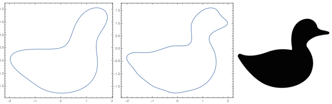
On Moment Problems with Holonomic Functions

F. Bréhard, M. Joldes, J-B. Lasserre



Moment Problems and Applications

Moments of a measure

$$m_\alpha = \int_{\mathbb{R}^n} x^\alpha \mathrm{d}\mu \qquad \text{for} \quad \alpha \in \mathbb{N}^n$$ [a]

[a] $\alpha = (\alpha_1, \dots, \alpha_n)$, $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$, $|\alpha| = \alpha_1 + \dots + \alpha_n$, $\mathbb{K}[x]_d$ = polynomials of total degree at most d

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- n-dim (compact) semi-algebraic set G, with $g \in \mathbb{K}[x]$ vanishing on ∂G
- holonomic $f : \mathbb{R}^n \to \mathbb{R}$ i.e., it satisfies a "complete" system of linear PDEs with polynomial coefficients

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$\to$ Direct problem: knowing G and f, find a *complete* system of recurrences for (m_α)

- $\rightsquigarrow$ **Finite determinancy** of such measures
- $\rightsquigarrow$ Solved with **Creative Telescoping**, e.g., [Oaku2013] + Takayama's algorithm

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→ Applications in Statistics, Signal-processing, Combinatorics...

Inverse problem: Data recovery from moments

Reconstruction of a shape $G \subset \mathbb{R}^n$ (convex or not)

from the knowledge of finitely many moments

$$m_\alpha = \int_G x^\alpha \mathrm{d}x, \quad |\alpha| \leqslant N,$$

for some given integer N, is a challenging problem.

*Lasserre, J.-B., Putinar M., *Algebraic–exponential Data Recovery from Moments.*, Discrete & Comp Geometry 54.4 (2015): 993-1012.

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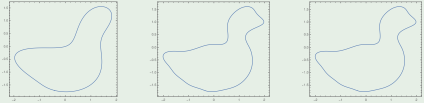
A Toulouse Duck?

m[0,0] =0.25371; m[0,1] = 0.10738; m[1,0] = 0.13670; m[0,2] = 0.05205; m[1,1] = 0.06143; m[2,0] = 0.08248; ...

$$m_{i,j} = \int_{\text{Duck}} x^i y^j \mathrm{d}x\mathrm{d}y$$



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  - Numerical methods, e.g.: convex polytopes [GolubMilanfarVarah1999] [GravinLasserrePasechnikRobins2012]; planar quadrature domains [EbenfeltEtAl2005]; sublevel set of homogeneous polynomials [Lasserre2013]; shape and Gaussian Mixture reconstruction[diDioKummer2019]

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  - Symbolic/algebraic methods:

    Multivariate extensions of Prony's method, e.g. [Mourrain2018] (Reconstruction of sparse exponential functions ($\sum_{\alpha \in I} \lambda_\alpha e^{\alpha x}$) from evaluations, moments of Dirac measures); reconstruction of univariate piecewise D-finite densities [Batenkov2009]

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Exact Support and/or Density reconstruction

- Lasserre and Putinar's exact reconstruction algorithm (2015)

Theorem 1 (Inverse Problem: Lebesgue measure, Algebraic support)

Let $G \subset \mathbb{R}^n$, bounded open set, whose algebraic boundary is included in the zero set of a polynomial $g \in \mathbb{K}[x]_d$. Given d and a finite number of power moments m_α, up to order $|\alpha| = 3d$, the coefficients of g can be exactly recovered.

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Our contribution: a computer algebra approach

- generalization in the framework of holonomic distributions
- exact recovery of both support and Exp-Polynomial density $f = \exp(p)$, with explicit bound on the required number of moments
- similar algorithm for holonomic density, but no a priori bound on the required number of moments

Table of Contents

Operator Algebras, Differential Equations and Recurrences

Differential equations/recurrences are translated to skew polynomials:

1. Differential Ore Algebras

– $\mathbb{K}[x]\langle\partial_x\rangle$ polynomial Ore algebra

$$\partial_{x_i} x_i = x_i \partial_{x_i} + 1$$

– $\mathbb{K}(x)\langle\partial_x\rangle$ rational Ore algebra

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$\Rightarrow$ f is D-finite iff $\mathbb{K}(x)\langle\partial_x\rangle/\mathfrak{Ann}(f)$ has finite dimension over the ∂_{x_i}

Example:

$f(x) = c\exp(p(x))$ with $p \in \mathbb{K}_s[x]$

$f'_{x_i} - p'_{x_i} f = 0$

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$\langle x\delta, f\rangle = 0$
$\Rightarrow$ $x \in \mathfrak{Ann}(\delta)$, but $1 \notin \mathfrak{Ann}(\delta)$

$\Rightarrow$ δ is holonomic

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Ore Algebras, Differential Equations and Recurrences

2. Difference Ore Algebras

– Difference operators: non-commutative, spanned by $\alpha_1,\ S_{\alpha_1},\ \ldots,\ \alpha_n,\ S_{\alpha_n}$

$$\begin{aligned} (\alpha_i u)_\alpha &= \alpha_i u_\alpha \\ (S_{\alpha_i} u)_\alpha &= u_{\alpha_1,\ldots,\alpha_i+1,\ldots,\alpha_n} \\ S_{\alpha_i}\alpha_i &= (\alpha_i+1)S_{\alpha_i} \end{aligned}$$

