $NRP_{d_q}(\mathcal{N})$ and $NRP_{d_{q^{-1}}}\left(\mathcal{F}_{q^+}(\mathcal{N})\right)$). Then $\mathcal{F}_{q\#}$ (resp. $\mathcal{F}_{q^+}$) acts in the following way on the slopes of the Newton-Ramis polygon:

$$\begin{array}{ccc} \{\text{slopes of } NRP_{\varphi_q}(\mathcal{N})\} & \longrightarrow & \left\{\text{slopes of } NRP_{\varphi_{q^{-1}}}\left(\mathcal{F}_{q\#}(\mathcal{N})\right)\right\} \\ \lambda & \longmapsto & -\frac{\lambda}{1+\lambda} \\ \infty & \longmapsto & -1 \end{array}$$

$$\left(\begin{array}{lccc} & \left\{\text{slopes of } NRP_{d_q}(\mathcal{N})\right\} & \longrightarrow & \left\{\text{slopes of } NRP_{d_{q^{-1}}}\left(\mathcal{F}_{q^\#}\left(\mathcal{N}\right)\right)\right\} \\ \text{resp.} & \lambda & \longmapsto & -\frac{\lambda}{1+\lambda} \\ & \infty & \longmapsto & -1 \end{array}\right)$$

Proof. As far as $\mathcal{F}_{q^\#}$ is concerned, it is enough to notice that

$$\mathcal{F}_{q^\#}\left(\sum_{i=0}^{\mu}\sum_{j=0}^{\nu} b_{i,j}x^j\varphi_q^i\right) = \sum_{i=0}^{\mu}\sum_{j=0}^{\nu} \frac{b_{i,j}}{q^{j(j-3)/2}q^i}\frac{1}{x^j}\varphi_{q^{-1}}^{i+j} .$$

Let

$$\mathcal{N} = \sum_{i=0}^{\mu}\sum_{j=0}^{\nu} a_{i,j}x^j {d_q}^i .$$

As for $\mathcal{F}_{q^\#}$:

$$\begin{aligned}\mathcal{F}_{q^+}\left(\mathcal{N}\right) &= \sum_{i=0}^{\mu}\sum_{j=0}^{\nu} \frac{(-1)^j a_{i,j}}{q^j}{d_q}^j \circ x^i \\ &= \sum_{i=0}^{\mu}\sum_{j=0}^{\mu}\sum_{h=0}^{j} \frac{(-1)^j a_{i,j}}{q^j}\binom{j}{h}_q \frac{i_q!}{(h-i)_q!} q^{(j-h)(i-h)} x^{i-h} d_{q^{-1}}^{j-h} .\end{aligned}$$

Then if $(i, j-i) \in NRP_{d_q}(\mathcal{N})$ we have

$$(j-h, i-j) \in NRP_{d_{q^{-1}}}\left(\mathcal{F}_{q^+}\left(\mathcal{N}\right)\right) \text{ for all } h = 0, \dots, j.$$

The statement easily follows by this remark. ∎

With the help of the Newton-Ramis polygon, we have described what happens at zero and at ∞ under the Fourier transforms. Now we want to describe the situation at a point $\xi \in \mathbb{P}^1(K) \smallsetminus \{0, \infty\}$ when we consider the inverse image of a regular singular q-difference operator by $\mathcal{F}_{q^\#}$ or $\mathcal{F}_{q^+}$.

Proposition 1.4.2. *Let $\mathcal{N} \in K[x, d_q]$ be a linear q-difference operator such that $NRP_{d_q}(\mathcal{N})$ has only the zero slope at ∞; then the operator $\mathcal{F}_{q^+}\mathcal{N}$ has a basis of solution in*

$$K[\![x-\xi]\!]_q = \left\{\sum_{n=0}^{\infty} a_n(x-\xi)_n : a_n \in K\right\} ,$$

with $(x-\xi)_n = (x-\xi)(x-q\xi)\cdots(x-q^{n-1}\xi)$, for all $\xi \in \mathbb{P}^1(K) \smallsetminus \{0, \infty\}$.

Remark 1.4.3. Since $\mathcal{F}_{q^+}^{-1}(\mathcal{N}) = \lambda \circ \mathcal{F}_{(q^{-1})^+}(\mathcal{N})$, an analogous statement is true for $\mathcal{F}_{q^+}^{-1}(\mathcal{N})$.

