RISC

JOHANNES KEPLER UNIVERSITY LINZ

# *On a sequence of polynomials generated by a Kapteyn series of the second kind*

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LAND OBERÖSTERREICH

Séminaire différentiel, October 19, 2021, online

## *Kapteyn series*

Kapteyn series were introduced around 1900 by *Willem Kapteyn*

- Kapteyn series of the first kind:

$$\sum_{k\geq 0} \alpha_k^{\nu} J_{\nu+k}\big((\nu+k)z\big)$$

- Kapteyn series of the *second kind*:

$$\sum_{k\geq 0} \alpha_k^{\mu,\nu} J_{\mu+k}\big((\mu+\nu+2k)z\big) J_{\nu+k}\big((\mu+\nu+2k)z\big)$$

where

- $\mu, \nu \in \mathbb{C}$
- $J_\nu(z) = \sum_{n\geq 0} \frac{(-1)^n}{\Gamma(\nu+n+1)n!} \left(\frac{z}{2}\right)^{\nu+2n}$ the *Bessel function* of the first kind

## A bit of history

### Kepler's equation

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- solved 1771 by Lagrange *Sur le Probléme de Képler* using Lagrange inversion obtaining

$$E(M) = M + \sum_{n=1}^{\infty} \frac{\varepsilon^n}{n!} \frac{d^{n-1}}{DM^{n-1}} \sin^n(M)$$

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- solved 1819 by Bessel *Analytische Auflösung der Kepler'schen Aufgabe* using a different method introducing Bessel functions of the first kind

$$J_n(z) = \frac{1}{\pi} \int_0^{\pi} \cos(nE - z \sin E) dE, \qquad n \in \mathbb{Z}.$$

$$\hookrightarrow \qquad E(M) = M + \sum_{n=1}^{\infty} \frac{2}{n} J_n(n\varepsilon) \sin(nM)$$

## *Today's topic*

- in 2007 Lerche and Tautz (Astrophysics, Electrodynamics) studied

$$S_1(a) = \sum_{k=1}^{\infty} k^4 J_k^2(ka) = \frac{a^2(64 + 592a^2 + 472a^4 + 27a^6)}{256(1-a^2)^{13/2}}$$

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- In 2011 Dominici considered the following Kapteyn series of the second kind

$$g_n(z) = \sum_{k \geq 0} k^{2n} J_k^2(2kz) = \sum_{k \geq 0} b_{n,k} z^{2k}, \quad n \geq 0.$$

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Recall the definition of Kapteyn series of the second kind

$$\sum_{k\geq 0} \alpha_k^{\mu,\nu} J_{\mu+k}\big((\mu+\nu+2k)z\big) J_{\nu+k}\big((\mu+\nu+2k)z\big)$$

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Some particular cases:

$$g_0(z) = \frac{1}{2} + \frac{1}{2\sqrt{1-4z^2}}, \quad g_1(z) = \frac{z^2(1+z^2)}{(1-4z^2)^{7/2}},$$

$$g_2(z) = \frac{z^2(1 + 37z^2 + 118z^4 + 27z^6)}{(1-4z^2)^{13/2}}$$

## *Conjecture*

$$g_n(z) = \frac{P_n(z^2)}{(1-4z^2)^{3n+1/2}} + \frac{1}{2}\delta_{n,0}, \quad n \geq 0,$$

where $P_n(x)$ is a polynomial of degree $2n$.