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Goals

Recurrences for the moments $m_\alpha = \int_G x^\alpha f(x)\mathrm{d}x$:

- Direct problem: $\mathfrak{I} \subseteq \mathfrak{Ann}(f) \xrightarrow{?} \mathfrak{I} \subseteq \mathfrak{Ann}(m_\alpha)$
- Inverse problem: Reconstruct G and $\mathfrak{I} \subseteq \mathfrak{Ann}(f)$ from sufficiently many m_α

Holonomic Measures

- Measure $\mu = f\mathbf{1}_G$ as a linear functional:

$$\langle f\mathbf{1}_G, \varphi\rangle = \int_{\mathbb{R}^n} \varphi(x) f(x) \mathbf{1}_G(x) \mathrm{d}x = \int_G \varphi(x) f(x) \mathrm{d}x$$

- Action of Ore polynomials: $L\,\mu = ?$

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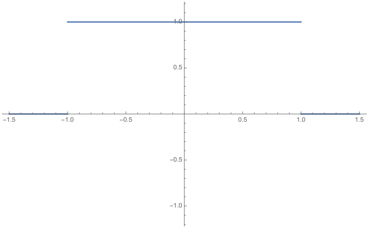
Example: Lebesgue measure over a segment

Let $G = [-1, 1]$, $f = 1$, and $\mu = \mathbf{1}_G$

$$\langle \mathbf{1}_G, \varphi\rangle = \int_{-1}^{1} \varphi(x)\mathrm{d}x$$



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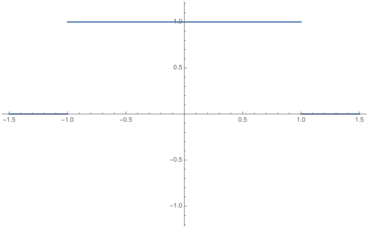

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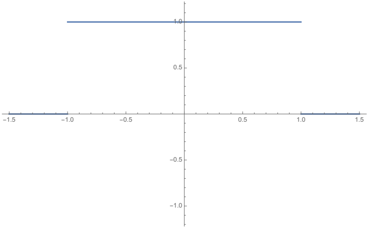

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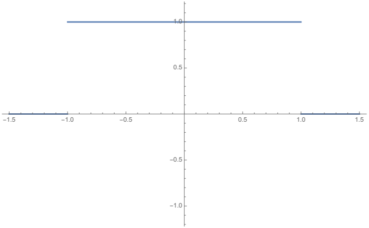

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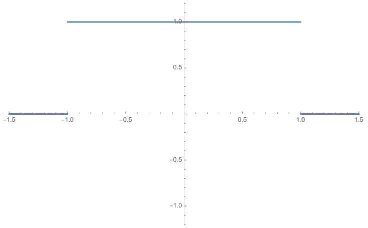

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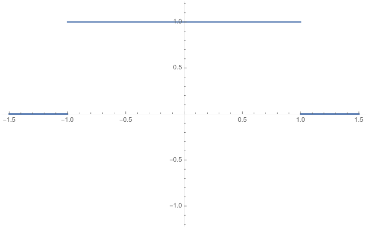

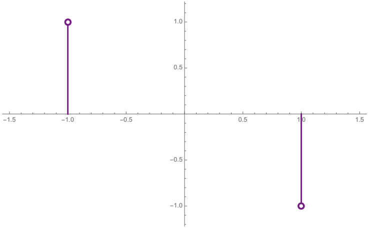

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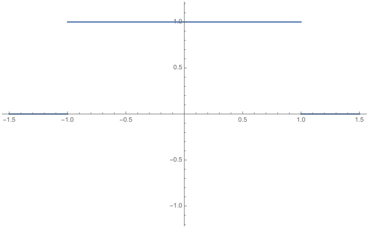

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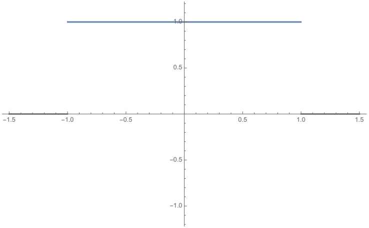

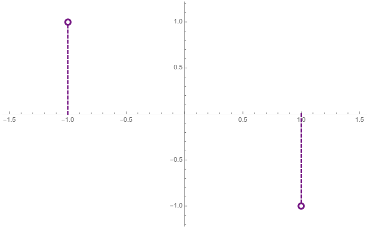

Holonomic Measures

- Measure $\mu = f\mathbf{1}_G$ as a linear functional:

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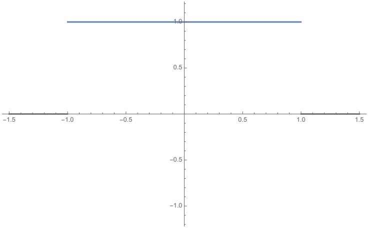

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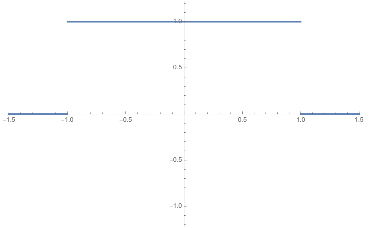