Proof. Let

$$\mathcal{N} = \sum_{j=0}^{\mu} a_i(x){d_q}^i = \sum_{i=0}^{\mu}\sum_{j=0}^{\nu} a_{i,j}x^j {d_q}^i \ , \ \ i.e. \ a_i(x) = \sum_{j=0}^{\nu} a_{i,j}x^j$$

be a linear q-difference operator satisfying the hypothesis. Considering the Newton polygon of $\mathcal{N}$ at ∞ we deduce that

$$\deg a_i(x) < \deg a_\mu(x) \ , \ \forall\, i = 0, \dots, \mu - 1,$$

thus $a_{i,\nu} = 0$ for all $i = 0, \dots, \mu - 1$. Since the coefficient of $d_{q^{-1}}^{\nu}$ in

$$\begin{aligned}\mathcal{F}_{q^+}(\mathcal{N}) &= \sum_{i=0}^{\mu}\sum_{j=0}^{\nu} a_{i,j}\left(-\frac{1}{q}d_{q^{-1}}\right)^j \circ x^i \\ &= \sum_{j=0}^{\nu-1}\sum_{i=0}^{\mu} c_{j,i}x^i d_{q^{-1}}^j + a_{\mu,\nu}(-q)^{\mu-\nu}x^\mu d_{q^{-1}}^{\nu}\end{aligned}$$

does not vanish on $\{q^n\xi : n \in \mathbb{Z}_{>0}\}$, we can conclude using the following lemma:

Lemma 1.4.4. *Let $\mathcal{L} = {d_q}^{\mu} + \sum_{i=0}^{\mu-1} a_i(x){d_q}^i \in K(x)[d_q]$ be a q-difference operator and let $\xi \in K$. If $a_1(x), \dots, a_{\mu-1}(x)$ have no pole in the set $\{q^n\xi : n \in \mathbb{Z}_{>0}\}$, then $\mathcal{L}$ has a basis of formal solutions (i.e. μ linearly independent solutions) in $K[\![x-\xi]\!]_q$.*

Proof. The q-difference equation $\mathcal{L}y = 0$ is equivalent to the linear q-difference system

(1.4.4.1)
$$d_q\begin{pmatrix} y \\ d_q y \\ \vdots \\ {d_q}^{\mu-1}y\end{pmatrix} = \left(\begin{array}{c|ccc} 0 & & & \\ \vdots & & \mathbb{I}_{\mu-1} & \\ 0 & & & \\ \hline a_0(x) & a_1(x) & \cdots & a_{\mu-1}(x)\end{array}\right)\begin{pmatrix} y \\ d_q y \\ \vdots \\ {d_q}^{\mu-1}y\end{pmatrix} =: G(x)\begin{pmatrix} y \\ d_q y \\ \vdots \\ {d_q}^{\mu-1}y\end{pmatrix}.$$

Let $G_0(x) = 1$, $G_1(x) = G(x)$ and $G_{n+1}(x) = G_1(qx)G_n(x) + d_qG_n(x)$ for all $n > 1$. Then the $G_n(x)$'s don't have any pole in ξ and $Y = \sum_{n\geq 0}\frac{G_n(\xi)}{[n]!_q}(x-\xi)_n$ is a formal fundamental solution for (1.4.4.1). ∎

Proposition 1.4.5. *Let $\mathcal{L} = \sum_{i=0}^{\mu} a_i\left(\frac{1}{x}\right)\varphi_{q^{-1}}^i \in K\left[\frac{1}{x}, \varphi_{q^{-1}}\right]$ such that $\deg_{\frac{1}{x}} a_i(\frac{1}{x}) \leq i$. We suppose that $NRP_{\varphi_{q^{-1}}}(\mathcal{L})$ has no negative slope at ∞, then $\mathcal{F}_{q^\#}^{-1}(\mathcal{L})$ has a basis of solutions in $K[\![x-\xi]\!]_q$ for every $\xi \in \mathbb{P}^1(K) \smallsetminus \{0, \infty\}$.*

Proof. By hypothesis we have

$$\nu = \mathrm{ord}_{\frac{1}{x}} a_\mu\left(\frac{1}{x}\right) \leq \mathrm{ord}_{\frac{1}{x}} a_i\left(\frac{1}{x}\right) \ ,$$

for all $i = 0, \dots, \mu - 1$. We set

$$\mathcal{L} = \sum_{i=0}^{\mu}\sum_{j=\nu}^{\nu'} a_{i,j}\frac{1}{x^j}\varphi_{q^{-1}}^i \ .$$

Then we have

$$\begin{aligned}\mathcal{F}_{q^\#}^{-1}(\mathcal{L}) &= \mathcal{F}_{q^\#}^{-1}\left(\sum_{i=0}^{\mu}\sum_{j=\nu}^{\nu'} a_{i,j}\frac{\varphi_q^j}{x^j}\varphi_{q^{-1}}^{i-j}\right) \\ &= \sum_{h=0}^{\mu-\nu-1} b_h(x)\varphi_q^i + a_{\mu,\nu}x^\nu\varphi_q^{\mu-\nu} \in K[x, \varphi_q] \ .\end{aligned}$$

We conclude by the previous lemma, writing $\mathcal{F}_{q^\#}^{-1}(\mathcal{L})$ as an operator in d_q. ∎

Appendix 2.
On the definition of the arithmetic q-Gevrey series

In [DVIII, 8.4] we have presented a proof of the fact that a "G-function", which is solution of a q-difference equation (instead of a differential equation as usual), is necessarily the Taylor expansion of a rational function $\in \overline{\mathbb{Q}}(x)$. In [A3], Y. André asks for a good definition of a q-analogue of G-functions, applying to the so-called basic hypergeometric series and allowing us to construct an arithmetic q-Gevrey series theory.