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## *The coefficients $b_{n,k}$*

Let $b_{n,k}$ be defined by

$$g_n(z) = \sum_{k\geq 0} k^{2n} J_k^2(2kz) = \sum_{k\geq 0} b_{n,k} z^{2k}, \quad |z| < \tfrac{1}{2}.$$

Then

$$b_{n,k} = \frac{1}{2}\binom{2k}{k} \sum_{j=0}^{2k} \frac{(-1)^{2k-j}}{(2k-j)!j!}(k-j)^{2k+2n} + \frac{1}{2}\delta_{n+k,0}, \quad n,k \geq 0.$$

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We want to show that

$$g_n(z) = \frac{P_n(z^2)}{(1-4z^2)^{3n+1/2}}$$

## *Univariate holonomic functions*

A formal power series $f(x) = \sum_{n\geq 0} a_n x^n$ is called *holonomic* if there exist polynomials $p_0, \ldots, p_d$, not all zero, such that,

$$p_0(x)f(x) + p_1(x)f'(x) + \cdots + p_d(x)f^{(d)}(x) = 0.$$

A sequence $(a_n)_{n\geq 0}$ is called *holonomic* if there exist polynomials $q_0, \ldots, q_r$, not all zero, such that,

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- Maple: *gfun* (Salvy+Zimmermann)
- Mathematica: *GeneratingFunctions* (Mallinger),
  *Guess* (Kauers)
- Sage: *ore_algebra* (Kauers et al)

## *Multivariate holonomic functions*

Consider now functions $f(n_1, \ldots, n_r, x_1, \ldots, x_s)$ ($r, s$ fixed), where

- $n_1, \ldots, n_r$ are discrete variables
- $x_1, \ldots, x_s$ are continuous variables

$f$ is called *holonomic*, if it satisfies a certain *linear system of difference/differential equations* with polynomial coefficients.

*Theorem* A holonomic function is uniquely determined by a holonomic system of equations and a finite number of initial values.

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*Example* Bessel function of the first kind:

$$\begin{aligned} xJ_{n+2}(x) - 2(n+1)J_{n+1}(x) + xJ_n(x) &= 0, \\ x^2 J_n''(x) + xJ_n'(x) - (n^2 - x^2)J_n(x) &= 0, \\ xJ_{n+1}(x) + xJ_n'(x) - nJ_n(x) &= 0 \end{aligned}$$

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*Example* Binomial coefficient:

$$(n-k)\binom{n}{k} = (1+k)\binom{n}{k+1}, \quad (n+1)\binom{n}{k} = (n-k+1)\binom{n+1}{k}$$

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*Counterexample* Stirling numbers of the second kind $S_2(n, k) = \left\{ {n \atop k} \right\}$

$$S_2(n+1, k+1) - (k+1)S_2(n, k+1) - S_2(n, k) = 0$$

## *Multivariate holonomic functions*

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- Maple: *mgfun* (Chyzak)
- Mathematica: *HolonomicFunctions* (Koutschan),
  *Guess* (Kauers)

## *The coefficients $b_{n,k}$*

The goal is to prove (for $n \geq 1, k \geq 0$)

$$g_n(z) = \sum_{k \geq 0} b_{n,k} z^{2k} \stackrel{!}{=} \frac{P_n(z^2)}{(1 - 4z^2)^{3n+1/2}},$$

with $P_n$ a polynomial of degree $2n$ and where

$$b_{n,k} = \frac{1}{2} \binom{2k}{k} \sum_{j=0}^{2k} \frac{(-1)^{2k-j}}{(2k-j)!j!} (k-j)^{2k+2n}, \quad n \geq 1, k \geq 0.$$

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It can be shown that

$$b_{n,k} = \binom{2k}{k} \frac{(2k+1)_{2n}}{2} \sum_{j=0}^{2n} \left\{ {2k+j \atop 2k} \right\} \frac{(2k)!}{(2k+j)!} \frac{(-k)^{2n-j}}{(2n-j)!},$$

with the *Pochhhammer symbol* $(a)_n = a(a+1)\cdots(a+n-1)$.