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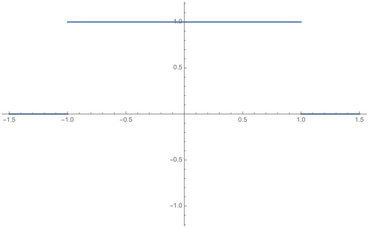

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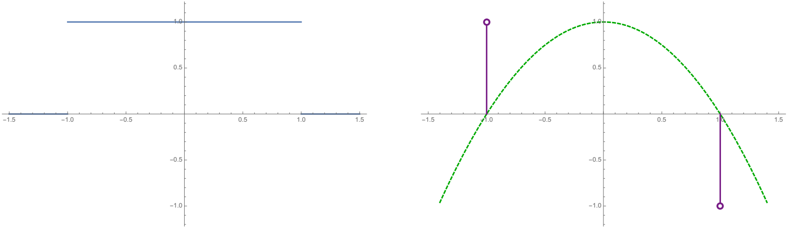

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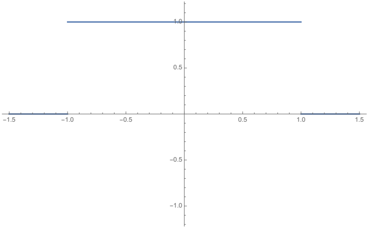

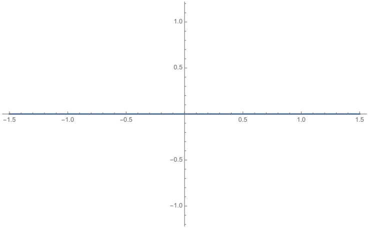

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- $\mathfrak{Ann}(T)$ in $\mathbb{K}[x]\langle\partial_x\rangle$ $\Rightarrow$ holonomic instead of D-finite

From Holonomic Measures to Recurrences on Moments

Example: Lebesgue measure over a segment (continued)

Let $G = [-1, 1]$, $f = 1$, $\mu = \mathbf{1}_G$ and $\varphi = x^k$:

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$\Rightarrow$ Recurrence satisfied by the moments (m_k):

$$(k+2) m_{k+1} - k m_{k-1} = 0$$

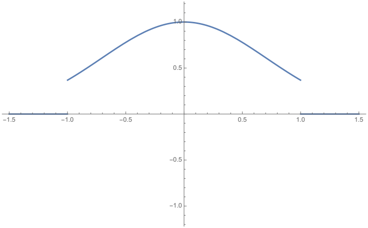
This is indeed true...

$$m_k = \int_{-1}^{1} x^k \mathrm{d}x = \begin{cases} \frac{2}{k+1} & \text{if } k \text{ even} \\ 0 & \text{if } k \text{ odd} \end{cases}$$

Using Integration by Parts

Example: Exp-Poly density over a segment

Let $G = [-1, 1]$, $\quad f = \exp(-x^2)$, $\quad \mu = f\mathbf{1}_G$. Recall: $\langle L\mu, \varphi\rangle = \int_{-1}^{1} (L^*\varphi) f \mathrm{d}x$



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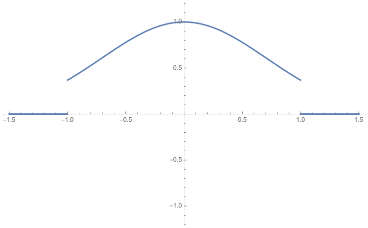

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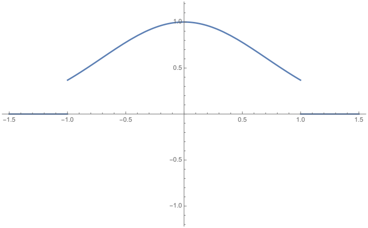

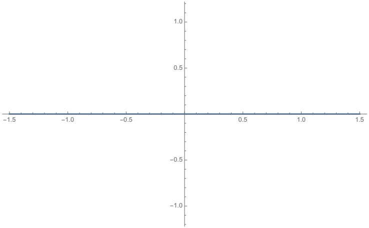

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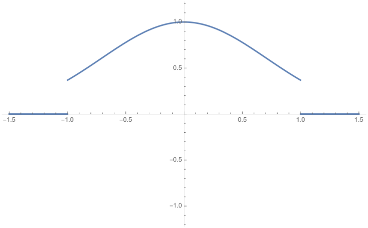

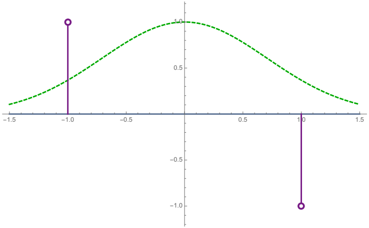

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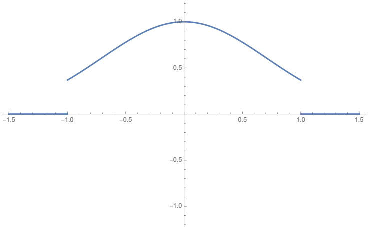

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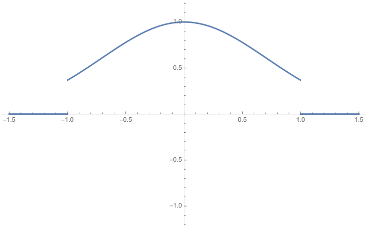