In this appendix we propose a tentative definition and prove that some basic expected properties actually hold.

2.1. Definition of q^α-size.

Let K be a number field, $\mathcal{V}_K$ its ring of integers, v a place of K. In the non-archimedean case, we normalize $|\ |_v$ as follows:

$$|p|_v = p^{-[K_v:\mathbb{Q}_p]/[K:\mathbb{Q}]}\ ,$$

where K_v is the v-adic completion of K and $v|p$. Similarly, in the archimedean case, we normalize $|\ |_v$ by setting

$$|x|_v = \begin{cases} |x|_{\mathbb{R}}^{1/[K:\mathbb{Q}]} & \text{if } K_v = \mathbb{R} \\ |x|_{\mathbb{C}}^{2/[K:\mathbb{Q}]} & \text{if } K_v = \mathbb{C} \end{cases}\ ,$$

where $|\ |_{\mathbb{R}}$ and $|\ |_{\mathbb{C}}$ are the usual euclidean absolute values. We denote by Σ_f the set of finite places K and by Σ_∞ the set of archimedean places of K.

We fix a non zero element $q \in K$, which is not a root of unity. *We suppose that for all immersion $K \hookrightarrow \mathbb{C}$, the image of q in $\mathbb{C}$ do not have norm 1.*

Notation 2.1.1. For every $\alpha \in \mathbb{Q}$, we set

$$\Sigma(q^\alpha) = \{v \in \Sigma_f : |q|_v = 1,\ v|p \text{ and } p \text{ splits totally in } \mathbb{Q}(q^\alpha)/\mathbb{Q}\} \cup \Sigma_\infty \cup \{v \in \Sigma_f : |q|_v \neq 1\}\ .$$

We remark that if $q^\alpha \in \mathbb{Q}$ then $\Sigma(q^\alpha) = \Sigma_f \cup \Sigma_\infty$.

Definition 2.1.2. *Let $y(x)=\sum_{n=0}^{\infty}a_nx^n \in K[\![x]\!]$ be a formal power series. Let $\alpha\in\mathbb{Q}$, $\alpha\neq 0$; we call q^α-size of y the number*

$$\sigma_{q^\alpha}(y)=\limsup_{n\to\infty}\frac{1}{n}\sum_{v\in\Sigma(q^\alpha)}\log^+\left(\sup_{s\leq n}|a_s|_v\right)\ ,$$

where $\log^+(x)=\log\sup(1,x)$, for all $x\in\mathbb{R}$.

If $q^\alpha\in\mathbb{Q}$, we will say size instead of q^α-size and write σ instead of σ_{q^α}.

Remark 2.1.3.
1) If $q^\alpha\in\mathbb{Q}$, the definition of q^α-size coincides with the classical definition of the size coming from G-function theory (*cf.* [A1] or [DGS]).
2) Let us consider the basic hypergeometric series

$${}_l\Phi_k(\underline{a},\underline{b};q,x)=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{l}(q^{a_i};q)_n}{\prod_{j=1}^{k}(q^{b_j};q)_n(q;q)_n}\left((-1)^nq^{n(n-1)/2}\right)^{1+k-l}x^n\ ,$$

where $l,k\in\mathbb{Z}_{\geq 0}$, $\underline{a}=(a_1,\dots,a_l)\in\overline{\mathbb{Q}}^l$, $\underline{b}=(b_1,\dots,b_k)\in\overline{\mathbb{Q}}^k$ and $(q^a;q)_n=(1-q^a)\cdots(1-q^{a+n-1})$ is the q-Pochhammer symbol. We think that ${}_l\Phi_k(\underline{a},\underline{b};q,x)$ has finite q^γ-size, for a choice of $\gamma\in\mathbb{Q}$ such that $q^{a_i},q^{b_j}\in\mathbb{Q}(q^\gamma)$ for any $i=1,\dots,l$ and $j=1,\dots,k$. A direct proof seems difficult, but a proof of the type [DGS, IV, 8], relying on the Dwork-Robba theorem, should work: this is work in progress.