## *The non-holonomic systems approach*

- *Summation algorithms for Stirling number identities*, Kauers (JSC'07)
- *Automated proofs for some Stirling number identities*, Kauers+Schneider (EJC'08)
- *A non-holonomic systems approach to special functions identities*, Chyzak+Kauers+Salvy (ISSAC'09)

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*Example identities:*

$$\sum_{k=0}^{n} \binom{n}{k} i(k+i)^{k-1}(n-k+j)^{n-k} = (n+i+j)^n$$

$$\sum_{k=0}^{n} (-1)^{m-k} k! \binom{n-k}{m-k} \left\{ {n+1 \atop k+1} \right\} = E_1(n,m)$$

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*Implementations:*

- Mathematica: *Stirling* (Kauers),
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$$\sum_{k=0}^{n} \binom{n}{k} i(k+i)^{k-1}(n-k+j)^{n-k} = (n+i+j)^n$$

## *Example (Abel's summation identity)*

In[1]:= $\mathbf{ann = Annihilator[\sum_{k=0}^{n} \binom{n}{k} i(k+i)^{k-1}(n-k+j)^{n-k}, \{S[n], S[i], S[j]\}]}$

Annihilator::nondf : The expression $(i+k)^{-1+k}$ is not recognized to be $\partial$-finite. The result might not generate a zero-dimensional ideal.

Annihilator::nondf : The expression $(j-k+n)^{-k+n}$ is not recognized to be $\partial$-finite. The result might not generate a zero-dimensional ideal.

Out[1]= $\{(i+1)S_n^2 - i(i+n+2)S_n S_i - (i+1)(j+n+2)S_n S_j + i(n+1)(i+j+n+2)S_i S_j\}$

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In[2]:= $\mathbf{FullSimplify}[\mathbf{ApplyOreOperator}[\mathbf{ann}, (n+i+j)^n]]$

Out[2]= $\{0\}$

## *Exampe:* $b_{n,k}$

$$b_{n,k} = \frac{1}{2}\binom{2k}{k}\sum_{j=0}^{2k}\frac{(-1)^{2k-j}}{(2k-j)!j!}(k-j)^{2k+2n}$$

$$\overset{!}{=} \binom{2k}{k}\frac{(2k+1)_{2n}}{2}\sum_{j=0}^{2n}\left\{\begin{matrix}2k+j\\2k\end{matrix}\right\}\frac{(2k)!}{(2k+j)!}\frac{(-k)^{2n-j}}{(2n-j)!}$$

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In[3]:= $\mathbf{Annihilator}[\sum_{j=0}^{2k}\frac{(-1)^{2k-j}}{(2k-j)!j!}(k-j)^{2k+2n}, \{S[n], S[k]\}]$

Annihilator::nondf : The expression $(-j+k)^{2k+2n}$ is not recognized to be $\partial$-finite. The result might not generate a zero-dimensional ideal.

Out[3]= $\{S_n S_k - S_n - (k+1)^2 S_k\}$

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In[4]:= $\mathbf{Annihilator}\left[(2k+1)_{2n}\sum_{j=0}^{2n}\left\{{2k+j \atop 2k}\right\}\frac{(2k)!}{(2k+j)!}\frac{(-k)^{2n-j}}{(2n-j)!}, \{S[n], S[k]\}\right]$

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Annihilator::nondf : The expression $\left\{{2k+j \atop 2k}\right\}$ is not recognized to be $\partial$-finite. The result might not generate a zero-dimensional ideal.

Out[4]= $\{\,\}$

## *Classical proof (skipping some steps)*

We want to prove

$$g_n(z) = \sum_{k \geq 0} b_{n,k} z^{2k} = \frac{P_n(z^2)}{(1-4z^2)^{3n+1/2}}$$

and one can show that

$$b_{n,k} = \binom{2k}{k} \frac{(2k+1)_{2n}}{2} \sum_{j=0}^{2n} \left\{ {2k+j \atop 2k} \right\} \frac{(2k)!}{(2k+j)!} \frac{(-k)^{2n-j}}{(2n-j)!}.$$

Let

$$r_n(k) = \sum_{j=0}^{2n} \left\{ {2k+j \atop 2k} \right\} \frac{(2k)!}{(2k+j)!} \frac{(-k)^{2n-j}}{(2n-j)!}.$$

## *Almost there...*

$$r_n(k) = \sum_{j=0}^{2n} \left\{ {2k+j \atop 2k} \right\} \frac{(2k)!}{(2k+j)!} \frac{(-k)^{2n-j}}{(2n-j)!},$$

is a polynomial with $\deg(r_n(k)) = n$.