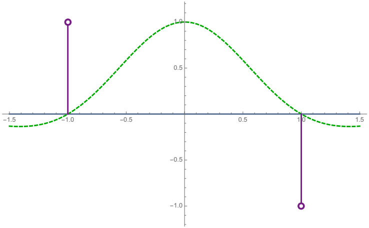

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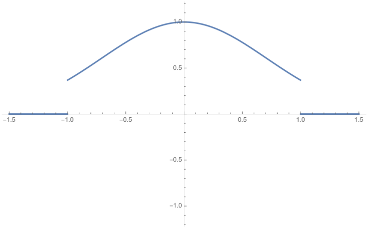

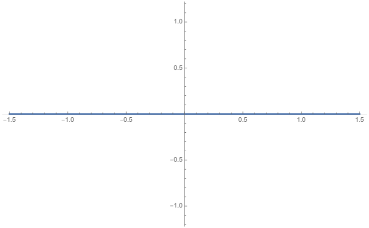

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$\Rightarrow$ Recurrence for the m_k:

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The General Case

$\mu = f\mathbf{1}_G$, $\quad L \in \mathbb{K}[x]\langle\partial_x\rangle$ of order r,

- Use **Lagrange identity**:

$$\varphi\,(L\,f) \;-\; (L^*\,\varphi)\,f \;=\; \partial_x\,\mathcal{L}_L(f,\varphi)$$

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$\rightarrow$ if $L \in \mathfrak{Ann}(f)$ $\rightarrow$ where $g = 0$ on ∂G $\rightarrow$ use **Stokes' theorem**

$\Rightarrow$ $\overline{L} = g^r L \in \mathfrak{Ann}(\mu)$

From Differential Equations to Recurrences

– Translate $\overline{L} = g^r L \in \mathfrak{Ann}(\mu)$ into a recurrence on (m_α):

$$x_i \quad \to \quad S_{\alpha_i} \qquad\qquad \partial_{x_i} \quad \to \quad -\alpha_i S_{\alpha_i}^{-1}$$

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If $f(x) = \exp(p(x))$ and $g = 0$ on ∂_G s.t. $\{x \in \mathbb{C}^n \mid g(x) = 0 \text{ and } \nabla g(x) = 0\} = \varnothing$, then the recurrences system is holonomic.

$\Rightarrow$ Conjecture for the general case?

From Differential Equations to Recurrences

– Translate $\overline{L} = g^r L \in \mathfrak{Ann}(\mu)$ into a recurrence on (m_α):

$$x_i \quad \to \quad S_{\alpha_i} \qquad\qquad \partial_{x_i} \quad \to \quad -\alpha_i S_{\alpha_i}^{-1}$$

Direct Problem

1. $\{L_1, \ldots, L_k\} \subseteq \mathfrak{Ann}(f)$ D-finite
2. $\{\overline{L}_1, \ldots, \overline{L}_k\} \subseteq \mathfrak{Ann}(\mu)$
3. Translate into $\{R_1, \ldots, R_k\} \subseteq \mathfrak{Ann}(m_\alpha)$
4. Gröbner basis algo on $\{R_1, \ldots, R_k\}$

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⇒ Conjecture for the general case?

Inverse Problem

∘ Reconstruct $\overline{L}_i$, then g and L_i from the given moments m_α

⇒ Translation $\overline{L}_i \leftrightarrow R_i$ is linear

Note: Actual proof of holonomicity of $\{R_1, \ldots, R_k\}$ not needed

Table of Contents

Inverse Problem — Roadmap and Issues

– To reconstruct g vanishing on ∂G and $L \in \mathfrak{Ann}(f)$ of order r:

1. Make an ansatz $\widetilde{L}$ for $\overline{L} = g^r L \in \mathfrak{Ann}(\mu)$
2. Find the coefficients of $\widetilde{L}$ by solving the linear system:

$$\langle \widetilde{L}\,\mu, x^\alpha \rangle = \langle \mu, \widetilde{L}^* x^\alpha \rangle = \int_G (\widetilde{L}^* x^\alpha) f(x) \mathrm{d}x = 0, \qquad |\alpha| \leqslant N \qquad (LS_N)$$

   requiring moments m_α for $|\alpha| \leqslant N + \dots$

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– Issues to be handled:

- False solutions in (LS_N): $\widetilde{L} \notin \mathfrak{Ann}(\mu)$?
- How many moments m_α: a priori bounds on N?
- Can g and L be always extracted from $\widetilde{L} \in \mathfrak{Ann}(\mu)$?

