Integrality of instanton numbers

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Mirror symmetry is a relationship between of Calabi–Yau manifolds. Two such manifolds may look very different geometrically but are nevertheless equivalent when employed as 'extra dimensions' to describe interaction of particles in string theory.

Beginnings of mirror symmetry

P. Candelas, X. de la Ossa, P. Green, L. Parkes, *An exactly soluble superconformal theory from a mirror pair of Calabi–Yau manifolds*, Phys. Lett. B 258 (1991), no.1–2, 118–126

$$L = \theta^4 - 5^5 t \left(\theta + \tfrac{1}{5}\right)\left(\theta + \tfrac{2}{5}\right)\left(\theta + \tfrac{3}{5}\right)\left(\theta + \tfrac{4}{5}\right), \quad \theta = t\tfrac{d}{dt}$$

The differential equation $Ly = 0$ has solutions

$$y_0(t) = \sum_{n=0}^{\infty} \frac{(5n)!}{n!^5} t^n = 1 + 120t + 113400t^2 + \ldots =: f_0(t) \in \mathbb{Z}[\![t]\!]$$

and

$$y_1(t) = f_0(t)\log(t) + f_1(t), \quad f_1(t) := \sum_{n=1}^{\infty} \frac{(5n)!}{n!^5} \left(\sum_{j=1}^{5k} \frac{5}{j}\right) t^k \in t\mathbb{Q}[\![t]\!]$$

Observation: $q(t) := \exp\left(\frac{y_1(t)}{y_0(t)}\right) = t\exp\left(\frac{f_1(t)}{f_0(t)}\right) \in t\mathbb{Z}[\![t]\!]$
(proved by B.-H.Lian and S.-T.Yau in 1996)

Canonical coordinate and Yukawa coupling

$q(t) = \exp\left(y_1(t)/y_0(t)\right) = t + 770t^2 + 1014275t^3 +$
is called the *canonical coordinate* or *mirror map*.

Solutions to $Ly = 0$:

$$y_0(t) = f_0, \quad y_1(t) = f_0 \log(t) + f_1,$$
$$y_2(t) = f_0 \frac{\log(t)^2}{2!} + f_1 \log(t) + f_2, \quad f_2 \in t\mathbb{Q}[\![t]\!]$$

Express the ratios y_i/y_0 in terms of $q = q(t)$:

$$\frac{y_0}{y_0} = 1, \quad \frac{y_1}{y_0} = \log(q),$$
$$\frac{y_2}{y_0} = \frac{1}{2}\log(q)^2 + 575q + \frac{975375}{4}q^2 + + \frac{1712915000}{9}q^3 + \dots$$

$Y(q) := \left(q\frac{d}{dq}\right)^2 \frac{y_2}{y_0} = 1 + 575q + 975375q^2 + \dots$
is called the *Yukawa coupling*.

Physics wins!

$$Y(q) = \left(q\frac{d}{dq}\right)^2 \frac{y_2}{y_0} = 1 + 575q + \ldots = \frac{1}{5}\sum_{d\geq 0} n_d d^3 \frac{q^d}{1-q^d}$$

$$n_0 = 5, \quad n_1 = 2875, \quad n_2 = 609250,$$

$$n_3 = 317206375, \quad n_4 = 242467530000, \ldots$$

are called *instanton numbers*.

Observation / prediction: The numbers n_d coincide with the numbers of degree d rational curves that lie on a generic threefold of degree 5 in $\mathbb{P}^4$.

Only the first two numbers were known at that time! The number 2875 of lines on a general quintic was determined by H. Schubert in 1886. The number 609250 of conics was determined by S. Katz in 1986. In 1993 G.Ellingsrud and S.Strømme computed the number of cubic curves on the quintic threefold. Their result served as a crucial cross-check for the above physicists' prediction.

