Institut für Algebra

Journée du Séminaire Différentiel

Discrete harmonic functions in the three-quarter plane

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- Institut for Algebra -

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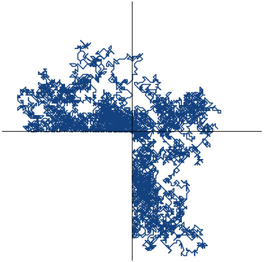

Content

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Continuous harmonic functions

A continuous harmonic function annihilates the standard Laplacian Δf

if f is a harmonic function on an open set $\mathcal{U} \subset \mathbb{R}^2$, then f is twice differentiable and

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

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Applications and properties

- analysis – resolution of partial differential equations
- infinitely differentiable in open sets
- maximum principle; mean value properties; Harnack's inequalities (in the case of non-negative harmonic functions)

Discrete harmonic functions

Simplest discrete Laplacian in dimension 2

$$\Delta f(x,y) = f(x-1,y) + f(x+1,y) + f(x,y-1) + f(x,y+1) - 4f(x,y).$$

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Example $f(x,y) = \frac{xy}{360}\left(3x^4 - 10x^2y^2 + 3y^4 - 5x^2 - 5y^2 + 14\right)$ is harmonic in $\mathbb{Z}^2$.

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Applications and properties

- discrete complex analysis
- probability of absorption at absorbing states of Markov chains or the Ising model
- satisfy multivariate linear recurrence relations (ubiquitous in combinatorics)

Gambler Ruin Problem and Harmonic Functions

S_n denotes the fortune at time n, with initial fortune of i gold coins.

At every step, the gambler bets 1 gold coin, and

- ➙ wins with probability p; ➙ loses with probability $q = 1 - p$

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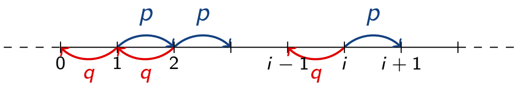

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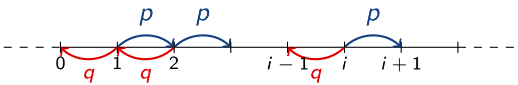

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Markov property

$$h_i = \mathbb{P}_i[\exists n \geq 0 : S_n = 0]$$

$$\begin{cases} h_0 & = & 1 \\ h_i & = & p h_{i+1} + q h_{i-1} \end{cases}$$

Transition matrix and harmonicity

$$P = \begin{pmatrix} 1 & 0 & 0 & \cdots & \cdots & \cdots \\ q & 0 & p & 0 & \cdots & \cdots \\ 0 & q & 0 & p & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

$h = (h_i)_{i\geq 0}$ satisfies $Ph = h$.

Application to random walks

Doob transform

Standard procedure in probability. From a Markov process and an associated harmonic function it defines a new random process.

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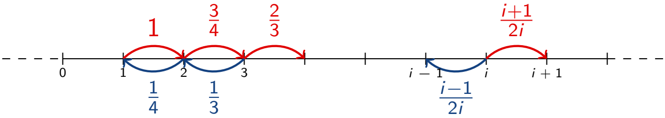
Example

Let $(S_n)_{n\in\mathbb{N}}$ be a simple random walk over $\mathbb{Z}$.
The function $V(i) = i \, (i \in \mathbb{N})$ is discrete harmonic for $(S_n)_{n\in\mathbb{N}}$.

Let $(T_n)_{n\in\mathbb{N}}$ be defined form $(S_n)_{n\in\mathbb{N}}$ by

$$\begin{cases} \mathbb{P}[T_{n+1} = i+1 \mid T_n = i] & = & \frac{V(i+1)}{2V(i)} & = & \frac{i+1}{2i}, \\ \mathbb{P}[T_{n+1} = i-1 \mid T_n = i] & = & \frac{V(i-1)}{2V(i)} & = & \frac{i-1}{2i}. \end{cases}$$

The process $(T_n)_{n\in\mathbb{N}}$ is a random walk over $\mathbb{N}^*$.



Applications to random walks

Asymptotic behavior of the number of excursions

Let $e_{(0,0)\to(i,j)}(n)$ be the number of n-excursions from the origin to (i,j).

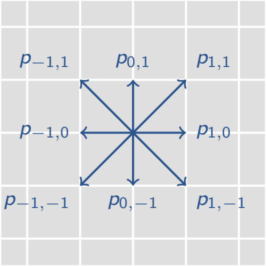
$$e_{(0,0)\to(i,j)}(n) \sim \kappa \cdot V(i,j) \cdot \rho^n \cdot n^{-\alpha}, \quad n \longrightarrow \infty$$

- V **is a harmonic function**
- ρ is the exponential growth
- α is the critical exponent

Furthermore, the growth of the harmonic function V is directly related to the critical exponent α

Content

Definition and properties



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discrete harmonic function

$$\widetilde{f} = (\widetilde{f}(i,j))_{(i,j)\in\mathcal{Q}}$$

associated to random walks which satisfy the following properties:

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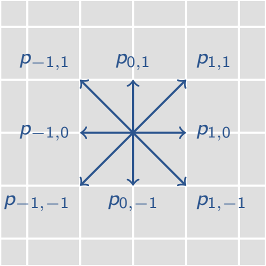

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Properties of the random walks