Reconstruction of Exp-Poly Densities

– $\mu = f\mathbf{1}_G$ with $f(x) = \exp(p(x))$ for $p \in \mathbb{K}[x]_s$ and $g \in \mathbb{K}[x]_d$ vanishing on ∂G

$$\overline{L}_i \quad = \quad g(\partial_{x_i} - p'_{x_i}) \quad \in \quad \mathfrak{Ann}(\mu)$$

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Algorithm ReconstructExpPoly

Input: Moments m_α of μ for $|\alpha| \leqslant N + d + s - 1$
Output: Polynomials $\widetilde{g}$ and $\widetilde{p}$

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Theorem — Correctness of RECONSTRUCTEXPPOLY

If $N \geqslant 3d + s - 1$, then RECONSTRUCTEXPPOLY computes:

- $\widetilde{g} = \lambda g$ with $\lambda \neq 0$
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Reconstruction of Exp-Poly Densities — Proof

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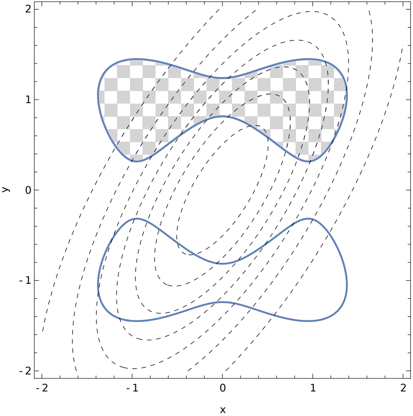
Example — Algebraic Support, Gaussian Measure

→ Reconstruction of:

$f(x,y) = \exp(-x^2 + xy - y^2/2)$ and $g(x,y) = \left(x^2 - 9/10\right)^2 + \left(y^2 - 11/10\right)^2 - 1$



Moments $(m_{ij})_{i+j \leqslant 18}$ with $\mathbf{10}$ digits of accuracy

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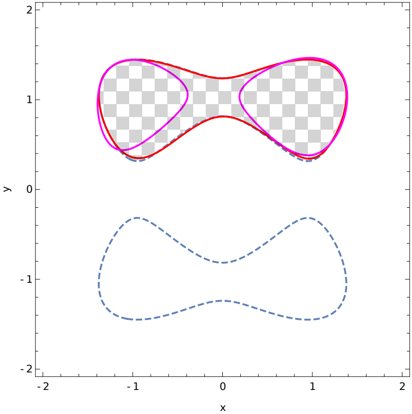

Moments $(m_{ij})_{i+j \leqslant 18}$ with $\mathbf{4}, \mathbf{6}, \mathbf{8}$ digits of accuracy

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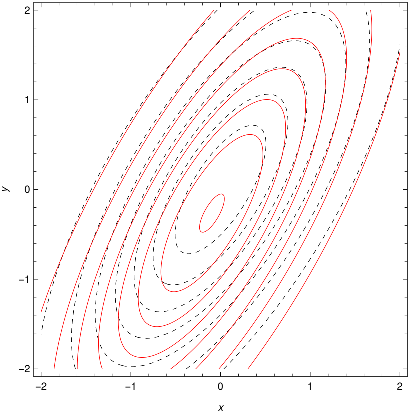

Moments $(m_{ij})_{i+j \leqslant 18}$ with 8 digits of accuracy

Density and Support Reconstruction in the General Case

– $\mu = f\mathbf{1}_G$ with $g \in \mathbb{K}[x]_d$ vanishing on ∂G, and $\{L_1, \dots, L_n\}$ rectangular system for f:

$$L_i = q_{ir_i}\partial_{x_i}^{r_i} + \cdots + q_{i1}\partial_{x_i} + q_{i0} \in \mathfrak{Ann}(f) \cap \mathbb{K}[x]\langle\partial_{x_i}\rangle$$

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Algorithm ReconstructDensity

Input: Moments m_α of μ for $|\alpha| \leqslant N + s$

Output: A rectangular system $\{\widetilde{L}_1, \ldots, \widetilde{L}_n\}$ for f

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$$\langle \mu, \widetilde{L}_i^* x^\alpha \rangle = 0, \qquad 1 \leqslant i \leqslant n, \quad |\alpha| \leqslant N$$
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Algorithm ReconstructSupport

Input: Rectangular $\{L_1, \ldots, L_n\}$ and m_α for $|\alpha| \leqslant N + dr + \max_{ij}\{\deg(q_{ij}) - j\}$
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Density and Support Reconstruction in the General Case

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Input: Moments m_α of μ for $|\alpha| \leqslant N + s$
Output: A rectangular system $\{\widetilde{L}_1, \dots, \widetilde{L}_n\}$ for f

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Algorithm ReconstructSupport

Input: Rectangular $\{L_1, \dots, L_n\}$ and m_α for $|\alpha| \leqslant N + dr + \max_{ij}\{\deg(q_{ij}) - j\}$
Output: Polynomial $\widetilde{g} \in \mathbb{K}[x]_d$

1. Compute coefficients of ansatz $\widetilde{h} \in \mathbb{K}[x]_{dr}$ with nontrivial solution of
$$\langle \mu, (\widetilde{h}L_i)^* x^\alpha \rangle = 0, \qquad 1 \leqslant i \leqslant n, \quad |\alpha| \leqslant N$$
2. $\widetilde{g}^r \leftarrow \widetilde{h}$ $\qquad (\tilde{g} \leftarrow \widetilde{h}/\ \mathrm{GCD}(\widetilde{h}, \widetilde{h}'_{x_1}, \dots, \widetilde{h}'_{x_n}))$

Density and Support Reconstruction in the General Case

Theorem — Correctness of ReconstructDensity

For N large enough, the rectangular system $\{\widetilde{L}_1, \ldots, \widetilde{L}_n\}$ computed by ReconstructDensity is in $\mathfrak{Ann}(f)$.

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For N large enough, the rectangular system $\{\widetilde{L}_1, \dots, \widetilde{L}_n\}$ computed by ReconstructDensity is in $\mathfrak{Ann}(f)$.