Integrality of instanton numbers

$$L = \theta^4 - 5^5 t \left(\theta + \tfrac{1}{5}\right)\left(\theta + \tfrac{2}{5}\right)\left(\theta + \tfrac{3}{5}\right)\left(\theta + \tfrac{4}{5}\right), \quad \theta = t\tfrac{d}{dt}$$

$$y_0 = f_0,\ y_1 = f_0 \log(t) + f_1,\ y_2 = f_0 \tfrac{\log(t)^2}{2!} + f_1 \log(t) + f_2$$

$$q = \exp(y_1/y_0), \quad Y(q) = \left(q\tfrac{d}{dq}\right)^2 (y_2/y_0) = \frac{1}{5}\sum_{d\geq 0} n_d d^3 \tfrac{q^d}{1-q^d}$$

In 1990s *Gromov–Witten theory* was developed to provide a rigorous basis for counting curves on general manifolds (Kontsevich–Manin). Subsequently, Givental and Lian–Liu–Yau proved the *mirror theorem* which justified the equality of instanton numbers and genus zero Gromov–Witten invariants.
Conjecture: $n_d \in \mathbb{Z}$ for every d.

Theorem (MV–Frits Beukers, 2020) For the quintic case, the denominators of instanton numbers n_d can only have prime divisors 2, 3, 5.

More differential operators like this?

A 4th order differential operator $L \in \mathbb{Q}[t, \frac{d}{dt}]$ is called a *Calabi–Yau operator* if:

- its singularities are regular
- $t = 0$ is a point of maximally unipotent monodromy (MUM)

$$L = \theta^4 + \sum_{j=1}^{4} a_j(t)\theta^{4-j}, \quad \theta = t\frac{d}{dt}, \quad a_j(0) = 0,\ 1 \le j \le 4$$

- it is self-dual
- it satisfies the integrality conditions:
  - the holomorphic solution $y_0(t) \in \mathbb{Z}[\![t]\!]$
  - the canonical coordinate $q = \exp(y_1/y_0) \in \mathbb{Z}[\![t]\!]$
  - the instanton numbers $n_d \in \mathbb{Z}$

If one allows N-integrality instead of integrality, about 500 such operators were found *experimentally*:

G. Almkvist, C. van Enckevort, D. van Straten, W. Zudilin, Tables of Calabi–Yau operators (arXiv:math/0507430) *"AESZ tables"* (2010)

D. van Straten, *Calabi–Yau operators* in Adv. Lect. Math. 42 (2018)

Towards the proof of integrality of instanton numbers

Lemma. For a power series $Y(q) \in \mathbb{Q}[\![q]\!]$, consider the Lambert expansion

$$Y(q) = \sum_{d \geq 0} a_d \frac{q^d}{1 - q^d}.$$

Take a prime number p. Suppose $\exists\, \phi \in \mathbb{Z}_p[\![q]\!]$ such that

$$Y(q^p) - Y(q) = \left(q \frac{d}{dq}\right)^s \phi(q).$$

Then $a_d / d^s \in \mathbb{Z}_p$ for all $d \geq 1$.

Towards the proof of integrality of instanton numbers

Take $s = 3$ and write the respective $\phi \in \mathbb{Q}[\![q]\!]$ explicitly:

$$\sum_{d \geq 1} n_d d^3 \frac{q^d}{1 - q^d} = \left(q \frac{d}{dq}\right)^3 Z, \quad Z(q) = \sum_{d \geq 1} n_d Li_3(q^d) \in \mathbb{Q}[\![q]\!]$$

$$\phi := p^{-3} Z(q^p) - Z(q) \overset{??}{\in} \mathbb{Z}_p[\![q]\!]$$

J. Stienstra, *Ordinary Calabi–Yau–3 Crystals*, Fields Inst. Commun., 38 (2003): one can prove p-integrality of ϕ by relating it to a matrix coefficient of the p-adic *Frobenius structure* for the differential operator L

M. Kontsevich, A. Schwarz, V. Vologodsky, *Integrality of instanton numbers and p-adic B-model*, Phys. Lett. B 637 (2006), no. 1–2

