On small height and local degrees

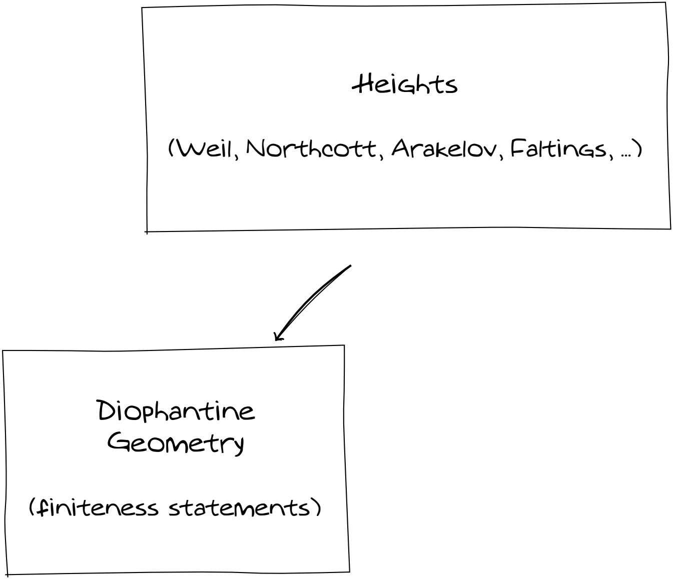
Sara Checcoli (*)
(joint work with Arno Fehm)

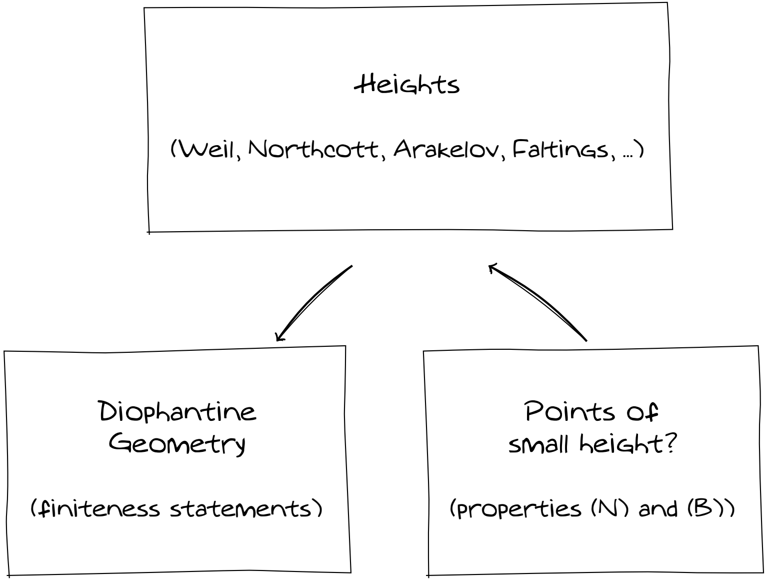
(*) Institut Fourier, Université Grenoble-Alpes

Séminaire différentiel
October 19th, 2021

Heights

(Weil, Northcott, Arakelov, Faltings, ...)


Heights
(Weil, Northcott, Arakelov, Faltings, ...)
Diophantine Geometry
(finiteness statements)


Heights
(Weil, Northcott, Arakelov, Faltings, ...)
Diophantine Geometry
(finiteness statements)
Points of small height?
(properties (N) and (B))

§1. Introduction

§1.1 The Weil height

Def. Let $\alpha \in \overline{\mathbb{Q}}^*$ and let $p_\alpha(x) = a(x-\alpha_1)\cdots(x-\alpha_d) \in \mathbb{Z}[x]$ be its minimal polynomial (with coprime coefficients).

§1.1 The Weil height

Def. Let $\alpha \in \overline{\mathbb{Q}}^*$ and let $p_\alpha(x) = a(x-\alpha_1)\cdots(x-\alpha_d) \in \mathbb{Z}[x]$ be its minimal polynomial (with coprime coefficients).
The (absolute logarithmic) Weil height of α is the number

$$h(\alpha) = \frac{1}{d} \cdot \log \left(|a| \prod_{i=1}^{d} \max(1, |\alpha_i|) \right).$$

§1.1 The Weil height

Def. Let $\alpha \in \overline{\mathbb{Q}}^*$ and let $p_\alpha(x) = a(x - \alpha_1) \cdots (x - \alpha_d) \in \mathbb{Z}[x]$ be its minimal polynomial (with coprime coefficients).
The (absolute logarithmic) Weil height of α is the number

$$h(\alpha) = \frac{1}{d} \cdot \log \left(|a| \prod_{i=1}^{d} \max(1, |\alpha_i|) \right).$$

$\to$ Measure of 'arithmetic complexity' of a number:

§1.1 The Weil height

Def. Let $\alpha \in \overline{\mathbb{Q}}^*$ and let $p_\alpha(x) = a(x-\alpha_1)\cdots(x-\alpha_d) \in \mathbb{Z}[x]$ be its minimal polynomial (with coprime coefficients).
The (absolute logarithmic) Weil height of α is the number

$$h(\alpha) = \frac{1}{d} \cdot \log\left(|a| \prod_{i=1}^{d} \max(1, |\alpha_i|)\right).$$

$\rightarrow$ Measure of 'arithmetic complexity' of a number:

- If $\alpha = p/q \in \mathbb{Q}^*$

§1.1 The Weil height

Def. Let $\alpha \in \overline{\mathbb{Q}}^*$ and let $p_\alpha(x) = a(x-\alpha_1)\cdots(x-\alpha_d) \in \mathbb{Z}[x]$ be its minimal polynomial (with coprime coefficients).
The (absolute logarithmic) Weil height of α is the number

$$h(\alpha) = \frac{1}{d} \cdot \log\left(|a| \prod_{i=1}^{d} \max(1, |\alpha_i|)\right).$$

$\rightarrow$ Measure of 'arithmetic complexity' of a number:

- If $\alpha = p/q \in \mathbb{Q}^* \Rightarrow h(\alpha) = \log(\max(|p|, |q|))$

§1.1 The Weil height

Def. Let $\alpha \in \overline{\mathbb{Q}}^*$ and let $p_\alpha(x) = a(x-\alpha_1)\cdots(x-\alpha_d) \in \mathbb{Z}[x]$ be its minimal polynomial (with coprime coefficients).
The (absolute logarithmic) Weil height of α is the number

$$h(\alpha) = \frac{1}{d} \cdot \log\left(|a| \prod_{i=1}^{d} \max(1, |\alpha_i|)\right).$$

$\rightarrow$ Measure of 'arithmetic complexity' of a number:
• If $\alpha = p/q \in \mathbb{Q}^* \Rightarrow h(\alpha) = \log(\max(|p|, |q|))$
So $h(2021/2020) = \log(2021)$ while $h(1) = 0$

§1.1 The Weil height

Def. Let $\alpha \in \overline{\mathbb{Q}}^*$ and let $p_\alpha(x) = a(x-\alpha_1)\cdots(x-\alpha_d) \in \mathbb{Z}[x]$ be its minimal polynomial (with coprime coefficients).
The (absolute logarithmic) Weil height of α is the number

$$h(\alpha) = \frac{1}{d} \cdot \log\left(|a| \prod_{i=1}^{d} \max(1, |\alpha_i|)\right).$$

$\rightarrow$ Measure of 'arithmetic complexity' of a number:

- If $\alpha = p/q \in \mathbb{Q}^* \Rightarrow h(\alpha) = \log(\max(|p|, |q|))$

So $h(2021/2020) = \log(2021)$ while $h(1) = 0$

- For d fixed: if $h(\alpha)$ 'big' $\Leftrightarrow$ max coefficient of $p_\alpha(x)$ is 'big'.

§1.1 The Weil height

Def. Let $\alpha \in \overline{\mathbb{Q}}^*$ and let $p_\alpha(x) = a(x-\alpha_1)\cdots(x-\alpha_d) \in \mathbb{Z}[x]$ be its minimal polynomial (with coprime coefficients).
The (absolute logarithmic) Weil height of α is the number

$$h(\alpha) = \frac{1}{d} \cdot \log\left(|a| \prod_{i=1}^{d} \max(1, |\alpha_i|)\right).$$

→ Measure of 'arithmetic complexity' of a number:

- If $\alpha = p/q \in \mathbb{Q}^* \Rightarrow h(\alpha) = \log(\max(|p|, |q|))$
So $h(2021/2020) = \log(2021)$ while $h(1) = 0$
- For d fixed: if $h(\alpha)$ 'big' ⇔ max coefficient of $p_\alpha(x)$ is 'big'.

→ 2 definitions of height:

§1.1 The Weil height

Def. Let $\alpha \in \overline{\mathbb{Q}}^*$ and let $p_\alpha(x) = a(x-\alpha_1)\cdots(x-\alpha_d) \in \mathbb{Z}[x]$ be its minimal polynomial (with coprime coefficients).
The (absolute logarithmic) Weil height of α is the number

$$h(\alpha) = \frac{1}{d} \cdot \log\left(|a| \prod_{i=1}^{d} \max(1, |\alpha_i|)\right).$$

$\rightarrow$ Measure of 'arithmetic complexity' of a number:

- If $\alpha = p/q \in \mathbb{Q}^* \Rightarrow h(\alpha) = \log(\max(|p|, |q|))$
So $h(2021/2020) = \log(2021)$ while $h(1) = 0$
- For d fixed: if $h(\alpha)$ 'big' $\Leftrightarrow$ max coefficient of $p_\alpha(x)$ is 'big'.

$\rightarrow$ 2 definitions of height:
- above one, with 'Mahler measure' (usefull for computations)

§1.1 The Weil height

Def. Let $\alpha \in \overline{\mathbb{Q}}^*$ and let $p_\alpha(x) = a(x-\alpha_1)\cdots(x-\alpha_d) \in \mathbb{Z}[x]$ be its minimal polynomial (with coprime coefficients).
The (absolute logarithmic) Weil height of α is the number

$$h(\alpha) = \frac{1}{d} \cdot \log\left(|a| \prod_{i=1}^{d} \max(1, |\alpha_i|) \right).$$

$\rightarrow$ Measure of 'arithmetic complexity' of a number:
- If $\alpha = p/q \in \mathbb{Q}^* \Rightarrow h(\alpha) = \log(\max(|p|, |q|))$
So $h(2021/2020) = \log(2021)$ while $h(1) = 0$
- For d fixed: if $h(\alpha)$ 'big' $\Leftrightarrow$ max coefficient of $p_\alpha(x)$ is 'big'.

$\rightarrow$ 2 definitions of height:
- above one, with 'Mahler measure' (usefull for computations)
- height=sum of local contributions (usefull for proving statements).