Theorem — Correctness of ReconstructSupport

ReconstructSupport computes $\widetilde{g} = \lambda g$ with $\lambda \neq 0$ whenever $q_{ir} \neq 0$ on ∂G and $N \geqslant (2r-1)d + (d-1)b + s$ where:

- $r = \max_{1 \leqslant i \leqslant n} r_i$, orders of the L_i
- $b = r \mod 2$
- $s = \max_{1 \leqslant i \leqslant n} \{\deg(q_{ir})\}$ maximal degree of the head coefficients

Support Reconstruction — Proof

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\mathcal{L}_{L_i}(f, \widetilde{h}\varphi) = f & \left[q_{i1} \widetilde{h}\varphi - \partial_{x_i}(q_{i2} \widetilde{h}\varphi) + \cdots + (-1)^{r-1} \partial_{x_i}^{r-1}(q_{ir} \widetilde{h}\varphi) \right] \\
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ReconstructSupport computes $\widetilde{g} = \lambda g$ with $\lambda \neq 0$ whenever:

- $N \geqslant (2r-1)d + (d-1)b + s$
- $q_{ir} \neq 0$ on ∂G

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⇒ Contradiction: $h_0 = 0$ on ∂_G, hence $g \mid h_0$

The Singular Case — Example in Combinatorics

→ Express Catalan numbers as moments of a measure μ:

$$C_n = \frac{1}{n+1}\binom{2n}{n} \stackrel{?}{=} \int_I x^n f(x)\mathrm{d}x$$

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$$C_n = \lambda \int_{-\infty}^{+\infty} x^n \sqrt{\frac{4-x}{x}}\mathrm{d}x \quad ?$$

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$$C_n = \frac{1}{2\pi}\int_0^4 x^n \sqrt{\frac{4-x}{x}}\mathrm{d}x \quad ?$$

Table of Contents

Some Limits and Perspectives

- A priori bounds for N in the general case with unknown D-finite density?
- Full determination of the density, including initial conditions
- Extracting the component of $V(g)$ corresponding to ∂G

Bounds for the Number of Moments?

– Is there an explicit bound N_0 on N s.t. for ansatz $\widetilde{L}$ of $\overline{L} = g^r L$:

$$\langle \widetilde{L}\,\mu, \varphi \rangle = 0 \quad \text{for all } \varphi \in \mathbb{K}[x]_N \qquad \Rightarrow \qquad \widetilde{L}\,\mu = 0 \qquad \text{when } N \geqslant N_0 \qquad ?$$

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– Such a bound N_0 depending only on the structure of $\widetilde{L}$ cannot exist:

Example [Batenkov2009] — Legendre Polynomials P_n over $[-1,1]$

$P_n(x)$ annihilated by $L_n = (1-x^2)\partial_x^2 - 2x\partial_x + n(n+1)$ $\Rightarrow$ common ansatz $\widetilde{L}$

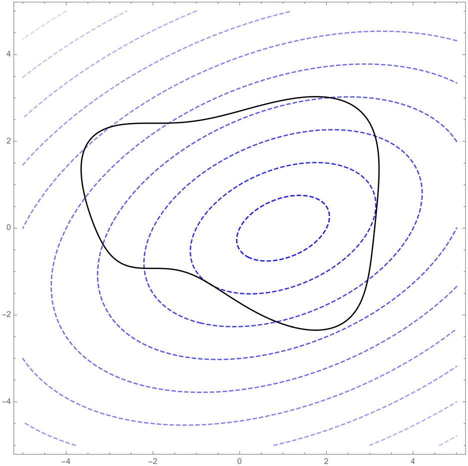
but $\quad m_k^{(n)} = \int_{-1}^{1} x^k P_n(x)\mathrm{d}x = 0$ for $k < n$ and $m_n^{(n)} > 0$

→ Explicit bounds depending on upper bounds on the coefficients of $\widetilde{L}$?

Reconstructing Initial Conditions of the Density

– Algorithm ReconstructDensity only computes a system $\widetilde{\mathfrak{I}} = \{\widetilde{L}_1, \ldots, \widetilde{L}_n\}$ but not the initial conditions that fully characterize f

Reconstructing Initial Conditions of the Density



$$f(x,y) = \lambda_1 e^{p_1(x,y)}$$

$$p_1 = -\frac{1}{2}\begin{pmatrix} x-\mu_{x1} \\ y-\mu_{y1} \end{pmatrix}^T \Sigma_1^{-1} \begin{pmatrix} x-\mu_{x1} \\ y-\mu_{y1} \end{pmatrix}$$

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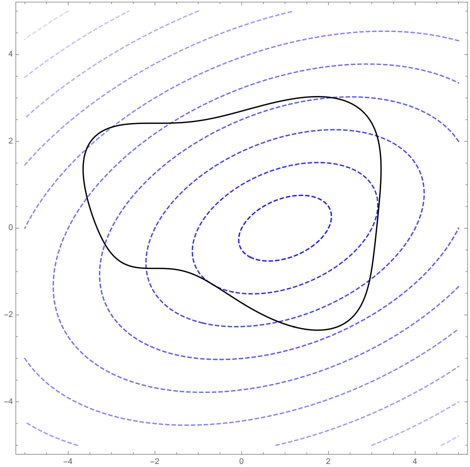