V. Vologodsky, *On the N-integrality of instanton numbers*, arXiv:0707.4617

Frobenius structure

A p-adic Frobenius structure is an equivalence between the differential system corresponding to L and its pullback under the change of variable $t \mapsto t^p$, over the field $E_p = \widehat{\mathbb{Q}(t)}$ of p-adic analytic elements (Dwork).

$$L = \theta^4 + \sum_{j=1}^{4} a^j(t)\theta^{4-j} \qquad \text{with MUM point at } t = 0$$

$$y_0 = f_0,\ y_1 = f_0 \log(t) + f_1, y_2 = f_0 \frac{\log(t)^2}{2!} + f_1 \log(t) + f_2$$

$$y_3 = f_0 \tfrac{\log(t)^3}{3!} + f_1 \tfrac{\log(t)^2}{2!} + f_2 \log(t) + f_3,\ f_i \in \mathbb{Q}[\![t]\!]$$

$$U = \left(\theta^i y_j\right)_{i,j=0}^{3} \text{ fundamental solution matrix}$$

Are there constants $\alpha_0, \alpha_1, \alpha_2, \alpha_3 \in \mathbb{Q}_p$ such that

$$\Phi(t) = U(t) \begin{pmatrix} \alpha_0 & p\alpha_1 & p^2\alpha_2 & p^3\alpha_3 \\ 0 & p\alpha_0 & p^2\alpha_1 & p^3\alpha_2 \\ 0 & 0 & p^2\alpha_0 & p^3\alpha_2 \\ 0 & 0 & 0 & p^3\alpha_0 \end{pmatrix} U(t^p)^{-1} \in E_p \quad ?$$

Frobenius structure

$$U = \left(\theta^i y_j\right)_{i,j=0}^{3} \text{ fundamental solution matrix for } L$$

$$\Phi(t) = U(t) \begin{pmatrix} \alpha_0 & p\alpha_1 & p^2\alpha_2 & p^3\alpha_3 \\ 0 & p\alpha_0 & p^2\alpha_1 & p^3\alpha_2 \\ 0 & 0 & p^2\alpha_0 & p^3\alpha_2 \\ 0 & 0 & 0 & p^3\alpha_0 \end{pmatrix} U(t^p)^{-1} \in \mathbb{Q}[\![t]\!]^{4\times 4}$$

Definition. We say that L has a p-adic Frobenius structure if there exist p-adic constants $\alpha_0 = 1, \alpha_1, \alpha_2, \alpha_3 \in \mathbb{Z}_p$ such that

$$\Phi_{ij} \in p^j \mathbb{Z}_p[\![t]\!], \quad 0 \le i, j \le 3.$$

Conjecturably, Calabi-Yau differential operators have p-adic Frobenius structure for almost all p.

D. van Straten, *CY-operators and L-functions*, in proceedings of the workshop *Hypergeometric Motives and Calabi–Yau Differential Equations*, 2017 MATRIX Annals, 491–503

p-Integrality of instanton numbers

$L = \theta^4 + a_1(t)\theta^3 + a_2(t)\theta^2 + a_3(t)\theta + a_4(t)$
$a_i(0) = 0, i = 1, \ldots, 4$ (MUM point at $t = 0$)

Theorem (MV-Frits Beukers, 2020). Suppose that a p-adic Frobenius structure exists for L. Then

- the holomorphic solution is p-integral: $y_0 \in \mathbb{Z}_p[\![t]\!]$
- the canonical coordinate is p-integral: $q = \exp(y_1/y_0) \in \mathbb{Z}_p[\![t]\!]$
- if in addition L is self-dual and $\alpha_1 = 0$, then the instanton numbers of L are p-integral: $n_d \in \mathbb{Z}_p$ for all $d \geq 1$

In the latter case, the series ϕ such that $Y(q^p) - Y(q) = (q\frac{d}{dq})^3\phi$ is basically given by the top right Frobenius matrix entry: $\phi \approx p^{-3}\Phi_{03}$.