§1.1 The Weil height

Def. Let $\alpha \in \overline{\mathbb{Q}}^*$ and let $p_\alpha(x) = a(x-\alpha_1)\cdots(x-\alpha_d) \in \mathbb{Z}[x]$ be its minimal polynomial (with coprime coefficients).
The (absolute logarithmic) Weil height of α is the number

$$h(\alpha) = \frac{1}{d} \cdot \log\left(|a| \prod_{i=1}^{d} \max(1, |\alpha_i|) \right).$$

Properties:

§1.1 The Weil height

Def. Let $\alpha \in \overline{\mathbb{Q}}^*$ and let $p_\alpha(x) = a(x-\alpha_1)\cdots(x-\alpha_d) \in \mathbb{Z}[x]$ be its minimal polynomial (with coprime coefficients).
The (absolute logarithmic) Weil height of α is the number

$$h(\alpha) = \frac{1}{d} \cdot \log\left(|a| \prod_{i=1}^{d} \max(1, |\alpha_i|)\right).$$

Properties:

- $h(\alpha) \geq 0$ for all α;

§1.1 The Weil height

Def. Let $\alpha \in \overline{\mathbb{Q}}^*$ and let $p_\alpha(x) = a(x-\alpha_1)\cdots(x-\alpha_d) \in \mathbb{Z}[x]$ be its minimal polynomial (with coprime coefficients).
The (absolute logarithmic) Weil height of α is the number

$$h(\alpha) = \frac{1}{d} \cdot \log\left(|a| \prod_{i=1}^{d} \max(1, |\alpha_i|)\right).$$

Properties:

- $h(\alpha) \geq 0$ for all α;
- $h(\alpha^m) = |m| h(\alpha)$ for all $m \in \mathbb{Z}$;

§1.1 The Weil height

Def. Let $\alpha \in \overline{\mathbb{Q}}^*$ and let $p_\alpha(x) = a(x-\alpha_1)\cdots(x-\alpha_d) \in \mathbb{Z}[x]$ be its minimal polynomial (with coprime coefficients).
The (absolute logarithmic) Weil height of α is the number

$$h(\alpha) = \frac{1}{d} \cdot \log\left(|a| \prod_{i=1}^{d} \max(1, |\alpha_i|)\right).$$

Properties:

- $h(\alpha) \geq 0$ for all α;
- $h(\alpha^m) = |m| h(\alpha)$ for all $m \in \mathbb{Z}$;
- $h(\sigma(\alpha)) = h(\alpha)$ for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$;

§1.1 The Weil height

Def. Let $\alpha \in \overline{\mathbb{Q}}^*$ and let $p_\alpha(x) = a(x-\alpha_1)\cdots(x-\alpha_d) \in \mathbb{Z}[x]$ be its minimal polynomial (with coprime coefficients).
The (absolute logarithmic) Weil height of α is the number

$$h(\alpha) = \frac{1}{d} \cdot \log\left(|a| \prod_{i=1}^{d} \max(1, |\alpha_i|)\right).$$

Properties:

- $h(\alpha) \geq 0$ for all α;
- $h(\alpha^m) = |m| h(\alpha)$ for all $m \in \mathbb{Z}$;
- $h(\sigma(\alpha)) = h(\alpha)$ for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$;
- Kronecker's theorem: $h(\alpha) = 0 \Leftrightarrow \alpha = 0$ or α is a root of unity.

§1.1 The Weil height

Def. Let $\alpha \in \overline{\mathbb{Q}}^*$ and let $p_\alpha(x) = a(x-\alpha_1)\cdots(x-\alpha_d) \in \mathbb{Z}[x]$ be its minimal polynomial (with coprime coefficients).
The (absolute logarithmic) Weil height of α is the number

$$h(\alpha) = \frac{1}{d} \cdot \log\left(|a| \prod_{i=1}^{d} \max(1, |\alpha_i|)\right).$$

Properties:

- $h(\alpha) \geq 0$ for all α;
- $h(\alpha^m) = |m| h(\alpha)$ for all $m \in \mathbb{Z}$;
- $h(\sigma(\alpha)) = h(\alpha)$ for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$;
- Kronecker's theorem: $h(\alpha) = 0 \Leftrightarrow \alpha = 0$ or α is a root of unity.

Q: What about points of non-zero small height?

§1.2. Small height

Northcott's theorem (or, why the height is so usefull):

§1.2. Small height

Northcott's theorem (or, why the height is so usefull):
$\#\{\alpha \in \overline{\mathbb{Q}} \mid h(\alpha) \leq B, [\mathbb{Q}(\alpha) : \mathbb{Q}] \leq d\} < \infty$ for all $B \geq 0$, $d \geq 1$.

§1.2. Small height

Northcott's theorem (or, why the height is so usefull):

$$\#\{\alpha \in \overline{\mathbb{Q}} \mid h(\alpha) \leq B, [\mathbb{Q}(\alpha) : \mathbb{Q}] \leq d\} < \infty \text{ for all } B \geq 0, d \geq 1.$$

- Effective: you can explicitly find the elements

§1.2. Small height

> Northcott's theorem (or, why the height is so usefull):
> $$\#\{\alpha \in \overline{\mathbb{Q}} \mid h(\alpha) \leq B, [\mathbb{Q}(\alpha) : \mathbb{Q}] \leq d\} < \infty \text{ for all } B \geq 0,\ d \geq 1.$$

- Effective: you can explicitly find the elements
- Diophantine geometry: bound height + bound degree

$\Rightarrow$ finiteness (e.g. rational points on high genus curves).

§1.2. Small height

Northcott's theorem (or, why the height is so usefull):
$\#\{\alpha \in \overline{\mathbb{Q}} \mid h(\alpha) \leq B, [\mathbb{Q}(\alpha) : \mathbb{Q}] \leq d\} < \infty$ for all $B \geq 0$, $d \geq 1$.

- Effective: you can explicitly find the elements
- Diophantine geometry: bound height + bound degree

$\Rightarrow$ finiteness (e.g. rational points on high genus curves).

Lehmer's conjecture (or, small height $\Rightarrow$ big degree):

§1.2. Small height

Northcott's theorem (or, why the height is so usefull):
$\#\{\alpha \in \overline{\mathbb{Q}} \mid h(\alpha) \leq B, [\mathbb{Q}(\alpha) : \mathbb{Q}] \leq d\} < \infty$ for all $B \geq 0$, $d \geq 1$.

- Effective: you can explicitly find the elements
- Diophantine geometry: bound height + bound degree

$\Rightarrow$ finiteness (e.g. rational points on high genus curves).

Lehmer's conjecture (or, small height $\Rightarrow$ big degree):
there exists a constant $c > 0$ such that for all $\alpha \in \overline{\mathbb{Q}}$
then $h(\alpha) \cdot [\mathbb{Q}(\alpha) : \mathbb{Q}]$ is either 0 or bigger than c.

§1.2. Small height

Northcott's theorem (or, why the height is so usefull):
$\#\{\alpha \in \overline{\mathbb{Q}} \mid h(\alpha) \leq B, [\mathbb{Q}(\alpha):\mathbb{Q}] \leq d\} < \infty$ for all $B \geq 0$, $d \geq 1$.

- Effective: you can explicitly find the elements
- Diophantine geometry: bound height + bound degree

$\Rightarrow$ finiteness (e.g. rational points on high genus curves).

Lehmer's conjecture (or, small height $\Rightarrow$ big degree):
there exists a constant $c > 0$ such that for all $\alpha \in \overline{\mathbb{Q}}$
then $h(\alpha) \cdot [\mathbb{Q}(\alpha):\mathbb{Q}]$ is either 0 or bigger than c.

- Open (known for algebraic numbers which are not algebraic integers, roots of non-reciprocal polynomials, generators of Galois extensions,... Best unconditional result by Dobrowolski)

§1.2. Small height

Northcott's theorem (or, why the height is so usefull):
$\#\{\alpha \in \overline{\mathbb{Q}} \mid h(\alpha) \leq B, [\mathbb{Q}(\alpha) : \mathbb{Q}] \leq d\} < \infty$ for all $B \geq 0$, $d \geq 1$.

- Effective: you can explicitly find the elements
- Diophantine geometry: bound height + bound degree

$\Rightarrow$ finiteness (e.g. rational points on high genus curves).

Lehmer's conjecture (or, small height $\Rightarrow$ big degree):
there exists a constant $c > 0$ such that for all $\alpha \in \overline{\mathbb{Q}}$
then $h(\alpha) \cdot [\mathbb{Q}(\alpha) : \mathbb{Q}]$ is either 0 or bigger than c.

- Open (known for algebraic numbers which are not algebraic integers, roots of non-reciprocal polynomials, generators of Galois extensions,...Best unconditional result by Dobrowolski)

Q. Are there sets of algebraic numbers where one can 'do better'?

§1.2. Small height

Northcott's theorem (or, why the height is so usefull):
$\#\{\alpha \in \overline{\mathbb{Q}} \mid h(\alpha) \leq B, [\mathbb{Q}(\alpha) : \mathbb{Q}] \leq d\} < \infty$ for all $B \geq 0$, $d \geq 1$.

- Effective: you can explicitly find the elements
- Diophantine geometry: bound height + bound degree

$\Rightarrow$ finiteness (e.g. rational points on high genus curves).

Lehmer's conjecture (or, small height $\Rightarrow$ big degree):
there exists a constant $c > 0$ such that for all $\alpha \in \overline{\mathbb{Q}}$
then $h(\alpha) \cdot [\mathbb{Q}(\alpha) : \mathbb{Q}]$ is either 0 or bigger than c.

- Open (known for algebraic numbers which are not algebraic integers, roots of non-reciprocal polynomials, generators of Galois extensions,... Best unconditional result by Dobrowolski)

Q. Are there sets of algebraic numbers where one can 'do better'? That is, get the same statements forgetting the degrees?

§1.3. Properties (N) and (B)

§1.3. Properties (N) and (B)

Def. (Bombieri and Zannier, 2001) A subfield $L \subseteq \overline{\mathbb{Q}}$ has:

- the Northcott property (N) if $\#\{\alpha \in L \mid h(\alpha) \leq B\} < \infty$ for all $B \geq 0$.

§1.3. Properties (N) and (B)

Def. (Bombieri and Zannier, 2001) A subfield $L \subseteq \overline{\mathbb{Q}}$ has:

- the Northcott property (N) if $\#\{\alpha \in L \mid h(\alpha) \leq B\} < \infty$ for all $B \geq 0$.
- the Bogomolov property (B) if $\exists c = c(L) > 0$ such that, for all $\alpha \in L$, either $h(\alpha) = 0$ or $h(\alpha) \geq c$

§1.3. Properties (N) and (B)

Def. (Bombieri and Zannier, 2001) A subfield $L \subseteq \overline{\mathbb{Q}}$ has:

- the Northcott property (N) if $\#\{\alpha \in L \mid h(\alpha) \leq B\} < \infty$ for all $B \geq 0$.
- the Bogomolov property (B) if $\exists c = c(L) > 0$ such that, for all $\alpha \in L$, either $h(\alpha) = 0$ or $h(\alpha) \geq c$

Remarks:

• Property (B) named after Bogomolov conjecture (on points of small height on curves).

§1.3. Properties (N) and (B)

Def. (Bombieri and Zannier, 2001) A subfield $L \subseteq \overline{\mathbb{Q}}$ has:

- the Northcott property (N) if $\#\{\alpha \in L \mid h(\alpha) \leq B\} < \infty$ for all $B \geq 0$.
- the Bogomolov property (B) if $\exists c = c(L) > 0$ such that, for all $\alpha \in L$, either $h(\alpha) = 0$ or $h(\alpha) \geq c$

Remarks:

• Property (B) named after Bogomolov conjecture (on points of small height on curves).