- homogeneous walk inside $\mathcal{Q}$
- zero drift

$$\sum_{-1\leq i,j\leq 1} ip_{i,j} = \sum_{-1\leq i,j\leq 1} jp_{i,j} = 0$$

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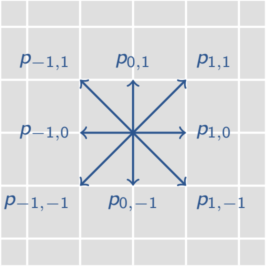

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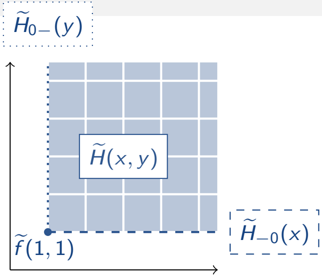
Properties of the harmonic functions

- For all $i \geq 1$ **and** $j \geq 1$, $\widetilde{f}(i,j) = \sum_{-1\leq i_0,j_0\leq 1} p_{i_0,j_0}\widetilde{f}(i+i_0, j+j_0)$
- If $i \geq 0$, $\widetilde{f}(i,0) = 0$ and if $j \geq 0$, then $\widetilde{f}(0,j) = 0$
- If $i > 0$ and $j > 0$ then $\widetilde{f}(i,j) > 0$

Strategy

Generating function

$$\widetilde{H}(x,y) = \sum_{(i,j)\in\mathcal{Q}} \widetilde{f}(i,j)x^{i-1}y^{j-1}.$$



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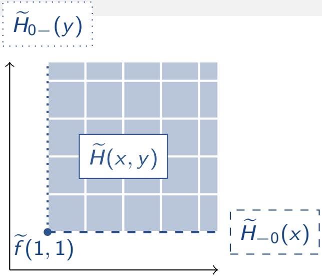

Functional equation

$$K(x,y)\widetilde{H}(x,y) = K(x,0)\widetilde{H}_{-0}(x) + K(0,y)\widetilde{H}_{0-}(y) - K(0,0)\widetilde{f}(1,1)$$

where

$$\widetilde{H}_{-0}(x) = \sum_{i\geq 1} \widetilde{f}(i,1)x^{i-1},$$

$$\widetilde{H}_{0-}(y) = \sum_{j\geq 1} \widetilde{f}(1,j)y^{j-1},$$

$$K(x,y) = xy\left[\sum_{-1\leq i,j\leq 1} p_{i,j}x^{-i}y^{-j} - 1\right].$$

Kernel of the random walks

$$K(x,y) = xy\left[\sum_{-1\leq i,j\leq 1} p_{i,j}x^{-i}y^{-j} - 1\right].$$

The Kernel: a polynomial of degree 2 in x and in y

$$K(x,y) = \widetilde{\alpha}(y)x^2 + \widetilde{\beta}(y)x + \widetilde{\gamma}(y) = \alpha(x)y^2 + \beta(x)y + \gamma(x).$$

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Zeros of the Kernel, $i = 0, 1$:

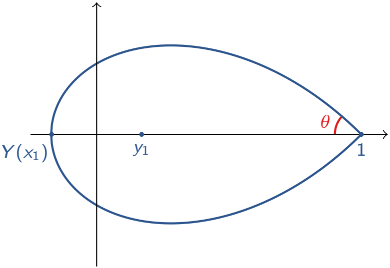
$$K(X_i(y), y) = 0 \qquad K(x, Y_i(x)) = 0$$

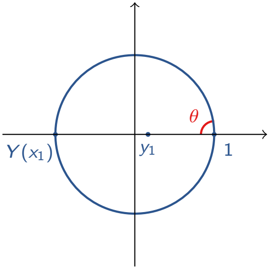
$$X_i(y) = \frac{-\widetilde{\beta}(y) \pm \sqrt{\widetilde{\delta}(y)}}{2\widetilde{\alpha}(y)} \qquad Y_i(x) = \frac{-\beta(x) \pm \sqrt{\delta(x)}}{2\alpha(x)}.$$

The roots of the kernel define analytic curves.

Branches of the Kernel

Curves $Y([x_1, 1]) = \{y \in \mathbb{C} : K(x, y) = 0 \text{ and } x \in [x_1, 1]\}$





Gouyou-Beauchamps model

$(p_{1,0} = p_{-1,1} = p_{-1,0} = p_{1,-1} = 1/4)$

$\theta = \pi/4$

Simple model

$(p_{1,0} = p_{0,1} = p_{-1,0} = p_{0,-1} = \frac{1}{4})$

$\theta = \pi/2$

Generating function $H_{-0}(x)$ stated as a BVP

Functional Equation

$$K(x,y)\widetilde{H}(x,y) = K(x,0)\widetilde{H}_{-0}(x) + K(0,y)\widetilde{H}_{0-}(y) - K(0,0)\widetilde{f}(1,1).$$

Boundary value problem

By evaluating the functional equation on the branch curves, we can transform the functional equation into the following **boundary value problem**:

For $x \in X([y_1, 1]) \setminus \{1\}$,

$$K(x,0)\widetilde{H}_{-0}(x) - K(\bar{x},0)\widetilde{H}_{-0}(\bar{x}) = 0.$$

Explicit expression for the generating function

Theorem [Raschel, 2014]