The hard part: working out explicit examples

Given $L = \theta^4 + \ldots$, we would like to construct Φ and show that $\alpha_1 = 0$. We need a geometric model, a family of hypersurfaces whose periods are solutions of L.

- Find $g(\mathbf{x}) \in \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ such that

$$y_0(t) = \frac{1}{(2\pi i)^n} \oint \cdots \oint \frac{1}{1 - tg(\mathbf{x})} \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}$$

  e.g. $n = 4$, $g(\mathbf{x}) = x_1 + x_2 + x_3 + x_4 + \frac{1}{x_1 x_2 x_3 x_4}$
  $L = \theta^4 - (5t)^5 (\theta + 1)(\theta + 2)(\theta + 3)(\theta + 4)$

More generally, consider a Laurent polynomial $f(\mathbf{x})$ with coefficients in $\mathbb{Z}[t]$.
$X_f = \{f(\mathbf{x}) = 0\} \subset \mathbb{T}^n$ toric hypersurface

- Look for a subquotient in the de Rham cohomology of $\mathbb{T}^n \setminus X_f$ which, after taking a suitable localization R of $\mathbb{Z}[t]$, is a free R-module generated by $\theta^j (\frac{1}{f(\mathbf{x})}) \frac{d\mathbf{x}}{\mathbf{x}}$, $j = 0, 1, 2, 3$.

Cohomology and differential forms

$f(\mathbf{x}) \in R[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, R is a localization of $\mathbb{Z}[t]$
$X_f = \{f(\mathbf{x}) = 0\} \subset \mathbb{T}^n$ toric hypersurface

$\Delta \subset \mathbb{R}^n$ Newton polytope of $f(\mathbf{x})$

$$\Omega_f := \left\{ \frac{h(\mathbf{x})}{f(\mathbf{x})^m} \;\middle|\; \begin{array}{l} m \geq 1, h \in R[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \\ supp(h) \subset m\Delta \end{array} \right\} \quad R\text{-module}$$

$H^n_{DR}(\mathbb{T}^n \setminus X_f) = \Omega_f \frac{d\mathbf{x}}{\mathbf{x}} / \{ \text{ exact forms } \}$ (Griffiths, Batyrev)

In our examples, we construct free subquotients generated by derivatives $\theta^i(1/f)$, $i = 0, 1, \dots$ by restricting to numerators $h(\mathbf{x})$ supported in the interior $m\Delta^\circ$ and, in some cases, by taking the G-invariant part, where $G \subset GL_n(\mathbb{Z})$ is a finite group of monomial substitutions preserving Δ and $f(\mathbf{x})$.

- $M := \Omega_f^\circ \frac{d\mathbf{x}}{\mathbf{x}} / \{ \text{ exact forms } \} = \Omega_f^\circ / \sum_{i=1}^n x_i \frac{\partial}{\partial x_i}(\Omega_f)$
  $= \Omega_f^\circ / \{ \textit{ partial derivatives } \}$
- $M \cong \oplus_{j=0}^3 R\, \theta^j(1/f)$ and $L(1/f) = 0$ in M

Frobenius structure

In "Dwork crystals I" we construct the R-linear Cartier operator

$$\mathcal{C}_p : \frac{h(\mathbf{x})}{f(\mathbf{x})^m} = \sum a_{\mathbf{u}} \mathbf{x}^{\mathbf{u}} \frac{d\mathbf{x}}{\mathbf{x}} \mapsto \sum a_{p\mathbf{u}} \mathbf{x}^{\mathbf{u}} \notin \Omega_f$$

These are formal expansions with respect to a vertex $\mathbf{b}$ of Δ, but the right-hand side usually doesn't belong to Ω_f.

Let $\sigma : R \to R$ be a Frobenius lift: a ring endomorphism such that $\sigma(r) \equiv r^p \mod p$ for any $r \in R$, like $\sigma : t \mapsto t^p$ on $\mathbb{Z}[t]$. Lemma: the above $\sum a_{p\mathbf{u}} \mathbf{x}^{\mathbf{u}}$ belongs to the p-adic completion of Ω_{f^σ}.