• (N) $\Rightarrow$ (B)

§1.3. Properties (N) and (B)

<u>Def.</u> (Bombieri and Zannier, 2001) A subfield $L \subseteq \overline{\mathbb{Q}}$ has:

- the <u>Northcott property (N)</u> if $\#\{\alpha \in L \mid h(\alpha) \leq B\} < \infty$ for all $B \geq 0$.
- the <u>Bogomolov property (B)</u> if $\exists c = c(L) > 0$ such that, for all $\alpha \in L$, either $h(\alpha) = 0$ or $h(\alpha) \geq c$

<u>Remarks</u>:

• Property (B) named after Bogomolov conjecture (on points of small height on curves).

• (N) $\Rightarrow$ (B)

<u>Pf.</u> If $\{\alpha \in L \mid 0 < h(\alpha) \leq 2\} = \{\alpha_1, \ldots, \alpha_m\}$ then, for $\alpha \in L$, $h(\alpha) = 0$ or $h(\alpha) \geq c = \min h(\alpha_i)$ $\square$

§1.3. Properties (N) and (B)

Def. (Bombieri and Zannier, 2001) A subfield $L \subseteq \overline{\mathbb{Q}}$ has:

- the Northcott property (N) if $\#\{\alpha \in L \mid h(\alpha) \leq B\} < \infty$ for all $B \geq 0$.
- the Bogomolov property (B) if $\exists c = c(L) > 0$ such that, for all $\alpha \in L$, either $h(\alpha) = 0$ or $h(\alpha) \geq c$

Remarks:

• Property (B) named after Bogomolov conjecture (on points of small height on curves).

• (N) $\Rightarrow$ (B)

Pf. If $\{\alpha \in L \mid 0 < h(\alpha) \leq 2\} = \{\alpha_1, \ldots, \alpha_m\}$ then, for $\alpha \in L$, $h(\alpha) = 0$ or $h(\alpha) \geq c = \min h(\alpha_i)$ □

• (B) $\not\Rightarrow$ (N)

Ex. If $\alpha \in L = \mathbb{Q}^{ab}$ then $h(\alpha) = 0$ or $h(\alpha) \geq \log(5)/12$ (Amoroso-Dvornicich, 2000).

§1.3. Properties (N) and (B)

Def. (Bombieri and Zannier, 2001) A subfield $L \subseteq \overline{\mathbb{Q}}$ has:

- the Northcott property (N) if $\#\{\alpha \in L \mid h(\alpha) \leq B\} < \infty$ for all $B \geq 0$.
- the Bogomolov property (B) if $\exists c = c(L) > 0$ such that, for all $\alpha \in L$, either $h(\alpha) = 0$ or $h(\alpha) \geq c$

Remarks:

• Property (B) named after Bogomolov conjecture (on points of small height on curves).

• (N) $\Rightarrow$ (B)
Pf. If $\{\alpha \in L \mid 0 < h(\alpha) \leq 2\} = \{\alpha_1, \ldots, \alpha_m\}$ then, for $\alpha \in L$, $h(\alpha) = 0$ or $h(\alpha) \geq c = \min h(\alpha_i)$ $\square$

• (B) $\not\Rightarrow$ (N)
Ex. If $\alpha \in L = \mathbb{Q}^{ab}$ then $h(\alpha) = 0$ or $h(\alpha) \geq \log(5)/12$ (Amoroso-Dvornicich, 2000). So $\mathbb{Q}^{ab}$ has (B), but not (N) (contains infinitely many roots of 1).

§1.3. Properties (N) and (B)

Def. (Bombieri and Zannier, 2001) A subfield $L \subseteq \overline{\mathbb{Q}}$ has:

- the Northcott property (N) if $\#\{\alpha \in L \mid h(\alpha) \leq B\} < \infty$ for all $B \geq 0$.
- the Bogomolov property (B) if $\exists c = c(L) > 0$ such that, for all $\alpha \in L$, either $h(\alpha) = 0$ or $h(\alpha) \geq c$

Remarks:

number fields
have both (N) and (B)
(by Northcott's theorem)

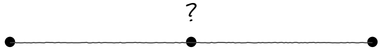
$\overline{\mathbb{Q}}$
has neither (B) nor (N)
(as $h(2^{1/n}) = \log(2)/n \to 0$)

§1.3. Properties (N) and (B)

Def. (Bombieri and Zannier, 2001) A subfield $L \subseteq \overline{\mathbb{Q}}$ has:

- the Northcott property (N) if $\#\{\alpha \in L \mid h(\alpha) \leq B\} < \infty$ for all $B \geq 0$.
- the Bogomolov property (B) if $\exists c = c(L) > 0$ such that, for all $\alpha \in L$, either $h(\alpha) = 0$ or $h(\alpha) \geq c$

Remarks:



number fields

have both (N) and (B)
(by Northcott's theorem)

$\overline{\mathbb{Q}}$

has neither (B) nor (N)
(as $h(2^{1/n}) = \log(2)/n \to 0$)

> Problem: Given a infinite extension $L/\mathbb{Q}$, decide whether it has (N) or (B). Hard in general!

§1.4. How much is known on property (B)?

Many examples of fields with (B):

§1.4. How much is known on property (B)?

Many examples of fields with (B):

1) Fields obtained 'adding torsion'

§1.4. How much is known on property (B)?

Many examples of fields with (B):

1) Fields obtained 'adding torsion'

- $\mathbb{Q}^{ab} = \mathbb{Q}(\mathbb{G}_m^{tor})$=field obtained adding to $\mathbb{Q}$ all torsion points of the torus i.e. all roots of unity (Amoroso-Dvornicich, 2000).

§1.4. How much is known on property (B)?

Many examples of fields with (B):

1) Fields obtained 'adding torsion'

- $\mathbb{Q}^{ab} = \mathbb{Q}(\mathbb{G}_m^{tor})$=field obtained adding to $\mathbb{Q}$ all torsion points of the torus i.e. all roots of unity (Amoroso-Dvornicich, 2000). More generally K^{ab}=maximal abelian extension of a number field K (Amoroso-Zannier 2000, 2010)

§1.4. How much is known on property (B)?

Many examples of fields with (B):

1) Fields obtained 'adding torsion'

• $\mathbb{Q}^{ab} = \mathbb{Q}(\mathbb{G}_m^{tor})$=field obtained adding to $\mathbb{Q}$ all torsion points of the torus i.e. all roots of unity (Amoroso-Dvornicich, 2000). More generally K^{ab}=maximal abelian extension of a number field K (Amoroso-Zannier 2000, 2010)

• $\mathbb{Q}(E_{tor})$=field obtained adding to $\mathbb{Q}$ all torsion points of an elliptic curve $E/\mathbb{Q}$ (Habegger, 2011)

§1.4. How much is known on property (B)?

Many examples of fields with (B):

1) Fields obtained 'adding torsion'

- $\mathbb{Q}^{ab} = \mathbb{Q}(\mathbb{G}_m^{tor})$=field obtained adding to $\mathbb{Q}$ all torsion points of the torus i.e. all roots of unity (Amoroso-Dvornicich, 2000). More generally K^{ab}=maximal abelian extension of a number field K (Amoroso-Zannier 2000, 2010)
- $\mathbb{Q}(E_{tor})$=field obtained adding to $\mathbb{Q}$ all torsion points of an elliptic curve $E/\mathbb{Q}$ (Habegger, 2011)

Idea (for $\mathbb{Q}^{ab}$): $\alpha \in K = \mathbb{Q}(\zeta_n)$

- write $h(\alpha) =$ sum local contributions

§1.4. How much is known on property (B)?

Many examples of fields with (B):

I) Fields obtained 'adding torsion'

- $\mathbb{Q}^{ab} = \mathbb{Q}(\mathbb{G}_m^{tor})$=field obtained adding to $\mathbb{Q}$ all torsion points of the torus i.e. all roots of unity (Amoroso-Dvornicich, 2000). More generally K^{ab}=maximal abelian extension of a number field K (Amoroso-Zannier 2000, 2010)

- $\mathbb{Q}(E_{tor})$=field obtained adding to $\mathbb{Q}$ all torsion points of an elliptic curve $E/\mathbb{Q}$ (Habegger, 2011)

Idea (for $\mathbb{Q}^{ab}$): $\alpha \in K = \mathbb{Q}(\zeta_n)$
- write $h(\alpha) =$ sum local contributions
- fix auxiliary prime p and use Frobenius (if $p \nmid n$)/variant of Frobenius at p (if $p \mid n$) $\Rightarrow$ each local contribution $\geq$ bound only depending on p (and not on n)

§1.4. How much is known on property (B)?

Many examples of fields with (B):

1) Fields obtained 'adding torsion'

Examples: $\mathbb{Q}^{ab}, K^{ab}, \mathbb{Q}(E_{tor})$

2) Fields with local conditions

§1.4. How much is known on property (B)?

Many examples of fields with (B):

1) Fields obtained 'adding torsion'

Examples: $\mathbb{Q}^{ab}, K^{ab}, \mathbb{Q}(E_{tor})$

2) Fields with local conditions

- $\mathbb{Q}^{tr}$=totally real numbers (Schinzel 1973)

§1.4. How much is known on property (B)?

Many examples of fields with (B):

1) Fields obtained 'adding torsion'

Examples: $\mathbb{Q}^{ab}, K^{ab}, \mathbb{Q}(E_{tor})$

2) Fields with local conditions

- $\mathbb{Q}^{tr}$=totally real numbers (Schinzel 1973)
- $L/\mathbb{Q}$ Galois with bounded local degrees at some prime p i.e. that can be embedded into a finite extension of $\mathbb{Q}_p$ (Bombieri-Zannier, 2001)

§1.4. How much is known on property (B)?

Many examples of fields with (B):

1) Fields obtained 'adding torsion'

Examples: $\mathbb{Q}^{ab}, K^{ab}, \mathbb{Q}(E_{tor})$

2) Fields with local conditions

- $\mathbb{Q}^{tr}$=totally real numbers (Schinzel 1973)
- $L/\mathbb{Q}$ Galois with bounded local degrees at some prime p i.e. that can be embedded into a finite extension of $\mathbb{Q}_p$ (Bombieri-Zannier, 2001)

Examples:

$\rightarrow$ L number field

§1.4. How much is known on property (B)?

Many examples of fields with (B):

1) Fields obtained 'adding torsion'

Examples: $\mathbb{Q}^{ab}, K^{ab}, \mathbb{Q}(E_{tor})$

2) Fields with local conditions

- $\mathbb{Q}^{tr}$=totally real numbers (Schinzel 1973)
- $L/\mathbb{Q}$ Galois with bounded local degrees at some prime p i.e. that can be embedded into a finite extension of $\mathbb{Q}_p$ (Bombieri-Zannier, 2001)

Examples:

$\rightarrow$ L number field

$\rightarrow$ $L = \mathbb{Q}^{tp}$ totally p-adic numbers

§1.4. How much is known on property (B)?

Many examples of fields with (B):

1) Fields obtained 'adding torsion'

Examples: $\mathbb{Q}^{ab}, K^{ab}, \mathbb{Q}(E_{tor})$

2) Fields with local conditions

- $\mathbb{Q}^{tr}$=totally real numbers (Schinzel 1973)
- $L/\mathbb{Q}$ Galois with bounded local degrees at some prime p i.e. that can be embedded into a finite extension of $\mathbb{Q}_p$ (Bombieri-Zannier, 2001)

Examples:

→ L number field

→ $L = \mathbb{Q}^{tp}$ totally p-adic numbers

→ $L = K^{(d)}$ compositum of all degree $\leq d$ extensions of a n.f. K

§1.4. How much is known on property (B)?