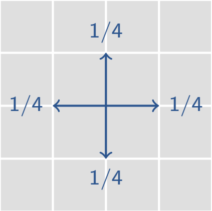
The function $\widetilde{H}_{-0}(x)$ has the following explicit expression

$$\widetilde{H}_{-0}(x) = \mu \frac{w(x) + \nu}{K(x,0)},$$

where w is a conformal mapping vanishing at 0, and the constants ν and μ are defined by

$$\nu = -w\left(X_0(0)\right), \qquad \mu = \widetilde{f}(1,1) \times \begin{cases} \frac{2p_{-1,1}}{w''(0)} & \text{if} \quad p_{1,1} = 0 \text{ and } p_{0,1} = 0, \\ \frac{p_{0,1}}{w'(0)} & \text{if} \quad p_{1,1} = 0 \text{ and } p_{0,1} \neq 0, \\ -\frac{p_{1,1}}{w(X_0(0))} & \text{if} \quad p_{1,1} \neq 0. \end{cases}$$

Example of the simple random walk



$$K(x,0) = \frac{x}{4}$$
$$w(x) = -\frac{2x}{(1-x)^2}$$
$$\nu = 0$$
$$\mu = -\frac{\widetilde{f}(1,1)}{8}$$

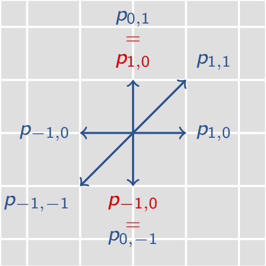
Expression for $H_{-0}(x)$

$$\widetilde{H}_{-0}(x) = \frac{\widetilde{f}(1,1)}{(1-x)^2} = \widetilde{f}(1,1)\sum_{i\geq 1} i x^{i-1}$$

The function $f(i,j) = ij$ is discrete harmonic for the simple random walk

Content

Definitions and properties



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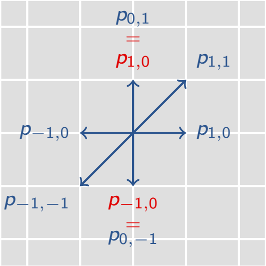

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$$\sum_{-1\leq i,j\leq 1} i p_{i,j} = \sum_{-1\leq i,j\leq 1} j p_{i,j} = 0$$

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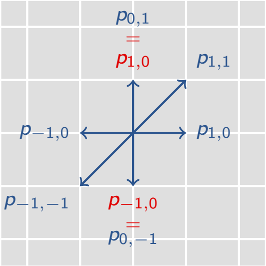

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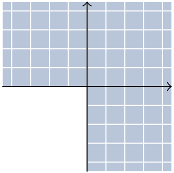
Properties of the harmonic functions

- For all $i \geq 1$ or $j \geq 1$,
  $f(i,j) = \sum_{-1\leq i_0,j_0\leq 1} p_{i_0,j_0} f(i+i_0, j+j_0)$
- If $i \leq 0$, $f(i,0) = 0$ and
  if $j \leq 0$, then $f(0,j) = 0$
- If $i > 0$ or $j > 0$ then $f(i,j) > 0$
- For all $(i,j) \in \mathcal{C}$, $f(i,j) = f(j,i)$

A first functional equation

Generating function

$$H(x,y) = \sum_{(i,j)\in\mathcal{C}} f(i,j)x^{i-1}y^{j-1}.$$



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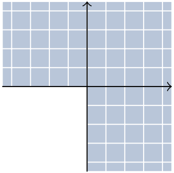

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$$K(x,y)H(x,y) = K(x,0)H_{-0}(x^{-1}) + K(0,y)H_{0-}(y^{-1}) - K(0,0)f(1,1)$$

where

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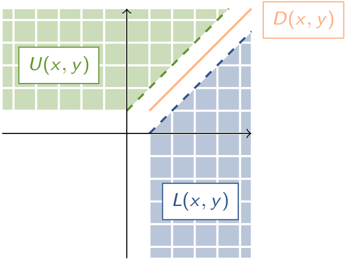


Decomposition of the generating function

$$H(x,y) = L(x,y) + D(x,y) + U(x,y)$$

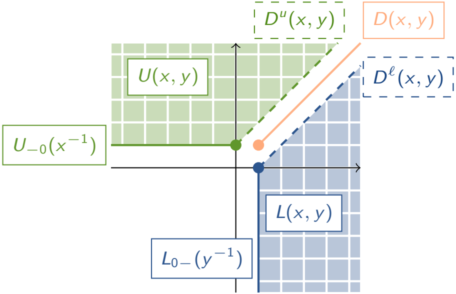
where

$$L(x,y) = \sum_{\substack{i\geq 1\\ j\leq i-1}} f(i,j)x^{i-1}y^{j-1},\ D(x,y) = \sum_{i\geq 1} f(i,i)x^{i-1}y^{i-1},\ U(x,y) = \sum_{\substack{j\geq 1\\ i\leq j-1}} f(i,j)x^{i-1}y^{j-1}.$$