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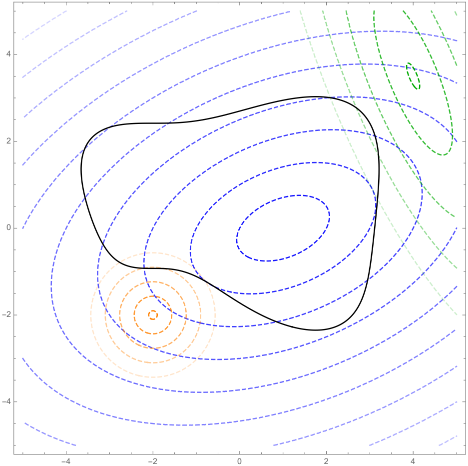

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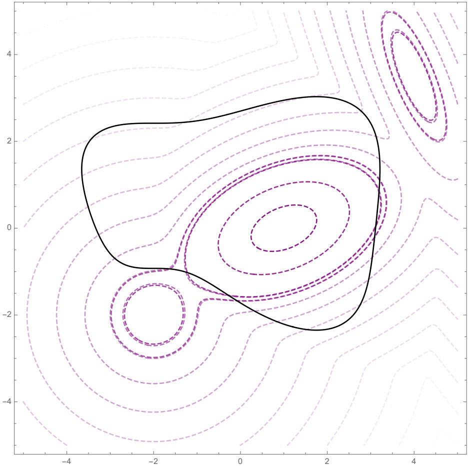

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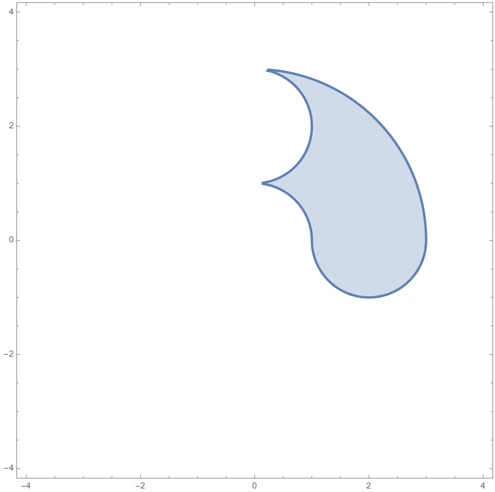
→ Solution: compute initial moments for a basis of solution densities of $\widetilde{\mathfrak{I}}$

- Optimization techniques, e.g., [HenrionLasserreSavorgnan2009]
- Computer algebra, e.g., [LairezMezzarobbaElDin2019]

Isolation of the Topological Boundary



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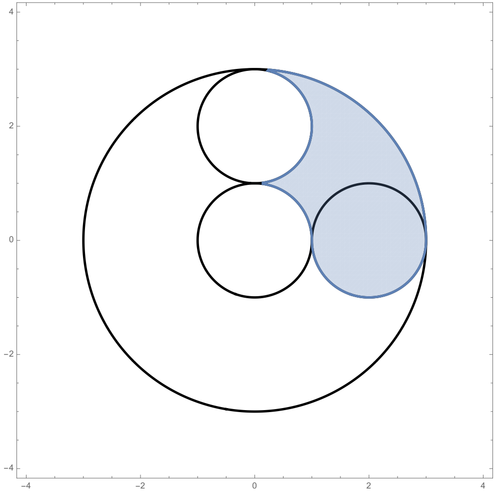

$I(\boldsymbol{\partial G}) = (g)$ with $g(x,y) =$
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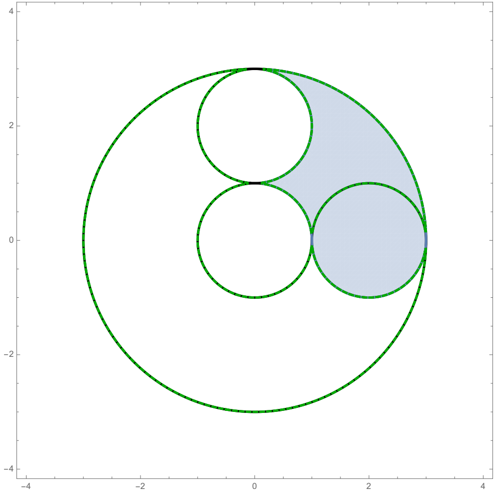

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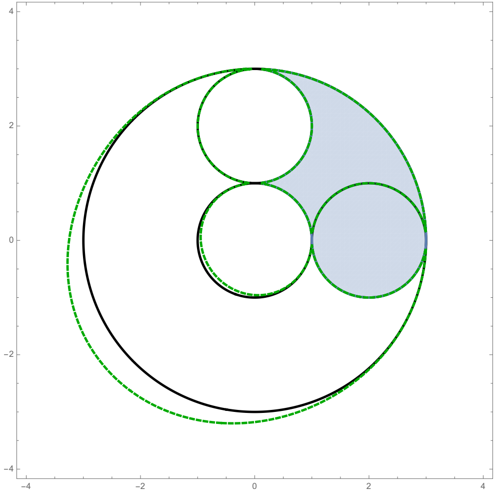