Many examples of fields with (B):

1) Fields obtained 'adding torsion'

Examples: $\mathbb{Q}^{ab}, K^{ab}, \mathbb{Q}(E_{tor})$

2) Fields with local conditions

- $\mathbb{Q}^{tr}$=totally real numbers (Schinzel 1973)
- $L/\mathbb{Q}$ Galois with bounded local degrees at some prime p i.e. that can be embedded into a finite extension of $\mathbb{Q}_p$ (Bombieri-Zannier, 2001)

Examples:

→ L number field

→ $L = \mathbb{Q}^{tp}$ totally p-adic numbers

→ $L = K^{(d)}$ compositum of all degree $\leq d$ extensions of a n.f. K
E.g. $\mathbb{Q}^{(2)} = \mathbb{Q}(\sqrt{-1}, \sqrt{2}, \sqrt{3}, \dots)$

§1.4. How much is known on property (B)?

Many examples of fields with (B):

1) Fields obtained 'adding torsion'

Examples: $\mathbb{Q}^{ab}, K^{ab}, \mathbb{Q}(E_{tor})$

2) Fields with local conditions

- $\mathbb{Q}^{tr}$=totally real numbers (Schinzel 1973)
- $L/\mathbb{Q}$ Galois with bounded local degrees at some prime p i.e. that can be embedded into a finite extension of $\mathbb{Q}_p$ (Bombieri-Zannier, 2001)

Examples:

→ L number field
→ $L = \mathbb{Q}^{tp}$ totally p-adic numbers
→ $L = K^{(d)}$ compositum of all degree $\leq d$ extensions of a n.f. K
E.g. $\mathbb{Q}^{(2)} = \mathbb{Q}(\sqrt{-1}, \sqrt{2}, \sqrt{3}, \ldots)$

Idea: use 'equidistribution'.

§1.4. How much is known on property (B)?

Many examples of fields with (B):

1) Fields obtained 'adding torsion'

Examples: $\mathbb{Q}^{ab}, K^{ab}, \mathbb{Q}(E_{tor})$

2) Fields with local conditions

Examples: $\mathbb{Q}^{tr}$, Galois extensions with bounded local degrees

3) Generalization of (1) and (2) (Amoroso, David and Zannier, 2014)

Rem. All the above examples do not satisfy property (N):

- $K^{ab}, \mathbb{Q}(E_{tor})$ contain infinitely many roots of 1
- $\mathbb{Q}^{tr}$ contains a sequence of elements with height $\to 0.27328\ldots$ (Smyth, 1980)
- $\mathbb{Q}^{tp}$: $\liminf_{\alpha\in\mathbb{Q}^{tp}} h(\alpha) \leq (\log p)/(p-1)$ (Bombieri, Zannier, 2001)

§1.5. How few is known on property (N)?

Essentially only 2 examples of fields with (N):

§1.5. How few is known on property (N)?

Essentially only 2 examples of fields with (N):

1) Bombieri & Zannier's criterion: $K_{ab}^{(d)}$

§1.5. How few is known on property (N)?

Essentially only 2 examples of fields with (N):

1) Bombieri & Zannier's criterion: $K_{ab}^{(d)}$

<u>Def.</u> $K_{ab}^{(d)}$=maximal subfield of $K^{(d)}$ which is abelian over K n.f.

§1.5. How few is known on property (N)?

Essentially only 2 examples of fields with (N):

1) Bombieri & Zannier's criterion: $K_{ab}^{(d)}$

Def. $K_{ab}^{(d)}$=maximal subfield of $K^{(d)}$ which is abelian over K n.f.

Example: $\mathbb{Q}_{ab}^{(2)} = \mathbb{Q}^{(2)} = \mathbb{Q}(\sqrt{-1}, \sqrt{2}, \sqrt{3}, \sqrt{5}, \ldots)$,

§1.5. How few is known on property (N)?

Essentially only 2 examples of fields with (N):

1) Bombieri & Zannier's criterion: $K_{ab}^{(d)}$

Def. $K_{ab}^{(d)}$=maximal subfield of $K^{(d)}$ which is abelian over K n.f.

Example: $\mathbb{Q}_{ab}^{(2)} = \mathbb{Q}^{(2)} = \mathbb{Q}(\sqrt{-1}, \sqrt{2}, \sqrt{3}, \sqrt{5}, \ldots)$, but $\mathbb{Q}_{ab}^{(3)} \neq \mathbb{Q}^{(3)}$

§1.5. How few is known on property (N)?

Essentially only 2 examples of fields with (N):

1) Bombieri & Zannier's criterion: $K_{ab}^{(d)}$

<u>Def.</u> $K_{ab}^{(d)}$=maximal subfield of $K^{(d)}$ which is abelian over K n.f.

Example: $\mathbb{Q}_{ab}^{(2)} = \mathbb{Q}^{(2)} = \mathbb{Q}(\sqrt{-1}, \sqrt{2}, \sqrt{3}, \sqrt{5}, \ldots)$, but $\mathbb{Q}_{ab}^{(3)} \neq \mathbb{Q}^{(3)}$

Idea: suppose $h(\alpha) \leq B$.

§1.5. How few is known on property (N)?

Essentially only 2 examples of fields with (N):

1) Bombieri & Zannier's criterion: $K_{ab}^{(d)}$

Def. $K_{ab}^{(d)}$=maximal subfield of $K^{(d)}$ which is abelian over K n.f.

Example: $\mathbb{Q}_{ab}^{(2)} = \mathbb{Q}^{(2)} = \mathbb{Q}(\sqrt{-1}, \sqrt{2}, \sqrt{3}, \sqrt{5}, \ldots)$, but $\mathbb{Q}_{ab}^{(3)} \neq \mathbb{Q}^{(3)}$

Idea: suppose $h(\alpha) \leq B$.

- By carefull study of ramification: if p ramifies in $K(\alpha)$

$\Rightarrow p \leq C(B)$

§1.5. How few is known on property (N)?

Essentially only 2 examples of fields with (N):

1) Bombieri & Zannier's criterion: $K_{ab}^{(d)}$

Def. $K_{ab}^{(d)}$=maximal subfield of $K^{(d)}$ which is abelian over K n.f.

Example: $\mathbb{Q}_{ab}^{(2)} = \mathbb{Q}^{(2)} = \mathbb{Q}(\sqrt{-1}, \sqrt{2}, \sqrt{3}, \sqrt{5}, \ldots)$, but $\mathbb{Q}_{ab}^{(3)} \neq \mathbb{Q}^{(3)}$

Idea: suppose $h(\alpha) \leq B$.

- By carefull study of ramification: if p ramifies in $K(\alpha)$
$\Rightarrow p \leq C(B)$
- Use that $K(\alpha)$=compositum of cyclic extensions of degree $\leq d!$ each with bounded discriminant + Hermite theorem

§1.5. How few is known on property (N)?

Essentially only 2 examples of fields with (N):

1) Bombieri & Zannier's criterion: $K_{ab}^{(d)}$

Def. $K_{ab}^{(d)}$=maximal subfield of $K^{(d)}$ which is abelian over K n.f.

Example: $\mathbb{Q}_{ab}^{(2)} = \mathbb{Q}^{(2)} = \mathbb{Q}(\sqrt{-1}, \sqrt{2}, \sqrt{3}, \sqrt{5}, \dots)$, but $\mathbb{Q}_{ab}^{(3)} \neq \mathbb{Q}^{(3)}$

Idea: suppose $h(\alpha) \leq B$.

- By carefull study of ramification: if p ramifies in $K(\alpha)$ $\Rightarrow p \leq C(B)$
- Use that $K(\alpha)$=compositum of cyclic extensions of degree $\leq d!$ each with bounded discriminant + Hermite theorem $\Rightarrow$ $[K(\alpha) : K] \leq D(B)$ $\Rightarrow$ finitely many α by Northcott

§1.5. How few is known on property (N)?

Essentially only 2 examples of fields with (N):

1) Bombieri & Zannier's criterion: $K_{ab}^{(d)}$

2) Widmer's criterion (discriminants growing 'fast')

§1.5. How few is known on property (N)?

Essentially only 2 examples of fields with (N):

1) Bombieri & Zannier's criterion: $K_{ab}^{(d)}$

2) Widmer's criterion (discriminants growing 'fast')

Theorem(Widmer, 2013): If $K = K_0 \subseteq K_1 \subseteq K_2 \subseteq \ldots$ is a tower of number fields such that

$$\inf_{K_{i-1} \subsetneq M \subseteq K_i} N_{K_{i-1}/\mathbb{Q}}(\mathrm{disc}(M/K_{i-1}))^{\frac{1}{[M:K_0][M:K_{i-1}]}} \to \infty$$

then $L = \bigcup_i K_i$ has property (N).

§1.5. How few is known on property (N)?

Essentially only 2 examples of fields with (N):

1) Bombieri & Zannier's criterion: $K_{ab}^{(d)}$

2) Widmer's criterion (discriminants growing 'fast')

Theorem(Widmer, 2013): If $K = K_0 \subseteq K_1 \subseteq K_2 \subseteq \ldots$ is a tower of number fields such that

$$\inf_{K_{i-1} \subsetneq M \subseteq K_i} N_{K_{i-1}/\mathbb{Q}}(\mathrm{disc}(M/K_{i-1}))^{\frac{1}{[M:K_0][M:K_{i-1}]}} \to \infty$$

then $L = \bigcup_i K_i$ has property (N).

- Roughly: if discriminants grow fast at each step in the tower $\Rightarrow$ L has (N).

§1.5. How few is known on property (N)?

Essentially only 2 examples of fields with (N):

1) Bombieri & Zannier's criterion: $K_{ab}^{(d)}$

2) Widmer's criterion (discriminants growing 'fast')

Theorem(Widmer, 2013): If $K = K_0 \subseteq K_1 \subseteq K_2 \subseteq \ldots$ is a tower of number fields such that

$$\inf_{K_{i-1} \subsetneq M \subseteq K_i} N_{K_{i-1}/\mathbb{Q}}\left(\mathrm{disc}(M/K_{i-1})\right)^{\frac{1}{[M:K_0][M:K_{i-1}]}} \to \infty$$

then $L = \bigcup_i K_i$ has property (N).

- Roughly: if discriminants grow fast at each step in the tower $\Rightarrow$ L has (N).

Idea: use bound of Silverman for minimal height of generators of number fields in terms of certain discriminants

§1.5. How few is known on property (N)?

Essentially only 2 examples of fields with (N):

1) Bombieri & Zannier's criterion: $K_{ab}^{(d)}$

2) Widmer's criterion (discriminants growing 'fast')

Question: Other examples of fields with property (N)?

§2. Results

§2.1. A 'new' criterion for (N)

§2.1. A new criterion for (N)?

Recall: $L/\mathbb{Q}$ Galois with bounded local degrees $\Rightarrow$ L has (B).

§2.1. A new criterion for (N)?

Recall: $L/\mathbb{Q}$ Galois with bounded local degrees $\Rightarrow$ L has (B).

More precisely:

Theorem (Bombieri-Zannier, 2001). $L/\mathbb{Q}$ Galois extension, $S(L) \neq \emptyset$ set of primes at which L has bounded local degrees.