Functional equations

Non-symmetric probability transitions & $p_{-1,1} = p_{1,-1} = 0$

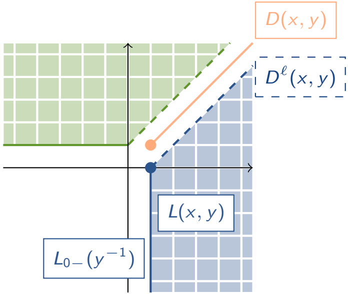
$$\begin{cases} K(x,y)U(x,y) &= -\left(p_{1,0}y + p_{0,-1}xy^2 + p_{1,1} + p_{-1,-1}x^2y^2 - xy\right) D(x,y) \\ & \quad + \left(p_{0,1}x + p_{1,1}\right) U_{-0}(x^{-1}) - \left(p_{1,0}y + p_{0,-1}xy^2\right) D^{\ell}(x,y) \\ & \quad + p_{1,1}f(1,1) + p_{1,0}f(1,0), \\ K(x,y)L(x,y) &= -\left(p_{0,1}x + p_{-1,0}x^2y + p_{1,1} + p_{-1,-1}x^2y^2 - xy\right) D(x,y) \\ & \quad + \left(p_{1,0}y + p_{1,1}\right) L_{0-}(y^{-1}) - \left(p_{0,1}x + p_{-1,0}x^2y\right) D^{u}(x,y) \\ & \quad + p_{1,1}f(1,1) + p_{0,1}f(0,1). \end{cases}$$



Functional equations

Symmetric probability transitions & $p_{-1,1} \neq 0$**;** $p_{1,-1} \neq 0$

$$K(x,y)L(x,y) = -\left(p_{0,1}x + p_{-1,1}x^2 + p_{-1,0}x^2y + \frac{1}{2}\left(p_{1,1} + p_{-1,-1}x^2y^2 - xy\right)\right)D(x,y)$$
$$+ p_{1,-1}\left(y^2 - xy\right)D^{\ell}(x,y) + \left(p_{1,0}y + p_{1,-1}y^2 + p_{1,1}\right)L_{0-}(y^{-1})$$
$$+ p_{1,-1}yf(1,0) + \frac{1}{2}p_{1,1}f(1,1)$$



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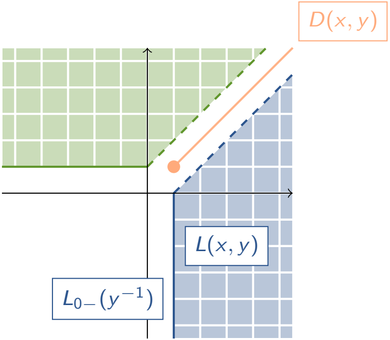
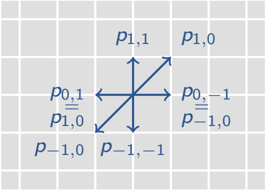
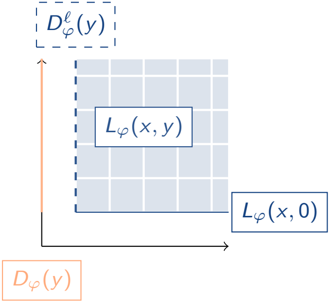


Transforming the cones

Change of variables $\varphi(x,y) = (xy, x^{-1})$





$$L(\varphi(x,y)) = xL_\varphi(x,y) = x\sum_{i,j\geq 1} f(j, j-i)x^{i-1}y^{j-1}$$

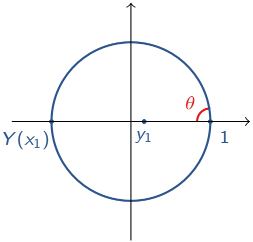
$$D(\varphi(x,y)) = D_\varphi(y) = \sum_{i\geq 1} f(i,i)y^{i-1}$$

$$D^\ell(\varphi(x,y)) = xD^\ell_\varphi(y) = x\sum_{i\geq 1} f(i,i-1)y^{i-1}$$

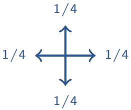
Transforming the probability transitions

$$K(\varphi(x,y)) = \frac{1}{x} K_\varphi(x,y)$$

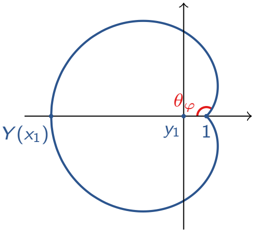
Curves $X([y_1, 1]) = \{x \in \mathbb{C} : K(x,y) = 0 \text{ and } y \in [y_1, 1]\}$



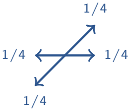
Simple model



$\theta = \pi/2$



Gessel model

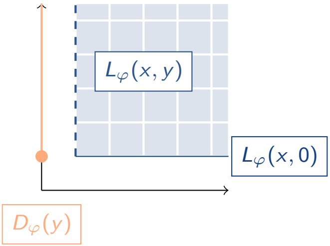


$\theta_\varphi = 3\pi/4$

New functional equation

Functional Equation

$$K_\varphi(x,y)L_\varphi(x,y) = -\left[x\widetilde{\alpha}_\varphi(y) + \frac{1}{2}\widetilde{\beta}_\varphi(y)\right] D_\varphi(y) + K_\varphi(x,0)L_\varphi(x,0) + \frac{1}{2}p_{1,1}f(1,1).$$