Proposition 2.1.4. *Let $y_1,y_2\in K[\![x]\!]$ and let α_1,α_2 be two rational numbers such that $\sigma_{q^{\alpha_1}}(y_1)<\infty$ and $\sigma_{q^{\alpha_2}}(y_2)<\infty$. Then there exists $\gamma\in\mathbb{Q}$ such that*

$$\sigma_{q^\gamma}(y_1+y_2)\leq\sigma_{q^\gamma}(y_1)+\sigma_{q^\gamma}(y_2)\ .$$

Moreover the sub-algebra of $K[\![x]\!]$ of power series, which have finite q^α-size for some $\alpha\in\mathbb{Q}$, is stable by Hadamard product.

Proof. We notice that it is enough to choose $\gamma\in\mathbb{Q}$ such that $\mathbb{Q}(q^\gamma)$ is an extension of $\mathbb{Q}(q^{\alpha_1})$ and $\mathbb{Q}(q^{\alpha_2})$ and adapt the classical proof concerning the case $q^\alpha\in\mathbb{Q}$ (*cf.* for instance [DGS, VIII, 1.3]). ■

2.2. Definition of arithmetic q-Gevrey series.

For any positive integer n, we set $(n)_q=1+\ldots+q^{n-2}+q^{n-1}=\frac{1-q^n}{1-q}$ and $[n]!_q=(n)_q\cdots(2)_q$. If $i\leq n$ is another positive integer, we set $\binom{n}{i}_q=\frac{[n]_q!}{[i]!_q[n-i]!_q}$.

Definition 2.2.1. *A series $y(x)=\sum_{n=0}^{\infty}a_nx^n\in K[\![x]\!]$ is arithmetic q-Gevrey of (double) order $(s_1,s_2)\in\mathbb{Q}^2$ if and only if there exists $\alpha\in\mathbb{Q}$ such that*

$$\sum_{n=0}^{\infty}\frac{a_n}{\left(q^{\frac{n(n-1)}{2}}\right)^{s_1}\left([n]!_q\right)^{s_2}}x^n$$

has finite q^α-size.

Remark 2.2.2.
1) We notice that being an arithmetic q-Gevrey series of order (s_1, s_2) implies being a q-Gevrey series of order $s_1 + s_2$ in the sense of [B] and [BB] for all $v \in \Sigma_\infty \cup \Sigma_f$ such that $|q|_v > 1$. If $v \in \Sigma_f$ and $|q|_v < 1$, then $|(n)_q|_v = 1$, therefore an arithmetic q-Gevrey series of order (s_1, s_2) is a q-Gevrey series of order s_1 in the sense of [BB].
2) The presence of a double order is motivated by the existence of two different analogues of the Laplace transform, which have a completely different arithmetical nature (*cf.* Appendix 1).

Definition 2.2.3. *We denote by $K\{x\}^A_{s_1,s_2;q}$ the set of all arithmetic q-Gevrey series of order (s_1, s_2) with coefficients in K solution of a q-difference equation with coefficients in $K(x)$.*

We note also $K\{x\}^{A,rat}_{s_1,s_2;q}$ the set of all $y(x) = \sum_{n\geq 0} a_n x^n \in K\{x\}^A_{s_1,s_2;q}$ such that

$$\sum_{n=0}^{\infty} \frac{a_n}{\left(q^{\frac{n(n-1)}{2}}\right)^{s_1} ([n]!_q)^{s_2}} x^n$$

has finite size.

Remark 2.2.4. The $K\{x\}^A_{s_1,s_2;q}$ is naturally a K-vector space, but not a K-algebra. In fact $\sum_{n=0}^{\infty} \frac{x^n}{[n]!_q}$ is an arithmetic q-Gevrey series of order $(0, -1)$ but

$$\left(\sum_{n=0}^{\infty} \frac{x^n}{[n]!_q}\right)^2 = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k}_q\right) \frac{x^n}{[n]!_q}$$

is *not* an arithmetic q-Gevrey series. For this reason, an arithmetic q-Gevrey theory may have applications to the irrationality theory but may hardly have applications to the transcendance theory, contrary to the case of arithmetic Gevrey theory in the differential case (*cf.* [A2] and [A3]).

We remark that, as in the differential case, the Hadamard product of q-Gevrey series of order (s_1, s_2) and (t_1, t_2) respectively is a q-Gevrey series of order $(s_1 + t_1, s_2 + t_2)$.

As far as $K\{x\}^{A,rat}_{s_1,s_2;q}$ is concerned the notation is motivated by [DVIII, (8.4)].

It is clear that we can always consider an arithmetic q-Gevrey series of order $(s, 0)$, with $s \neq 0$, as an arithmetic q^{-s}-Gevrey series of order $(-1, 0)$. This doesn't seem to be true that an arithmetic q-Gevrey series of order $(0, s)$ is an arithmetic q-Gevrey series of order $(0, -1)$.

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