§2.1. A new criterion for (N)?

Recall: $L/\mathbb{Q}$ Galois with bounded local degrees $\Rightarrow$ L has (B).

More precisely:

Theorem (Bombieri-Zannier, 2001). $L/\mathbb{Q}$ Galois extension, $S(L) \neq \emptyset$ set of primes at which L has bounded local degrees. Then

$$\liminf_{\alpha \in L} h(\alpha) \geq \beta(L) = \frac{1}{2} \sum_{p \in S(L)} \frac{\log p}{e_p(p^{f_p}+1)}$$

(e_p, f_p ramification index and inertial degree of L at p).

§2.1. A new criterion for (N)?

Recall: $L/\mathbb{Q}$ Galois with bounded local degrees $\Rightarrow$ L has (B).

More precisely:

Theorem (Bombieri-Zannier, 2001). $L/\mathbb{Q}$ Galois extension, $S(L) \neq \emptyset$ set of primes at which L has bounded local degrees. Then

$$\liminf_{\alpha \in L} h(\alpha) \geq \beta(L) = \frac{1}{2} \sum_{p \in S(L)} \frac{\log p}{e_p(p^{f_p} + 1)}$$

(e_p, f_p ramification index and inertial degree of L at p).

- If $\beta(L) = \infty \Rightarrow$ L has also (N) (Bolzano-Weierstrass)

§2.1. A new criterion for (N)?

Recall: $L/\mathbb{Q}$ Galois with bounded local degrees $\Rightarrow$ L has (B).

More precisely:

Theorem (Bombieri-Zannier, 2001). $L/\mathbb{Q}$ Galois extension, $S(L) \neq \emptyset$ set of primes at which L has bounded local degrees. Then

$$\liminf_{\alpha \in L} h(\alpha) \geq \beta(L) = \frac{1}{2} \sum_{p \in S(L)} \frac{\log p}{e_p(p^{f_p} + 1)}$$

(e_p, f_p ramification index and inertial degree of L at p).

- If $\beta(L) = \infty \Rightarrow L$ has also (N) (Bolzano-Weierstrass)
- L number field $\Rightarrow \beta(L) = \infty$ (Chebotarev density + PNT)

§2.1. A new criterion for (N)?

Recall: $L/\mathbb{Q}$ Galois with bounded local degrees $\Rightarrow$ L has (B).

More precisely:

Theorem (Bombieri-Zannier, 2001). $L/\mathbb{Q}$ Galois extension, $S(L) \neq \emptyset$ set of primes at which L has bounded local degrees. Then

$$\liminf_{\alpha \in L} h(\alpha) \geq \beta(L) = \frac{1}{2} \sum_{p \in S(L)} \frac{\log p}{e_p(p^{f_p} + 1)}$$

(e_p, f_p ramification index and inertial degree of L at p).

- If $\beta(L) = \infty \Rightarrow L$ has also (N) (Bolzano-Weierstrass)
- L number field $\Rightarrow \beta(L) = \infty$ (Chebotarev density + PNT)

> Q. $\exists$ infinite extensions $L/\mathbb{Q}$ such that $\beta(L) = \infty$?
> If so, is the divergence of $\beta(L)$ really
> a new criterion for property (N)?

§2.1. A new criterion for (N)

Theorem (C.-Fehm, 2020).

(1) The divergence of $\beta(L)$ is a new criterion for (N):

§2.1. A new criterion for (N)

Theorem (C.-Fehm, 2020).

(1) The divergence of $\beta(L)$ is a new criterion for (N):
$\exists$ infinite Galois extensions $L/\mathbb{Q}$ such that:

§2.1. A new criterion for (N)

Theorem (C.-Fehm, 2020).

(1) The divergence of $\beta(L)$ is a new criterion for (N):

$\exists$ infinite Galois extensions $L/\mathbb{Q}$ such that:

- $\beta(L) = \infty$;

§2.1. A new criterion for (N)

Theorem (C.-Fehm, 2020).

(I) The divergence of $\beta(L)$ is a new criterion for (N): $\exists$ infinite Galois extensions $L/\mathbb{Q}$ such that:

- $\beta(L) = \infty$;
- L does not satify Bombieri & Zannier's criterion (i.e. L is not of the form $\mathbb{Q}_{ab}^{(d)}$)

§2.1. A new criterion for (N)

Theorem (C.-Fehm, 2020).

(1) The divergence of $\beta(L)$ is a new criterion for (N):
$\exists$ infinite Galois extensions $L/\mathbb{Q}$ such that:

- $\beta(L) = \infty$;
- L does not satify Bombieri & Zannier's criterion (i.e. L is not of the form $\mathbb{Q}_{ab}^{(d)}$)
- L does not satify Widmer's criterion (on fast growth of discriminants in towers).

§2.1. A new criterion for (N)

Theorem (C.-Fehm, 2020).

(1) The divergence of $\beta(L)$ is a new criterion for (N):
$\exists$ infinite Galois extensions $L/\mathbb{Q}$ such that:

- $\beta(L) = \infty$;
- L does not satify Bombieri & Zannier's criterion (i.e. L is not of the form $\mathbb{Q}_{ab}^{(d)}$)
- L does not satify Widmer's criterion (on fast growth of discriminants in towers).

(2) (Some freedom in chosing the Galois group)

§2.1. A new criterion for (N)

Theorem (C.-Fehm, 2020).

(1) The divergence of $\beta(L)$ is a new criterion for (N):

$\exists$ infinite Galois extensions $L/\mathbb{Q}$ such that:

- $\beta(L) = \infty$;
- L does not satify Bombieri & Zannier's criterion (i.e. L is not of the form $\mathbb{Q}_{ab}^{(d)}$)
- L does not satify Widmer's criterion (on fast growth of discriminants in towers).

(2) (Some freedom in chosing the Galois group)

Given any infinite product $G = \prod_{i=1}^{\infty} G_i$ of finite solvable groups G_i,

§2.1. A new criterion for (N)

Theorem (C.-Fehm, 2020).

(1) The divergence of $\beta(L)$ is a new criterion for (N): $\exists$ infinite Galois extensions $L/\mathbb{Q}$ such that:

- $\beta(L) = \infty$;
- L does not satify Bombieri & Zannier's criterion (i.e. L is not of the form $\mathbb{Q}_{ab}^{(d)}$)
- L does not satify Widmer's criterion (on fast growth of discriminants in towers).

(2) (Some freedom in chosing the Galois group) Given any infinite product $G = \prod_{i=1}^{\infty} G_i$ of finite solvable groups G_i, $\exists$ $L/\mathbb{Q}$ Galois such that $\mathrm{Gal}(L/\mathbb{Q}) = G$ and $\beta(L) = \infty$.

→ Ideas in the proof of (I):

If $\beta(L) = \frac{1}{2}\sum_{p\in S(L)} \log p/(e_p(p^{f_p}+1))$ want to prove:

(i) if $\beta(L) = \infty \Rightarrow L$ not of the form $\mathbb{Q}_{ab}^{(d)}$

→ Ideas in the proof of (I):

If $\beta(L) = \frac{1}{2} \sum_{p \in S(L)} \log p/(e_p(p^{f_p}+1))$ want to prove:

(i) if $\beta(L) = \infty \Rightarrow L$ not of the form $\mathbb{Q}_{ab}^{(d)}$

• Proof (easy):

$\rightarrow$ Ideas in the proof of (I):

If $\beta(L) = \frac{1}{2} \sum_{p \in S(L)} \log p/(e_p(p^{f_p} + 1))$ want to prove:

(i) if $\beta(L) = \infty \Rightarrow L$ not of the form $\mathbb{Q}_{ab}^{(d)}$

• Proof (easy):

Remark:

- If $M \subseteq L \Rightarrow \beta(L) \leq \beta(M)$

$\rightarrow$ Ideas in the proof of (I):

If $\beta(L) = \frac{1}{2}\sum_{p \in S(L)} \log p/(e_p(p^{f_p}+1))$ want to prove:

(i) if $\beta(L) = \infty \Rightarrow L$ not of the form $\mathbb{Q}_{ab}^{(d)}$

• Proof (easy):

Remark:

- If $M \subseteq L \Rightarrow \beta(L) \leq \beta(M)$
- lots of p have 'BIG' inertia degree in $L \Rightarrow \beta(L)$ is small

→ Ideas in the proof of (I):

If $\beta(L) = \frac{1}{2}\sum_{p\in S(L)} \log p/(e_p(p^{f_p}+1))$ want to prove:

(i) if $\beta(L)=\infty \Rightarrow L$ not of the form $\mathbb{Q}_{ab}^{(d)}$

• Proof (easy):

Remark:

- If $M \subseteq L \Rightarrow \beta(L) \leq \beta(M)$
- lots of p have 'BIG' inertia degree in $L \Rightarrow \beta(L)$ is small
- If $L = \mathbb{Q}_{ab}^{(d)} \Rightarrow \beta(L) \leq \beta(\mathbb{Q}^{(2)})$ as $\mathbb{Q}^{(2)} \subseteq \mathbb{Q}_{ab}^{(d)}$

→ Ideas in the proof of (I):

If $\beta(L) = \frac{1}{2}\sum_{p\in S(L)} \log p/(e_p(p^{f_p}+1))$ want to prove:

(i) if $\beta(L) = \infty \Rightarrow L$ not of the form $\mathbb{Q}_{ab}^{(d)}$

• Proof (easy):

Remark:

- If $M \subseteq L \Rightarrow \beta(L) \leq \beta(M)$
- lots of p have 'BIG' inertia degree in $L \Rightarrow \beta(L)$ is small
- If $L = \mathbb{Q}_{ab}^{(d)} \Rightarrow \beta(L) \leq \beta(\mathbb{Q}^{(2)})$ as $\mathbb{Q}^{(2)} \subseteq \mathbb{Q}_{ab}^{(d)}$
- $\beta(\mathbb{Q}^{(2)}) < \infty$ as every prime has inertia degree 2 in $\mathbb{Q}^{(2)}$.

$\rightarrow$ Ideas in the proof of (I):

If $\beta(L) = \frac{1}{2} \sum_{p \in S(L)} \log p/(e_p(p^{f_p} + 1))$ want to prove:
(ii) $\exists L/\mathbb{Q}$ infinite s.t. $\beta(L) = \infty$, not satisfying Widmer's criterion.

- Proof (sketch):

- Remark: if lots of p split totally in $L \Rightarrow \beta(L)$ is big

$\rightarrow$ Ideas in the proof of (I):

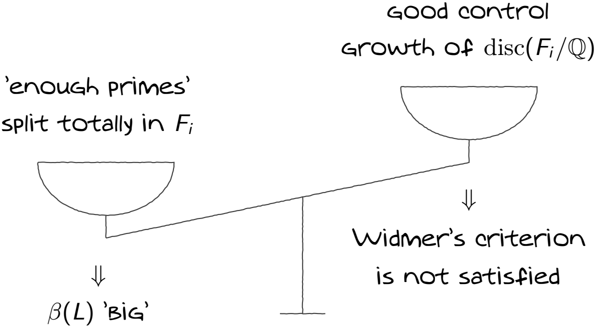
If $\beta(L) = \frac{1}{2} \sum_{p \in S(L)} \log p/(e_p(p^{f_p}+1))$ want to prove:
(ii) $\exists L/\mathbb{Q}$ infinite s.t. $\beta(L) = \infty$, not satisfying Widmer's criterion.