Generating function $D_\varphi(y)$ stated as a BVP

Functional Equation

$$K_\varphi(x,y)L_\varphi(x,y) = -\left[x\widetilde{\alpha}_\varphi(y) + \frac{1}{2}\widetilde{\beta}_\varphi(y)\right] D_\varphi(y) + K_\varphi(x,0)L_\varphi(x,0) + \frac{1}{2}p_{1,1}f(1,1).$$

Boundary value problem

By evaluating the functional equation on the branch curves, we can transform the functional equation into the following **boundary value problem**:

For $y \in Y_\varphi([x_1, 1]) \setminus \{1\}$,

$$\sqrt{\widetilde{\delta}_\varphi(y)}D_\varphi(y) - \sqrt{\widetilde{\delta}_\varphi(\bar{y})}D_\varphi(\bar{y}) = 0.$$

Anti-Tutte's invariant

Idea: write $\sqrt{\frac{\delta_\varphi(\bar{y})}{\delta_\varphi(y)}} = \frac{G_\varphi(\bar{y})}{G_\varphi(y)}$, such that: $G_\varphi(\bar{y}) = \overline{G_\varphi(y)}$;

$$\Leftrightarrow$$

Anti-Tutte's invariant

Idea: write $\sqrt{\dfrac{\delta_\varphi(\bar{y})}{\delta_\varphi(y)}} = \dfrac{G_\varphi(\bar{y})}{G_\varphi(y)}$, such that: $G_\varphi(\bar{y}) = \overline{G_\varphi(y)}$;

$$\Leftrightarrow$$

$G_\varphi = \dfrac{g_\varphi}{g'_\varphi}$ with g_φ such that $g_\varphi\left(\widetilde{Y}_{\varphi,+}(x)\right) g_\varphi\left(\widetilde{Y}_{\varphi,-}(x)\right) = 1$ for $x \in [x_1, 1)$;

$$\Leftrightarrow$$

Anti-Tutte's invariant

Idea: write $\sqrt{\dfrac{\delta_\varphi(\bar{y})}{\delta_\varphi(y)}} = \dfrac{G_\varphi(\bar{y})}{G_\varphi(y)}$, such that: $G_\varphi(\bar{y}) = \overline{G_\varphi(y)}$;

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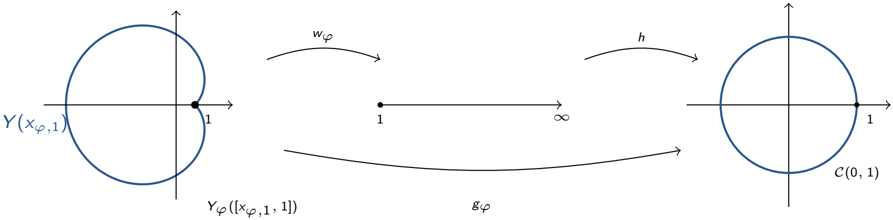
$$\Leftrightarrow$$

$$G_\varphi = \frac{g_\varphi}{g'_\varphi} \text{ with } g_\varphi \text{ such that } g_\varphi(y) g_\varphi(\bar{y}) = |g_\varphi(y)|^2 = 1 \text{ for } y \in Y([x_1, 1]).$$

Anti-Tutte's invariant



Conformal maps which transforms $Y([x_1, 1])$ into the unit circle with $\overline{g_\varphi(y)} = g_\varphi(\bar{y})$

$$g_\varphi = h\left(\frac{2\widetilde{w}_\varphi(Y_\varphi(x_{\varphi,1}))}{\widetilde{w}_\varphi(y)} - 1\right),$$

with: w_φ an explicit conformal map and $h(y) = \sqrt{y^2-1} - y$.

Generating function $D_\varphi(y)$ stated as a BVP

Functional Equation

$$K_\varphi(x,y)L_\varphi(x,y) = -\left[x\widetilde{\alpha}_\varphi(y) + \frac{1}{2}\widetilde{\beta}_\varphi(y)\right] D_\varphi(y) + K_\varphi(x,0)L_\varphi(x,0) + \frac{1}{2}p_{1,1}f(1,1).$$

Boundary value problem

By evaluating the functional equation on the branch curves, we can transform the functional equation into the following **boundary value problem**:

For $y \in Y_\varphi([x_1,1]) \setminus \{1\}$,

$$G_\varphi(y)D_\varphi(y) - G_\varphi(\bar{y})D_\varphi(\bar{y}) = 0.$$

Expression for the Diagonal Section

The diagonal section is defined by:

$$D(x, y) = D_\varphi(xy) = \sum_{i\geq 1} f(i, i)x^{i-1}y^{i-1}.$$

Theorem [T., 2019]

The diagonal section of discrete harmonic functions *not necessarily positive* can be expressed as

$$D(x, y) = \frac{P(\widetilde{w}_\varphi(xy))}{L_\varphi(xy)}, \quad P \in \mathbb{R}[y].$$

In particular, taking P of degree 1, we get the *unique positive* discrete harmonic function.

Expression for the Diagonal Section

Expression for the Diagonal Section

$$D_\varphi(y) = -\frac{f(1,1)}{\widetilde{w}'_\varphi(0)} \frac{\pi}{\theta_\varphi} \sqrt{-\frac{\widetilde{\delta}''_\varphi(1)}{2\widetilde{\delta}_\varphi(y)}} \sqrt{1-\widetilde{W}_\varphi(0)} \sqrt{\widetilde{W}_\varphi(y)},$$

with θ_φ an explicit angle, $\widetilde{w}_\varphi(y)$ and $\widetilde{W}_\varphi$ are a conformal mappings, all depending on the step set.