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is a polynomial with $\deg(r_n(k)) = n$.

Let $u_m(k)$ be a polynomial of degree $m$. Then

$$\sum_{k \geq 0} \binom{2k}{k} u_m(k) z^k = \frac{U_m(z)}{(1-4z)^{m+1/2}},$$

where $U_m(z)$ is a polynomial in $z$ with $\deg U_m(z) \leq m$.

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- $b_{n,k} = \frac{1}{2}\binom{2k}{k}(2k+1)_{2n} r_n(k)$

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- $b_{n,k} = \frac{1}{2}\binom{2k}{k}(2k+1)_{2n} r_n(k)$
- $\sum_{k\geq 0} b_{n,k} z^{2k} = \frac{P_n(z^2)}{(1-4z^2)^{3n+1/2}}$ for $n \geq 1$ with $\deg(P_n(x)) \leq 3n$

## *Tightening the degree bound*

Now, for $n \geq 1$, we have that

$$P_n(z) = (1-4z)^{3n+1/2} \sum_{k \geq 1} \binom{2k}{k} (2k+1)_{2n} r_n(k) z^k, \quad \deg(P_n(z)) \leq 3n.$$

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Let $P_n(z) = \sum_{k=1}^{3n} c(n,k) z^k$. Then

$$c(n,k) = \sum_{j=1}^{k} \frac{(-3n-\frac{1}{2})_{k-j}}{(k-j)!} 4^{k-j} \binom{2j}{j} (2j+1)_{2n} r_n(j).$$

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It can be shown that

$$c(n,k) = \frac{4^{n+k}}{k!} \left(\tfrac{1}{2}\right)_{3n+1} [\Delta_x^k C_{n,k}(x)]_{x=0}$$

with

$$C_{n,k}(x) = \left(x - k + 3n + \tfrac{3}{2}\right)_{k-2n-1} (x+1)_n r_n(x).$$

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Let $P_n(z) = \sum_{k=1}^{3n} c(n,k) z^k$. Then

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$$c(n,k) = \frac{4^{n+k}}{k!} \left(\tfrac{1}{2}\right)_{3n+1} [\Delta_x^k C_{n,k}(x)]_{x=0}$$

with

$$C_{n,k}(x) = \left(x - k + 3n + \tfrac{3}{2}\right)_{k-2n-1} (x+1)_n r_n(x).$$

## *Theorem*

$$g_n(z) = \frac{P_n(z^2)}{(1-4z^2)^{3n+1/2}} + \frac{1}{2}\delta_{n,0}, \quad n \geq 0,$$

where $P_n(x)$ is a polynomial of degree $2n$.

$$g_n(z) = \sum_{k\geq 0} b_{n,k} z^{2k}$$

$$b_{n,k} = \frac{1}{2}\binom{2k}{k}\sum_{j=0}^{2k} \frac{(-1)^{2k-j}}{(2k-j)!j!}(k-j)^{2k+2n}$$

$$(1-4z)^{3n+1/2} = \sum_{k\geq 0} a_{n,k} z^k$$

$$P_n(z) = \sum_{k\geq 0} c(n,k) z^k$$

## *Finding recurrences using SC*

$$g_n(z) = \sum_{k \geq 0} b_{n,k} z^{2k} = \frac{P_n(z^2)}{(1-4z^2)^{3n+1/2}}$$

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$$P_n(z^2) = \sum_{k \geq 0} c(n,k) z^{2k} = (1-4z^2)^{3n+1/2} g_n(z)$$

$$= \sum_{k \geq 0} \sum_{j=0}^{k} a_{n,k-j} b_{n,j} z^{2k}$$

## *Finding a recurrence for* $b_{n,k}$

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In[5]:= $\mathbf{Annihilator}[\boldsymbol{b}[\boldsymbol{n},\boldsymbol{k}],\{\boldsymbol{S}[\boldsymbol{n}],\boldsymbol{S}[\boldsymbol{k}]\}]$