• Proof (sketch):
- Remark: if lots of p split totally in $L \Rightarrow \beta(L)$ is big
- construct L=compositum of cyclic extensions $F_i/\mathbb{Q}$ of prime degrees p_i carefully chosen such that

→ Ideas in the proof of (I):

If $\beta(L) = \frac{1}{2}\sum_{p\in S(L)} \log p/(e_p(p^{f_p}+1))$ want to prove:
(ii) $\exists L/\mathbb{Q}$ infinite s.t. $\beta(L) = \infty$, not satisfying Widmer's criterion.

• Proof (sketch):
- Remark: if lots of p split totally in $L \Rightarrow \beta(L)$ is big
- construct L=compositum of cyclic extensions $F_i/\mathbb{Q}$ of prime degrees p_i carefully chosen such that



→ Ideas in the proof of (I):

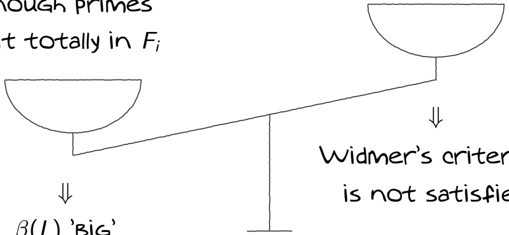
If $\beta(L) = \frac{1}{2}\sum_{p\in S(L)} \log p/(e_p(p^{f_p}+1))$ want to prove:
(ii) $\exists L/\mathbb{Q}$ infinite s.t. $\beta(L)=\infty$, not satisfying Widmer's criterion.

- Proof (sketch):

- Remark: if lots of p split totally in $L \Rightarrow \beta(L)$ is big
- construct L=compositum of cyclic extensions $F_i/\mathbb{Q}$ of prime degrees p_i carefully chosen such that

good control
growth of $\mathrm{disc}(F_i/\mathbb{Q})$

'enough primes'
split totally in F_i



⇓
Widmer's criterion
is not satisfied

⇓
$\beta(L)$ 'big'

(take $F_i \subseteq \mathbb{Q}(\zeta_{\ell_{i_1}},\ldots,\zeta_{\ell_{i_n}})$ with ℓ_{i_j} carefully chosen via Walfisz's thm on counting primes in arithmetic progressions)

→ Ideas in the proof of (2):

→ Ideas in the proof of (2):

To prove: given $G = \prod_i G_i$ direct product of finite solvable groups G_i,

$\to$ Ideas in the proof of (2):

To prove: given $G = \prod_i G_i$ direct product of finite solvable groups G_i, want to construct $L/\mathbb{Q}$ Galois with $\beta(L) = \infty$ and $\mathrm{Gal}(L/\mathbb{Q}) = G$.

$\rightarrow$ Ideas in the proof of (2):

To prove: given $G = \prod_i G_i$ direct product of finite solvable groups G_i, want to construct $L/\mathbb{Q}$ Galois with $\beta(L) = \infty$ and $\mathrm{Gal}(L/\mathbb{Q}) = G$.

• Recall:
Thm. (Shafarevich) Every finite solvable group occurs as the Galois group of an extension of $\mathbb{Q}$.

→ Ideas in the proof of (2):

To prove: given $G = \prod_i G_i$ direct product of finite solvable groups G_i, want to construct $L/\mathbb{Q}$ Galois with $\beta(L) = \infty$ and $\mathrm{Gal}(L/\mathbb{Q}) = G$.

• Recall:
Thm. (Shafarevich) Every finite solvable group occurs as the Galois group of an extension of $\mathbb{Q}$.

• We prove a sharper version of Shafarevich's theorem:
Thm. Given a finite solvable group G and a finite set of primes S, there exist infinitely many linearly disjoint extensions $L/\mathbb{Q}$ having $\mathrm{Gal}(L/\mathbb{Q}) = G$, which are totally real and in which all primes in S split totally.

→ Ideas in the proof of (2):

To prove: given $G = \prod_i G_i$ direct product of finite solvable groups G_i, want to construct $L/\mathbb{Q}$ Galois with $\beta(L) = \infty$ and $\mathrm{Gal}(L/\mathbb{Q}) = G$.

• Recall:
Thm. (Shafarevich) Every finite solvable group occurs as the Galois group of an extension of $\mathbb{Q}$.

• We prove a sharper version of Shafarevich's theorem:
Thm. Given a finite solvable group G and a finite set of primes S, there exist infinitely many linearly disjoint extensions $L/\mathbb{Q}$ having $\mathrm{Gal}(L/\mathbb{Q}) = G$, which are totally real and in which all primes in S split totally.

• Proof based on solution of split embedding problems with special local behaviour (Neukirch)

→ Ideas in the proof of (2):

To prove: given $G = \prod_i G_i$ direct product of finite solvable groups G_i, want to construct $L/\mathbb{Q}$ Galois with $\beta(L) = \infty$ and $\mathrm{Gal}(L/\mathbb{Q}) = G$.

• Recall:
Thm. (Shafarevich) Every finite solvable group occurs as the Galois group of an extension of $\mathbb{Q}$.

• We prove a sharper version of Shafarevich's theorem:
Thm. Given a finite solvable group G and a finite set of primes S, there exist infinitely many linearly disjoint extensions $L/\mathbb{Q}$ having $\mathrm{Gal}(L/\mathbb{Q}) = G$, which are totally real and in which all primes in S split totally.

• Proof based on solution of split embedding problems with special local behaviour (Neukirch)

• L constructed as compositum of suitable realisations of groups G_i's

§3.2 Property (N), local degrees and totally p-adic numbers of small height

§3.2. Property (N) and local degrees

Recall: $K_{ab}^{(d)}$ has property (N) (Bombieri, Zannier).

§3.2. Property (N) and local degrees

Recall: $K_{ab}^{(d)}$ has property (N) (Bombieri, Zannier).

- $K_{ab}^{(d)}/K$ is abelian and has uniformly bounded local degrees

i.e. there exists $M > 0$ such that, for all p, $K_{ab}^{(d)}$ can be embedded in an extension of $\mathbb{Q}_p$ of degree $\leq M$.

§3.2. Property (N) and local degrees

Recall: $K_{ab}^{(d)}$ has property (N) (Bombieri, Zannier).

- $K_{ab}^{(d)}/K$ is abelian and has uniformly bounded local degrees

i.e. there exists $M > 0$ such that, for all p, $K_{ab}^{(d)}$ can be embedded in an extension of $\mathbb{Q}_p$ of degree $\leq M$.

- L/K abelian with uniformly bounded local degrees $\Leftrightarrow$

$L = K_{ab}^{(d)}$ for some d (C. 2013)

§3.2. Property (N) and local degrees

Recall: $K_{ab}^{(d)}$ has property (N) (Bombieri, Zannier).

- $K_{ab}^{(d)}/K$ is abelian and has uniformly bounded local degrees i.e. there exists $M > 0$ such that, for all p, $K_{ab}^{(d)}$ can be embedded in an extension of $\mathbb{Q}_p$ of degree $\leq M$.
- L/K abelian with uniformly bounded local degrees $\Leftrightarrow$ $L = K_{ab}^{(d)}$ for some d (C. 2013)
- Bombieri-Zannier's thm can be stated as: every abelian extension of a n.f. with uniformly bounded local degrees has (N).

§3.2. Property (N) and local degrees

Recall: $K_{ab}^{(d)}$ has property (N) (Bombieri, Zannier).

• $K_{ab}^{(d)}/K$ is abelian and has uniformly bounded local degrees i.e. there exists $M > 0$ such that, for all p, $K_{ab}^{(d)}$ can be embedded in an extension of $\mathbb{Q}_p$ of degree $\leq M$.

• L/K abelian with uniformly bounded local degrees $\Leftrightarrow$ $L = K_{ab}^{(d)}$ for some d (C. 2013)

• Bombieri-Zannier's thm can be stated as: every abelian extension of a n.f. with uniformly bounded local degrees has (N).

Q1: Uniform boundedness of the local degrees $\Rightarrow$ (N)?

§3.2. Property (N) and local degrees

Recall: $K_{ab}^{(d)}$ has property (N) (Bombieri, Zannier).

- $K_{ab}^{(d)}/K$ is abelian and has uniformly bounded local degrees i.e. there exists $M > 0$ such that, for all p, $K_{ab}^{(d)}$ can be embedded in an extension of $\mathbb{Q}_p$ of degree $\leq M$.
- L/K abelian with uniformly bounded local degrees $\Leftrightarrow$ $L = K_{ab}^{(d)}$ for some d (C. 2013)
- Bombieri-Zannier's thm can be stated as: every abelian extension of a n.f. with uniformly bounded local degrees has (N).

Q1: Uniform boundedness of the local degrees $\Rightarrow$ (N)?

§3.2. Property (N) and local degrees

Recall: $K_{ab}^{(d)}$ has property (N) (Bombieri, Zannier).

• $K_{ab}^{(d)}/K$ is abelian and has uniformly bounded local degrees i.e. there exists $M > 0$ such that, for all p, $K_{ab}^{(d)}$ can be embedded in an extension of $\mathbb{Q}_p$ of degree $\leq M$.

• L/K abelian with uniformly bounded local degrees $\Leftrightarrow$ $L = K_{ab}^{(d)}$ for some d (C. 2013)

• Bombieri-Zannier's thm can be stated as: every abelian extension of a n.f. with uniformly bounded local degrees has (N).

Q2: ~~Uniform~~ Boundedness of the local degrees $\Rightarrow$ (N)?

§3.2. Property (N) and local degrees

Recall: $K_{ab}^{(d)}$ has property (N) (Bombieri, Zannier).

- $K_{ab}^{(d)}/K$ is abelian and has uniformly bounded local degrees i.e. there exists $M > 0$ such that, for all p, $K_{ab}^{(d)}$ can be embedded in an extension of $\mathbb{Q}_p$ of degree $\leq M$.
- L/K abelian with uniformly bounded local degrees $\Leftrightarrow$ $L = K_{ab}^{(d)}$ for some d (C. 2013)
- Bombieri-Zannier's thm can be stated as: every abelian extension of a n.f. with uniformly bounded local degrees has (N).

Q2: ~~Uniform~~ Boundedness of the local degrees $\Rightarrow$ (N)?
$\rightarrow$ $\exists$ extensions without (N) with local degrees bounded at infinitely many primes (but unbounded at infinitely many primes too) (Fehm, 2018).

§3.2. Property (N) and local degrees

Recall: $K_{ab}^{(d)}$ has property (N) (Bombieri, Zannier).

- $K_{ab}^{(d)}/K$ is abelian and has uniformly bounded local degrees i.e. there exists $M > 0$ such that, for all p, $K_{ab}^{(d)}$ can be embedded in an extension of $\mathbb{Q}_p$ of degree $\leq M$.
- L/K abelian with uniformly bounded local degrees $\Leftrightarrow$ $L = K_{ab}^{(d)}$ for some d (C. 2013)
- Bombieri-Zannier's thm can be stated as: every abelian extension of a n.f. with uniformly bounded local degrees has (N).

Q3: ~~Uniform~~ Boundedness of the local degrees at all primes$\Rightarrow$ (N)?

§3.2. Property (N) and local degrees

Recall: $K_{ab}^{(d)}$ has property (N) (Bombieri, Zannier).