Expression for the Diagonal Section

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$$D_\varphi(y) = -\frac{f(1,1)}{\widetilde{w}'_\varphi(0)}\frac{\pi}{\theta_\varphi}\sqrt{-\frac{\widetilde{\delta}''_\varphi(1)}{2\widetilde{\delta}_\varphi(y)}}\sqrt{1-\widetilde{W}_\varphi(0)}\sqrt{\widetilde{W}_\varphi(y)},$$

with θ_φ an explicit angle, $\widetilde{w}_\varphi(y)$ and $\widetilde{W}_\varphi$ are a conformal mappings, all depending on the step set.

Simple Walks

$$D_\varphi(y) = f(1,1)\frac{3}{8}\frac{\sqrt{\widetilde{W}_\varphi(y)}}{\sqrt{\widetilde{\delta}_\varphi(y)}} = f(1,1)\left(1 + \frac{44}{27}y + \frac{523}{243}y^2 + \frac{17168}{6561}y^3 + O\left(y^4\right)\right).$$

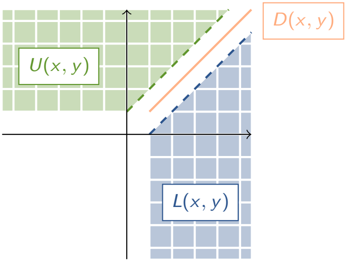
Expression for the generating functions

Remember - Functional Equation

$$K_\varphi(x,y)L_\varphi(x,y) = -\left[x\widetilde{\alpha}_\varphi(y) + \frac{1}{2}\widetilde{\beta}_\varphi(y)\right] D_\varphi(y) + K_\varphi(x,0)L_\varphi(x,0) + \frac{1}{2}p_{1,1}f(1,1).$$

Remember - Domain in three parts

$$H(x,y) = L(x,y) + D(x,y) + U(x,y).$$



Symmetry of the cut and the walk

$$\Rightarrow U(x,y) = L(y,x).$$

Expression for the generating functions

Remember - Functional Equation

$$K_\varphi(x,y)L_\varphi(x,y) = -\left[x\widetilde{\alpha}_\varphi(y) + \frac{1}{2}\widetilde{\beta}_\varphi(y)\right] D_\varphi(y) + K_\varphi(x,0)L_\varphi(x,0) + \frac{1}{2}p_{1,1}f(1,1).$$

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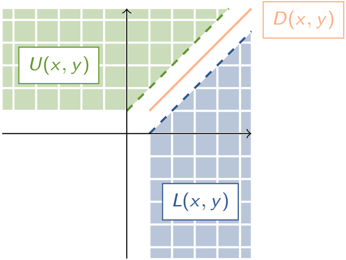

- We have an expression of $D_\varphi(y)$;

Expression for the generating functions

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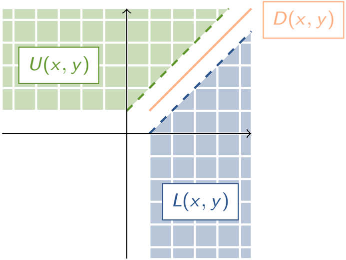

- We have an expression of $D_\varphi(y)$;
- With the functional equation we get an expression of $L_\varphi(x,y)$;

Expression for the generating functions

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$$K_\varphi(x,y)L_\varphi(x,y) = -\left[x\widetilde{\alpha}_\varphi(y) + \frac{1}{2}\widetilde{\beta}_\varphi(y)\right] D_\varphi(y) + K_\varphi(x,0)L_\varphi(x,0) + \frac{1}{2}p_{1,1}f(1,1).$$

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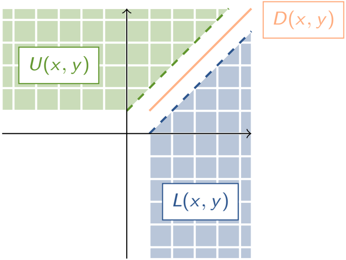


- We have an expression of $D_\varphi(y)$;
- With the functional equation we get an expression of $L_\varphi(x,y)$;
- With a change of variable we get an expression of $D(x,y)$ and $L(x,y)$;

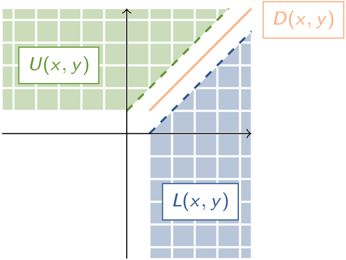
Expression for the generating functions

Remember - Functional Equation

$$K_\varphi(x,y)L_\varphi(x,y) = -\left[x\widetilde{\alpha}_\varphi(y) + \frac{1}{2}\widetilde{\beta}_\varphi(y)\right] D_\varphi(y) + K_\varphi(x,0)L_\varphi(x,0) + \frac{1}{2}p_{1,1}f(1,1).$$

Remember - Domain in three parts

$$H(x,y) = L(x,y) + D(x,y) + L(y,x).$$



- We have an expression of $D_\varphi(y)$;
- With the functional equation we get an expression of $L_\varphi(x,y)$;
- With a change of variable we get an expression of $D(x,y)$ and $L(x,y)$;
- Then we have an expression of $H(x,y)$.