Annihilator::nondf : The expression $(-j+k)^{(2*k+2*n)}$ is not recognized to be $\partial$-finite. The result might not generate a zero-dimensional ideal.

Out[5]= $\left\{(-k-1)S_n S_k + 2(2k+1)S_n + (k+1)^3 S_k\right\}$

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$$(k+1)b_{n+1,k+1} - 2(2k+1)b_{n+1,k} - (k+1)^3 b_{n,k+1} = 0,$$

for $n \geq 0, k \geq 1$ with initial values $b_{n,1} = 1, b_{0,k} = \frac{1}{2}\binom{2k}{k}$.

## *Computing a recurrence for $c(n,k)$*

$$c(n,k) = \sum_{j=1}^{k} \frac{(-3n-\frac{1}{2})_{k-j}}{(k-j)!} 4^{k-j} b_{n,j} = \sum_{j=1}^{k} a_{n,k-j} b_{n,j}.$$

using *Creative Telescoping*: given the summand

$$f(n,k,j) = a_{n,k-j} b_{n,j}$$

compute an annihilating operator of the form

$$\mathcal{A} + (S_j - 1)\mathcal{D} \qquad \text{with} \qquad \mathcal{A} = \sum_{a,b}{}' \gamma_{a,b}(n,k) S_n^a S_k^b.$$

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In[6]:= $\mathbf{annA = Annihilator}[\boldsymbol{a[n, k-j], S[n], S[k], S[j]}];$

In[7]:= $\mathbf{annB = Annihilator}[\boldsymbol{b[n, j], S[n], S[k], S[j]}];$

## *Guessing a recurrence for $c(n,k)$*

In[8]:= $\mathbf{data = Table[c[nn,kk],\{nn,40\},\{kk,40\}];}$

In[9]:= $\mathbf{GuessMultRE[data,\{c[n,k],c[n,1+k],c[n,2+k], c[n,3+k],c[1+n,k],c[1+n,1+k],c[1+n,2+k], c[1+n,3+k]\},n,k,3,StartPoint\to\{1,1\},Infolevel\to 3]}$

80 terms

collecting nonzero points...

modular system: 180 eqns, 80 vars

1 solutions predicted.

refined system: 135 eqns, 35 vars

Q.

1 solutions.

Out[9]= $$\{-4\left(12k^3-72k^2n+24k^2+108kn^2-144kn+9k+216n^2 -24n+2\right)c[n,k+1]+2\left(6k^3-18k^2n+33k^2-90kn +57k-114n+29\right)c[n,k+2]-(k+3)^3c[n,k+3] +(k+3)c[n+1,k+3]+8(2k-6n-1)^3c[n,k] -4(k-3n-1)c[n+1,k+2]\}$$

## *Proving a recurrence for $c(n, k)$*

- Use the command "CreativeTelescoping" adding the *support of the guessed recurrence* as additional information

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$$\mathcal{D} = \frac{8j(2j-2k+6n+3)(2j-2k+6n+5)(2j-2k+6n+7)}{(j-k-3)(j-k-2)(j-k-1)} S_n - \frac{24j^3(2n+1)(6n+5)(6n+7)}{(j-k-3)(j-k-2)(j-k-1)}$$

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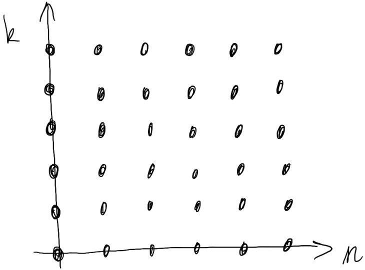
- Two difficulties:
  - NO natural boundaries
  - there are *poles* in the summation range