- $K_{ab}^{(d)}/K$ is abelian and has uniformly bounded local degrees i.e. there exists $M > 0$ such that, for all p, $K_{ab}^{(d)}$ can be embedded in an extension of $\mathbb{Q}_p$ of degree $\leq M$.
- L/K abelian with uniformly bounded local degrees $\Leftrightarrow$ $L = K_{ab}^{(d)}$ for some d (C. 2013)
- Bombieri-Zannier's thm can be stated as: every abelian extension of a n.f. with uniformly bounded local degrees has (N).

Q3: ~~Uniform~~ Boundedness of the local degrees at all primes $\Rightarrow$ (N)?

> Theorem (C.-Fehm, 2020). $\exists$ infinite Galois extensions $L/\mathbb{Q}$ without property (N), but having bounded local degrees at all primes.

§3.2. On a result of Fili

To prove the theorem we use special case of a result of Fili.

§3.2. On a result of Fili

To prove the theorem we use special case of a result of Fili.
Recall: for $L/\mathbb{Q}$ with bounded local degrees:

§3.2. On a result of Fili

To prove the theorem we use special case of a result of Fili.
Recall: for $L/\mathbb{Q}$ with bounded local degrees:
- Lower bound $\beta(L)$ for liminf $h(\alpha)$ by Bombieri-Zannier;

§3.2. On a result of Fili

To prove the theorem we use special case of a result of Fili.
Recall: for $L/\mathbb{Q}$ with bounded local degrees:
- Lower bound $\beta(L)$ for liminf $h(\alpha)$ by Bombieri-Zannier;
- Special case: $L = \cap_{i=1}^{n} \mathbb{Q}^{tp_i}$, also upper bound by Bombieri and Zannier:

§3.2. On a result of Fili

To prove the theorem we use special case of a result of Fili.
Recall: for $L/\mathbb{Q}$ with bounded local degrees:
- Lower bound $\beta(L)$ for liminf $h(\alpha)$ by Bombieri-Zannier;
- Special case: $L = \cap_{i=1}^{n} \mathbb{Q}^{tp_i}$, also upper bound by Bombieri and Zannier:

$$\frac{1}{2}\sum_{i=1}^{n}\frac{\log(p_i)}{(p_i+1)} \leq \liminf_{\alpha\in\cap_{i=1}^{n}\mathbb{Q}^{tp_i}} h(\alpha) \leq \sum_{i=1}^{n}\frac{\log(p_i)}{(p_i-1)}.$$

§3.2. On a result of Fili

To prove the theorem we use special case of a result of Fili.
Recall: for $L/\mathbb{Q}$ with bounded local degrees:
- Lower bound $\beta(L)$ for liminf $h(\alpha)$ by Bombieri-Zannier;
- Special case: $L = \cap_{i=1}^{n} \mathbb{Q}^{tp_i}$, also upper bound by Bombieri and Zannier:

$$\frac{1}{2} \sum_{i=1}^{n} \frac{\log(p_i)}{(p_i + 1)} \leq \liminf_{\alpha \in \cap_{i=1}^{n} \mathbb{Q}^{tp_i}} h(\alpha) \leq \sum_{i=1}^{n} \frac{\log(p_i)}{(p_i - 1)}.$$

- Generalisation by Fili (special case):

Theorem (Fili, 2014). Let $p_1, \ldots, p_n$ be distinct primes.

§3.2. On a result of Fili

To prove the theorem we use special case of a result of Fili.
Recall: for $L/\mathbb{Q}$ with bounded local degrees:
- Lower bound $\beta(L)$ for liminf $h(\alpha)$ by Bombieri-Zannier;
- Special case: $L = \cap_{i=1}^{n} \mathbb{Q}^{tp_i}$, also upper bound by Bombieri and Zannier:

$$\frac{1}{2} \sum_{i=1}^{n} \frac{\log(p_i)}{(p_i + 1)} \leq \liminf_{\alpha \in \cap_{i=1}^{n} \mathbb{Q}^{tp_i}} h(\alpha) \leq \sum_{i=1}^{n} \frac{\log(p_i)}{(p_i - 1)}.$$

- Generalisation by Fili (special case):

Theorem (Fili, 2014). Let $p_1, \ldots, p_n$ be distinct primes. For each i, fix $E_i/\mathbb{Q}_{p_i}$ finite Galois extension with ramification index e_i and inertia degree f_i.

§3.2. On a result of Fili

To prove the theorem we use special case of a result of Fili.
Recall: for $L/\mathbb{Q}$ with bounded local degrees:
- Lower bound $\beta(L)$ for liminf $h(\alpha)$ by Bombieri-Zannier;
- Special case: $L = \cap_{i=1}^{n} \mathbb{Q}^{tp_i}$, also upper bound by Bombieri and Zannier:

$$\frac{1}{2}\sum_{i=1}^{n} \frac{\log(p_i)}{(p_i+1)} \leq \liminf_{\alpha \in \cap_{i=1}^{n} \mathbb{Q}^{tp_i}} h(\alpha) \leq \sum_{i=1}^{n} \frac{\log(p_i)}{(p_i-1)}.$$

- Generalisation by Fili (special case):

Theorem (Fili, 2014). Let $p_1, \ldots, p_n$ be distinct primes. For each i, fix $E_i/\mathbb{Q}_{p_i}$ finite Galois extension with ramification index e_i and inertia degree f_i. Let $L_i/\mathbb{Q}$ maximal Galois extension contained in E_i.

§3.2. On a result of Fili

To prove the theorem we use special case of a result of Fili.
Recall: for $L/\mathbb{Q}$ with bounded local degrees:
- Lower bound $\beta(L)$ for liminf $h(\alpha)$ by Bombieri-Zannier;
- Special case: $L = \cap_{i=1}^{n} \mathbb{Q}^{tp_i}$, also upper bound by Bombieri and Zannier:

$$\frac{1}{2}\sum_{i=1}^{n} \frac{\log(p_i)}{(p_i+1)} \leq \liminf_{\alpha \in \cap_{i=1}^{n} \mathbb{Q}^{tp_i}} h(\alpha) \leq \sum_{i=1}^{n} \frac{\log(p_i)}{(p_i-1)}.$$

- Generalisation by Fili (special case):

Theorem (Fili, 2014). Let $p_1, \ldots, p_n$ be distinct primes. For each i, fix $E_i/\mathbb{Q}_{p_i}$ finite Galois extension with ramification index e_i and inertia degree f_i. Let $L_i/\mathbb{Q}$ maximal Galois extension contained in E_i. Then

$$\liminf_{\alpha \in \cap_{i=1}^{n} L_i} h(\alpha) \leq \sum_{i=1}^{n} \frac{\log(p_i)}{e_i(p_i^{f_i}-1)}.$$

§3.2. On a result of Fili

Our result follows:

§3.2. On a result of Fili

Our result follows:

- For each prime p_i, let $E_i/\mathbb{Q}_{p_i}$ unique unramified quadratic extension, L_i maximal Galois extension of $\mathbb{Q}$ inside E_i

§3.2. On a result of Fili

Our result follows:

- For each prime p_i, let $E_i/\mathbb{Q}_{p_i}$ unique unramified quadratic extension, L_i maximal Galois extension of $\mathbb{Q}$ inside E_i
- By Fili:

$$\liminf_{\alpha \in \cap_{i=1}^{n} L_i} h(\alpha) \leq \sum_{i=1}^{n} \frac{\log(p_i)}{p_i^2 - 1} \leq \sum_{k=1}^{\infty} \frac{\log(k)}{k^2} < T.$$

§3.2. On a result of Fili

Our result follows:

- For each prime p_i, let $E_i/\mathbb{Q}_{p_i}$ unique unramified quadratic extension, L_i maximal Galois extension of $\mathbb{Q}$ inside E_i
- By Fili:

$$\liminf_{\alpha \in \cap_{i=1}^{n} L_i} h(\alpha) \leq \sum_{i=1}^{n} \frac{\log(p_i)}{p_i^2 - 1} \leq \sum_{k=1}^{\infty} \frac{\log(k)}{k^2} < T.$$

- Pick iteratively $x_n \in \cap_{i=1}^{n} L_i \setminus \{x_1, \ldots, x_{n-1}\}$ with $h(x_n) < T$

§3.2. On a result of Fili

Our result follows:

- For each prime p_i, let $E_i/\mathbb{Q}_{p_i}$ unique unramified quadratic extension, L_i maximal Galois extension of $\mathbb{Q}$ inside E_i
- By Fili:

$$\liminf_{\alpha \in \cap_{i=1}^{n} L_i} h(\alpha) \leq \sum_{i=1}^{n} \frac{\log(p_i)}{p_i^2 - 1} \leq \sum_{k=1}^{\infty} \frac{\log(k)}{k^2} < T.$$

- Pick iteratively $x_n \in \cap_{i=1}^{n} L_i \setminus \{x_1, \ldots, x_{n-1}\}$ with $h(x_n) < T \Rightarrow$ the Galois closure of $\mathbb{Q}(x_1, x_2, \ldots)$ satisfies the claim.

§3.2. On a result of Fili

Our result follows:

- For each prime p_i, let $E_i/\mathbb{Q}_{p_i}$ unique unramified quadratic extension, L_i maximal Galois extension of $\mathbb{Q}$ inside E_i
- By Fili:

$$\liminf_{\alpha \in \cap_{i=1}^{n} L_i} h(\alpha) \leq \sum_{i=1}^{n} \frac{\log(p_i)}{p_i^2 - 1} \leq \sum_{k=1}^{\infty} \frac{\log(k)}{k^2} < T.$$

- Pick iteratively $x_n \in \cap_{i=1}^{n} L_i \setminus \{x_1, \ldots, x_{n-1}\}$ with $h(x_n) < T \Rightarrow$ the Galois closure of $\mathbb{Q}(x_1, x_2, \ldots)$ satisfies the claim.

→ Fili's result:
- more general (replace $\mathbb{Q}_p$ by $p-$adic field)

§3.2. On a result of Fili

Our result follows:

- For each prime p_i, let $E_i/\mathbb{Q}_{p_i}$ unique unramified quadratic extension, L_i maximal Galois extension of $\mathbb{Q}$ inside E_i
- By Fili:

$$\liminf_{\alpha\in\cap_{i=1}^n L_i} h(\alpha) \leq \sum_{i=1}^{n} \frac{\log(p_i)}{p_i^2-1} \leq \sum_{k=1}^{\infty} \frac{\log(k)}{k^2} < T.$$

- Pick iteratively $x_n \in \cap_{i=1}^n L_i \setminus \{x_1, \dots, x_{n-1}\}$ with $h(x_n) < T \Rightarrow$ the Galois closure of $\mathbb{Q}(x_1, x_2, \dots)$ satisfies the claim.

$\rightarrow$ Fili's result:
- more general (replace $\mathbb{Q}_p$ by $p-$adic field)
- not effective: no explicit bounds on degree and height of a sequence of integral elements in the liminf.