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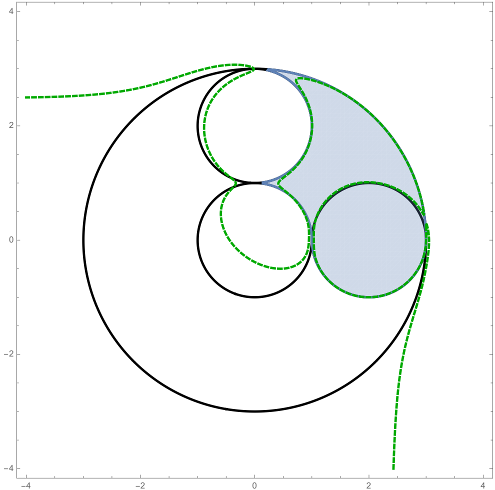
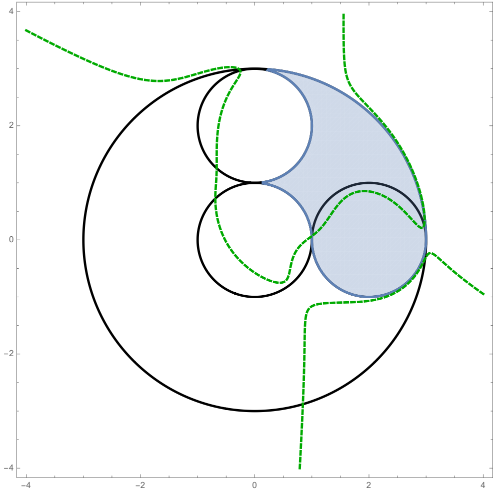


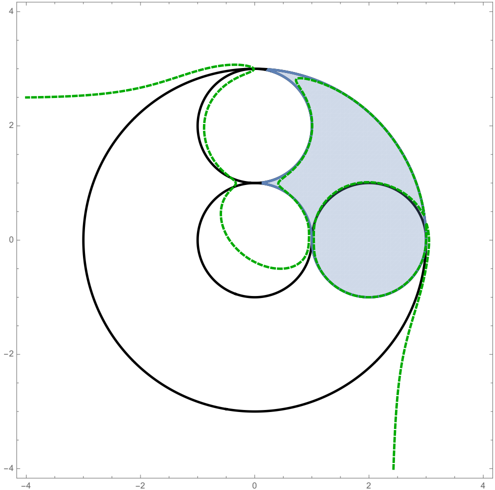
$I(\boldsymbol{\partial G}) = (g)$ with $g(x,y) =$
$(x^2+y^2-9)(x^2+y^2-1)((x-2)^2+y^2-1)(x^2+(y-2)^2-1)$
$\widetilde{g}$ reconstructed using **2** digits accuracy for the moments (m_α)

Isolation of the Topological Boundary



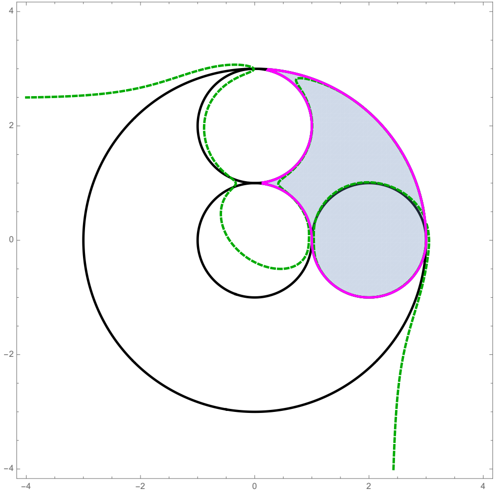
$I(\boldsymbol{\partial G}) = (g)$ with $g(x,y) =$
$(x^2+y^2-9)(x^2+y^2-1)((x-2)^2+y^2-1)(x^2+(y-2)^2-1)$
$\widetilde{g}$ reconstructed using **1** digit accuracy for the moments (m_α)

Isolation of the Topological Boundary



$I(\boldsymbol{\partial G}) = (g)$ with $g(x,y) =$
$(x^2+y^2-9)(x^2+y^2-1)((x-2)^2+y^2-1)(x^2+(y-2)^2-1)$
$\widetilde{g}$ reconstructed using **2** digits accuracy for the moments (m_α)

Isolation of the Topological Boundary



$I(\boldsymbol{\partial G}) = (g)$ with $g(x,y) =$
$(x^2+y^2-9)(x^2+y^2-1)((x-2)^2+y^2-1)(x^2+(y-2)^2-1)$
$\boldsymbol{\partial G} \approx \left\{(x,y) \mid g(x,y)=0 \text{ and } \mathbb{E}\left[\widetilde{g}(x,y)^2\right] \leqslant \varepsilon\right\}$, $\quad \widetilde{g} \leftarrow$ randomly perturbed $(\widetilde{m}_\alpha)$

Conclusion and Perspectives

Contributions:

- Extension of [LasserrePutinar2015] to reconstruction of unknown Exp-Poly density and unknown semi-algebraic support

→ Explicit bound for the number N of required moments

- Reconstruction algorithm for unknown holonomic density and unknown semi-algebraic support
- Numerical experiments using least-squares approximation when approximate moments are known

Future work:

- Generic bounds for N depending on the magnitude of the coefficients
- Numerical aspects: robustness w.r.t. approximate moments, or nonpolynomial boundary
- Isolation of the topological boundary via perturbation techniques
- Application to problems involving combinatorial sequences

Thank you for your attention!