## *The recurrence for $c(n,k)$*




$$
\begin{aligned}
&(k+3)c(n+1,k+3)\\
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&-4(k-3n-1)c(n+1,k+2)\\
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&+8(2k-6n-1)^3c(n,k)=0
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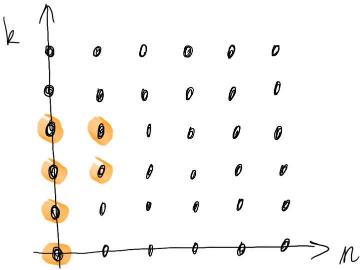


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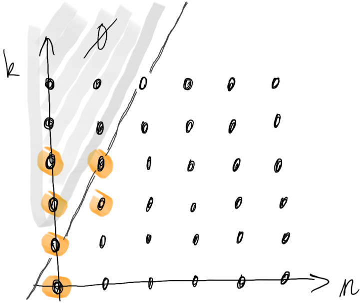


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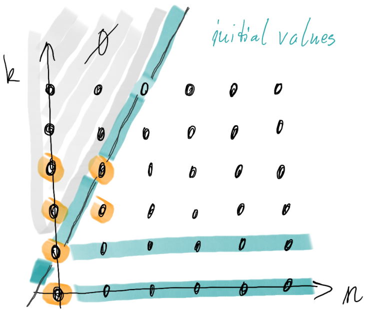


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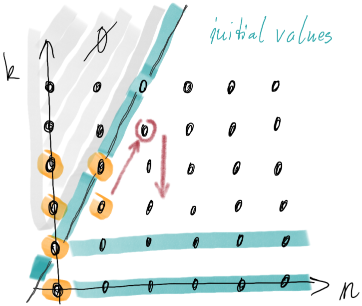


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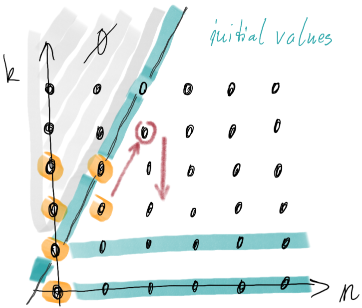


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$$

$$c_{n,1}=1,\quad c_{n,2}=2^{2n+2}-3(4n+1),\quad c_{n,2n}=2^{4n-1}\frac{(1/2)_n^3}{n!},\quad c_{n,k}=0,\quad k\geq 1.$$

## *Some concluding remarks*

The initial value $c_{n,2n}$ corresponds to the double sum evaluation

$$\sum_{j=1}^{2n} \frac{(-3n-\frac{1}{2})_{2n-j}}{(2n-j)!} 4^{2n-j} \binom{2j}{j} \sum_{i=0}^{j} \frac{(-1)^{j-i}}{(j+i)!\,(j-i)!} i^{2j+2n} = 2^{4n-1} \frac{(\frac{1}{2})_n^3}{n!}.$$

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*Conjecture* Let the polynomials $P_n(z)$ be defined by

$$\sum_{k=0}^{\infty} k^{2n}\ \mathrm{J}_k^2\left(2kz\right) = \frac{P_n\left(z^2\right)}{\left(1-4z^2\right)^{3n+\frac{1}{2}}}, \quad n \in \mathbb{N}.$$

Then, $P_n\left(x\right) \in \mathbb{N}\left[x\right]$.

## Some links

- Algorithmic Combinatorics at RISC

  https://combinatorics.risc.jku.at/
- Manuel Kauers

  http://kauers.de/
- Christoph Koutschan

  http://www.koutschan.de/software.php
- Calculations

  http://www3.risc.jku.at/people/vpillwei/kapteyn/

# *Thank you!*