Asymptotics

Expression for the Diagonal Section

$$D_\varphi(y) = -\frac{f(1,1)}{\widetilde{w}'_\varphi(0)}\frac{\pi}{\theta_\varphi}\sqrt{-\frac{\widetilde{\delta}''_\varphi(1)}{2\widetilde{\delta}_\varphi(y)}}\sqrt{1-\widetilde{W}_\varphi(0)}\sqrt{\widetilde{W}_\varphi(y)},$$

with θ_φ an explicit angle, $\widetilde{w}_\varphi$ and $\widetilde{W}_\varphi$ conformal mappings, all depending on the step set.

Asymptotics

Expression for the Diagonal Section

$$D_\varphi(y) = -\frac{f(1,1)}{\widetilde{w}'_\varphi(0)}\frac{\pi}{\theta_\varphi}\sqrt{-\frac{\widetilde{\delta}''_\varphi(1)}{2\widetilde{\delta}_\varphi(y)}}\sqrt{1-\widetilde{W}_\varphi(0)}\sqrt{\widetilde{W}_\varphi(y)},$$

with θ_φ an explicit angle, $\widetilde{w}_\varphi$ and $\widetilde{W}_\varphi$ conformal mappings, all depending on the step set.

$$D_\varphi(y) = C \cdot \frac{\sqrt{\widetilde{W}_\varphi(y)}}{\sqrt{\widetilde{\delta}_\varphi(y)}}$$

Asymptotics

Angle of the step set

$$\theta = \text{arccos}\left(- \frac{\sum_{-1\leq i,j\leq 1} ijp_{i,j}}{\sqrt{\left(\sum_{-1\leq i,j\leq 1} i^2 p_{i,j}\right)\cdot\left(\sum_{-1\leq i,j\leq 1} j^2 p_{i,j}\right)}} \right); \qquad \theta_\varphi = \pi - \frac{\theta}{2}$$

Example of the simple walks

$\theta = \frac{\pi}{2}; \qquad \theta_\varphi = \pi - \frac{\pi/2}{2} = \frac{3\pi}{4}$ (Gessel).

Asymptotics

Angle of the step set

$$\theta = \arccos\left(-\frac{\sum_{-1\le i,j\le 1} ijp_{i,j}}{\sqrt{\left(\sum_{-1\le i,j\le 1} i^2 p_{i,j}\right)\cdot\left(\sum_{-1\le i,j\le 1} j^2 p_{i,j}\right)}} \right); \qquad \theta_\varphi = \pi - \frac{\theta}{2}$$

Example of the simple walks

$\theta = \frac{\pi}{2}; \qquad \theta_\varphi = \pi - \frac{\pi/2}{2} = \frac{3\pi}{4}$ (Gessel).

$$W_\varphi(y) = \frac{c + o(1)}{(1-y)^{\pi/\theta_\varphi}} \Rightarrow D_\varphi(y) = -\frac{f(1,1)}{\widetilde{w}'_\varphi(0)}\frac{\pi}{\theta_\varphi}\sqrt{-\frac{\widetilde{\delta}''_\varphi(1)}{2\widetilde{\delta}_\varphi(y)}}\sqrt{1-\widetilde{W}_\varphi(0)}\sqrt{\widetilde{W}_\varphi(y)}$$

$$= \frac{c_D + o(1)}{(1-y)^{\pi/(2\pi-\theta)+1}} \text{ for } y \text{ close to } 1$$

Asymptotics

For y close to 1

In the three-quadrant

$$D_\varphi(y) = \frac{c_D + o(1)}{(1-y)^{\pi/(2\pi-\theta)+1}}$$

In the quadrant

$$\widetilde{D}_\varphi(y) = \frac{\widetilde{c}_D + o(1)}{(1-y)^{\pi/\theta+1}}$$

Asymptotics

For y close to 1

In the three-quadrant

$$D_\varphi(y) = \frac{c_D + o(1)}{(1-y)^{\pi/(2\pi-\theta)+1}}$$

In the quadrant

$$\widetilde{D}_\varphi(y) = \frac{\widetilde{c}_D + o(1)}{(1-y)^{\pi/\theta+1}}$$

Theorem [Mustapha, 2019]

Let $\alpha_\mathcal{Q} = \frac{\pi}{\theta}$ be the critical exponent of walks in the quadrant. Then the critical exponent $\alpha_\mathcal{C}$ of walks in the three-quadrant can be expressed as

$$\alpha_\mathcal{C} = \frac{\alpha_\mathcal{Q}}{2\alpha_\mathcal{Q}-1} = \frac{\pi}{2\pi-\theta}.$$

Content

Non-positive harmonic functions

Expression for the generating function

$$D(x,y) = \frac{P(\widetilde{w}_\varphi(xy))}{G_\varphi(xy)}, \quad P \in \mathbb{R}[y].$$

More generally, for **any polynomial** P of degree n we get discrete harmonic functions (but not necessarily positive).