§3.2. On a result of Fili

Our result follows:

- For each prime p_i, let $E_i/\mathbb{Q}_{p_i}$ unique unramified quadratic extension, L_i maximal Galois extension of $\mathbb{Q}$ inside E_i
- By Fili:

$$\liminf_{\alpha \in \cap_{i=1}^{n} L_i} h(\alpha) \leq \sum_{i=1}^{n} \frac{\log(p_i)}{p_i^2 - 1} \leq \sum_{k=1}^{\infty} \frac{\log(k)}{k^2} < T.$$

- Pick iteratively $x_n \in \cap_{i=1}^{n} L_i \setminus \{x_1, \ldots, x_{n-1}\}$ with $h(x_n) < T \Rightarrow$ the Galois closure of $\mathbb{Q}(x_1, x_2, \ldots)$ satisfies the claim.

$\rightarrow$ Fili's result:
- more general (replace $\mathbb{Q}_p$ by $p-$adic field)
- not effective: no explicit bounds on degree and height of a sequence of integral elements in the liminf.

Theorem (C.-Fehm, 2021). Effective version of Fili's theorem (explicit bounds on height and degree of elements in infinite sequence in the liminf)

Some open problems on property (N)

Some open problems on property (N)

Pb.1) Uniform boundedness of local degrees $\Rightarrow$ property (N) ?

Some open problems on property (N)

PB.I) Uniform boundedness of local degrees $\Rightarrow$ property (N) ?

Q. $\mathbb{Q}^{(3)}$ = compositum of quadratic and cubic extensions of degree 3, has property (N)?

Some open problems on property (N)

PB.1) Uniform boundedness of local degrees $\Rightarrow$ property (N) ?
Q. $\mathbb{Q}^{(3)}$ = compositum of quadratic and cubic extensions of degree 3, has property (N)?

PB.2) Siegel fields and property (N).

Some open problems on property (N)

PB.1) Uniform boundedness of local degrees $\Rightarrow$ property (N) ?
Q. $\mathbb{Q}^{(3)}$ = compositum of quadratic and cubic extensions of degree 3, has property (N)?

PB.2) Siegel fields and property (N).
Gaudron, Rémond (2017): A Siegel field is a a subfield $L \subseteq \overline{\mathbb{Q}}$ satisfying Siegel's lemma i.e.

Some open problems on property (N)

PB.1) Uniform boundedness of local degrees $\Rightarrow$ property (N) ?
Q. $\mathbb{Q}^{(3)}$ = compositum of quadratic and cubic extensions of degree 3, has property (N)?

PB.2) Siegel fields and property (N).
Gaudron, Rémond (2017): A Siegel field is a a subfield $L \subseteq \overline{\mathbb{Q}}$ satisfying Siegel's lemma i.e.
$\exists C(L) > 0$ such that for all $(a, b, c) \in L^3 \setminus \{(0, 0, 0)\}$ there exists $(x, y, z) \in L^3 \setminus \{(0, 0, 0)\}$ such that $ax + by + cz = 0$ and $H(x, y, z) \leq C(L)H(a, b, c)^{1/2}$ ($H = \exp(h)$).

Some open problems on property (N)

PB.1) Uniform boundedness of local degrees $\Rightarrow$ property (N) ?
Q. $\mathbb{Q}^{(3)}$ = compositum of quadratic and cubic extensions of degree 3, has property (N)?

PB.2) Siegel fields and property (N).
Gaudron, Rémond (2017): A Siegel field is a a subfield $L \subseteq \overline{\mathbb{Q}}$ satisfying Siegel's lemma i.e.
$\exists C(L) > 0$ such that for all $(a, b, c) \in L^3 \setminus \{(0, 0, 0)\}$ there exists $(x, y, z) \in L^3 \setminus \{(0, 0, 0)\}$ such that $ax + by + cz = 0$ and $H(x, y, z) \leq C(L)H(a, b, c)^{1/2}$ ($H = \exp(h)$).

- Number fields are Siegel fields

Some open problems on property (N)

PB.1) Uniform boundedness of local degrees $\Rightarrow$ property (N) ?
Q. $\mathbb{Q}^{(3)}$ = compositum of quadratic and cubic extensions of degree 3, has property (N)?

PB.2) Siegel fields and property (N).
Gaudron, Rémond (2017): A Siegel field is a a subfield $L \subseteq \overline{\mathbb{Q}}$ satisfying Siegel's lemma i.e.
$\exists C(L) > 0$ such that for all $(a, b, c) \in L^3 \setminus \{(0, 0, 0)\}$ there exists $(x, y, z) \in L^3 \setminus \{(0, 0, 0)\}$ such that $ax + by + cz = 0$ and $H(x, y, z) \leq C(L)H(a, b, c)^{1/2}$ ($H = \exp(h)$).

- Number fields are Siegel fields
- (Roy-Thunder, Zhang) $L = \overline{\mathbb{Q}}$ is a Siegel field

Some open problems on property (N)

PB.1) Uniform boundedness of local degrees $\Rightarrow$ property (N) ?
Q. $\mathbb{Q}^{(3)}$ = compositum of quadratic and cubic extensions of degree 3, has property (N)?

PB.2) Siegel fields and property (N).
Gaudron, Rémond (2017): A Siegel field is a a subfield $L \subseteq \overline{\mathbb{Q}}$ satisfying Siegel's lemma i.e.
$\exists C(L) > 0$ such that for all $(a, b, c) \in L^3 \setminus \{(0, 0, 0)\}$ there exists $(x, y, z) \in L^3 \setminus \{(0, 0, 0)\}$ such that $ax + by + cz = 0$ and $H(x, y, z) \leq C(L)H(a, b, c)^{1/2}$ ($H = \exp(h)$).

- Number fields are Siegel fields
- (Roy-Thunder, Zhang) $L = \overline{\mathbb{Q}}$ is a Siegel field
- (Gaudron, Rémond) If $L/\mathbb{Q}$ infinite with property (N) $\Rightarrow$ L not a Siegel field

Some open problems on property (N)

Pb.1) Uniform boundedness of local degrees $\Rightarrow$ property (N) ?
Q. $\mathbb{Q}^{(3)}$ = compositum of quadratic and cubic extensions of degree 3, has property (N)?

Pb.2) Siegel fields and property (N).
Gaudron, Rémond (2017): A Siegel field is a a subfield $L \subseteq \overline{\mathbb{Q}}$ satisfying Siegel's lemma i.e.
$\exists C(L) > 0$ such that for all $(a, b, c) \in L^3 \setminus \{(0, 0, 0)\}$ there exists $(x, y, z) \in L^3 \setminus \{(0, 0, 0)\}$ such that $ax + by + cz = 0$ and $H(x, y, z) \leq C(L)H(a, b, c)^{1/2}$ ($H = \exp(h)$).

- Number fields are Siegel fields
- (Roy-Thunder, Zhang) $L = \overline{\mathbb{Q}}$ is a Siegel field
- (Gaudron, Rémond) If $L/\mathbb{Q}$ infinite with property (N) $\Rightarrow$ L not a Siegel field

Q. Is the converse true? Are there infinite extensions $L/\mathbb{Q}$ which are neither Siegel fields nor fields with property (N)?

Some open problems on property (B)

Pb.3) Property (B) and generators of Galois extensions

Some open problems on property (B)

Pb.3) Property (B) and generators of Galois extensions

- (Amoroso, Masser, 2016) A strong Lehmer bound:

Some open problems on property (B)

Pb.3) Property (B) and generators of Galois extensions

- (Amoroso, Masser, 2016) A strong Lehmer bound:

for any $\epsilon > 0$, $\exists c(\epsilon) > 0$ such that $h(\alpha)[\mathbb{Q}(\alpha) : \mathbb{Q}]^{\epsilon} \geq c(\epsilon)$ for α not root of unity such that $\mathbb{Q}(\alpha)/\mathbb{Q}$ is Galois.

Pb.3) Property (B) and generators of Galois extensions

- (Amoroso, Masser, 2016) A strong Lehmer bound:

for any $\epsilon > 0$, $\exists c(\epsilon) > 0$ such that $h(\alpha)[\mathbb{Q}(\alpha) : \mathbb{Q}]^{\epsilon} \geq c(\epsilon)$ for α not root of unity such that $\mathbb{Q}(\alpha)/\mathbb{Q}$ is Galois.

The bound is so good that one might ask:

Pb.3) Property (B) and generators of Galois extensions

• (Amoroso, Masser, 2016) A strong Lehmer bound: for any $\epsilon > 0$, $\exists c(\epsilon) > 0$ such that $h(\alpha)[\mathbb{Q}(\alpha) : \mathbb{Q}]^{\epsilon} \geq c(\epsilon)$ for α not root of unity such that $\mathbb{Q}(\alpha)/\mathbb{Q}$ is Galois.

The bound is so good that one might ask:

Q. Does the set $\{\alpha \in \overline{\mathbb{Q}} \mid \mathbb{Q}(\alpha)/\mathbb{Q}$ is Galois$\}$ enjoy property (B)?

Pb.3) Property (B) and generators of Galois extensions

• (Amoroso, Masser, 2016) A strong Lehmer bound: for any $\epsilon > 0$, $\exists c(\epsilon) > 0$ such that $h(\alpha)[\mathbb{Q}(\alpha) : \mathbb{Q}]^{\epsilon} \geq c(\epsilon)$ for α not root of unity such that $\mathbb{Q}(\alpha)/\mathbb{Q}$ is Galois.

The bound is so good that one might ask:
Q. Does the set $\{\alpha \in \overline{\mathbb{Q}} \mid \mathbb{Q}(\alpha)/\mathbb{Q} \text{ is Galois}\}$ enjoy property (B)?

• (Amoroso, 2017) If α belongs to a certain class of generators of S_n-extensions, $h(\alpha) \geq c(n)$ with $c(n) \to \infty$ with n.

Pb.3) Property (B) and generators of Galois extensions
• (Amoroso, Masser, 2016) A strong Lehmer bound:
for any $\epsilon > 0$, $\exists c(\epsilon) > 0$ such that $h(\alpha)[\mathbb{Q}(\alpha) : \mathbb{Q}]^{\epsilon} \geq c(\epsilon)$ for α not root of unity such that $\mathbb{Q}(\alpha)/\mathbb{Q}$ is Galois.

The bound is so good that one might ask:
Q. Does the set $\{\alpha \in \overline{\mathbb{Q}} \mid \mathbb{Q}(\alpha)/\mathbb{Q} \text{ is Galois}\}$ enjoy property (B)?

• (Amoroso, 2017) If α belongs to a certain class of generators of S_n-extensions, $h(\alpha) \geq c(n)$ with $c(n) \to \infty$ with n.
Q. True for all generators of S_n-extensions?

Pb.3) Property (B) and generators of Galois extensions

• (Amoroso, Masser, 2016) A strong Lehmer bound: for any $\epsilon > 0$, $\exists c(\epsilon) > 0$ such that $h(\alpha)[\mathbb{Q}(\alpha) : \mathbb{Q}]^{\epsilon} \geq c(\epsilon)$ for α not root of unity such that $\mathbb{Q}(\alpha)/\mathbb{Q}$ is Galois.

The bound is so good that one might ask:
Q. Does the set $\{\alpha \in \overline{\mathbb{Q}} \mid \mathbb{Q}(\alpha)/\mathbb{Q} \text{ is Galois}\}$ enjoy property (B)?

• (Amoroso, 2017) If α belongs to a certain class of generators of S_n-extensions, $h(\alpha) \geq c(n)$ with $c(n) \to \infty$ with n.
Q. True for all generators of S_n-extensions?
What about other (non-abelian) groups?

Thank you!