Non-positive harmonic functions

Expression for the generating function

$$D(x, y) = \frac{P(\widetilde{w}_\varphi(xy))}{G_\varphi(xy)}, \quad P \in \mathbb{R}[y].$$

More generally, for **any polynomial** P of degree n we get discrete harmonic functions (but not necessarily positive).

Simple walks

$$P(y) = \frac{3}{4}y^2 - \frac{9}{16}$$

$$D_\varphi(y) = 1 + 4\,y + 9\,y^2 + 16\,y^3 + 25\,y^4 + O\left(y^5\right)$$

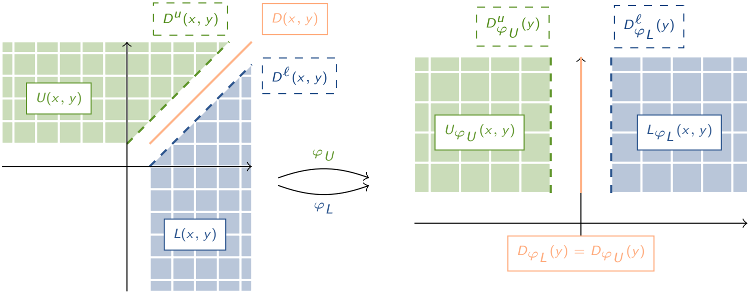
We find back the harmonic function $f(i, j) = ij$
(non-positive in the three-quadrant)

Non-positive harmonic functions

- ➙ Is every harmonic function completely determined by the polynomial P?
- ➙ What is the structure of non-positive harmonic functions?
- ➙ How does the cone of restriction affect this structure?
- ➙ What are the properties of non-positive harmonic functions?

É. Fusy, K. Raschel, P. Tarrago and A. T.

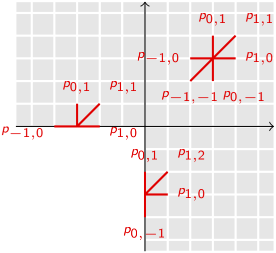
Non-symmetric case

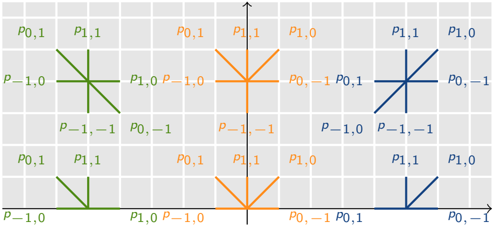


$$\begin{cases} K_{\varphi_U}(x,y)U_{\varphi_U}(x,y) &= -\left(p_{1,0}x^{-1}y + p_{0,-1}x^{-1}y^2 + p_{1,1} + p_{-1,-1}y^2 - y\right) D_{\varphi_U}(y) \\ &\quad + \left(p_{0,1}x^2 + p_{1,1}x\right) U_{\varphi_U}(x,0) - \left(p_{1,0} + p_{0,-1}y\right) D^\ell_{\varphi_L}(y) \\ &\quad + p_{1,1}f(1,1) + p_{1,0}f(1,0), \\ K_{\varphi_L}(x,y)L_{\varphi_L}(x,y) &= -\left(p_{0,1}xy + p_{-1,0}xy^2 + p_{1,1} + p_{-1,-1}y^2 - y\right) D_{\varphi_L}(y) \\ &\quad + \left(p_{1,0} + p_{1,1}x\right) L_{\varphi_L}(x,0) - \left(p_{0,1} + p_{-1,0}y\right) D^u_{\varphi_U}(y) \\ &\quad + p_{1,1}f(1,1) + p_{0,1}f(0,1). \end{cases}$$

Non-symmetric case

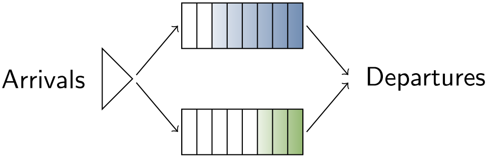
Change of variables $\varphi_L(x,y) = (xy, x^{-1})$ **&** $\varphi_U(x,y) = (x, x^{-1}y)$

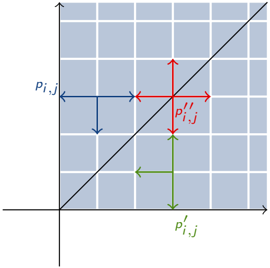




Random walks avoiding a quadrant can be seen as inhomogeneous walks in the half plane with probability transitions $\varphi_U((p_{i,j})_{-1\leq i,j\leq 1})$ in the left quadrant and $\varphi_L((p_{i,j})_{-1\leq i,j\leq 1})$ on the right quadrant.

Aside: Join-the-Shortest-Queue model (JSQ)





We consider a model with two queues in which the customers choose the shortest queue (if the two queues have same length, then the customers choose a queue according to a fixed probability law).

References

References

S. Mustapha, *Non-D-Finite Walks in a Three-Quadrant Cone*, Ann. Comb. **23** (2019), no. 1, 143–158. MR 3921340

K. Raschel, *Random walks in the quarter plane, discrete harmonic functions and conformal mappings*, Stochastic Process. Appl. **124** (2014), no. 10, 3147–3178, With an appendix by S. Franceschi. MR 3231615

A. Trotignon, *Discrete harmonic functions in the three-quarter plane*, arXiv **1906.08082** (2019), 1–26.