Our construction gives an R-linear operator on p-adic completions

$$\mathcal{C}_p : \widehat{\Omega}_f \to \widehat{\Omega}_{f^\sigma},$$

which commutes with $\theta = t\frac{d}{dt}$ and maps exact forms to exact forms. It descends to our rank 4 free modules generated by $\theta^i(1/f)$, $i = 0, 1, 2, 3$:

- Matrix of $\mathcal{C}_p : M \to M^\sigma$ is the Frobenius structure $\Phi \in \widehat{R}^{4\times 4}$ for our Picard–Fuchs differential operator L.

Evaluation of Φ at $t = 0$?

Our Frobenius matrix $\Phi \in \mathbb{Z}_p[\![t]\!]^{4\times 4}$ is defined explicitly. Still, a direct computation of $\Phi_{0j}(0) = p^j \alpha_j$ would be notoriously difficult.[1]

$$\begin{aligned}
\mathcal{C}_p\left(\frac{1}{f(\mathbf{x})}\right) &= \sum_{m\geq 1} \frac{h_m(\mathbf{x})}{f^\sigma(\mathbf{x})^m} \\
&\equiv \sum_{j=0}^{3} \lambda_j(t)\, \theta^j \left(\frac{1}{f}\right)^\sigma \quad \mathrm{mod} \quad \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}(\widehat{\Omega}_{f^\sigma}) \\
\lambda_j &= \Phi_{0j}
\end{aligned}$$

We will now explain a trick which allows to jump over the infinite reduction process in the right-hand side. It will be possible to access the coefficients λ_j directly, through certain p-adic congruences.

[1] A direct computation of this sort was done by I. Shapiro in *Frobenius map for quintic threefolds*, Int. Math. Res. Not. 2009, no. 13, 2519–2545.

Theorem (MV-Frits Beukers, 2019) Assume that R is p-adically complete and the k'th *Hasse–Witt condition* is satisfied. Then

$$\widehat{\Omega}_f = \Omega_f(k) \oplus \mathcal{F}_k,$$

where

$$\Omega_f(k) = \text{free } R\text{-module generated by } \frac{\mathbf{x}^{\mathbf{u}}}{f(\mathbf{x})^k},\ \mathbf{u} \in k\Delta$$

and

$$\begin{aligned}\mathcal{F}_k &= \left\{\omega = \sum a_{\mathbf{u}}\mathbf{x}^{\mathbf{u}} \in \widehat{\Omega}_f \;\middle|\; \forall \mathbf{u} \quad a_{\mathbf{u}} \in g.c.d.(u_1,\ldots,u_n)^k R\right\} \\ &= \left\{\omega = \sum a_{\mathbf{u}}\mathbf{x}^{\mathbf{u}} \in \widehat{\Omega}_f \;\middle|\; \forall s \geq 1 \quad \mathcal{C}_p^s(\omega) \in p^{ks}\widehat{\Omega}_{f^{\sigma^s}}\right\}.\end{aligned}$$

For $k=1$ this result is a version of N. Katz's *Internal reconstruction of unit-root F-crystals via expansion coefficients* (1985).

Note: $\mathcal{F}_k = \{\text{formal } k\text{th partial derivatives}\}$.

Proof of vanishing of α_1

$$\Omega_f^\circ = \left\{ \frac{h(\mathbf{x})}{f(\mathbf{x})^m} \;\middle|\; m \geq 1,\ supp(h) \subset m\Delta^\circ \right\}$$

$$M = \Omega_f^\circ / \{ \text{ partial derivatives } \} \cong \oplus_{j=0}^{3} R\, \theta^j (1/f)$$

$$\begin{array}{ll} \{ \text{ partial derivatives } \} \subset & \mathcal{F}_1 = \{ \text{ formal partial derivatives } \} \\ & \cup \\ & \mathcal{F}_2 = \{ \text{ formal 2nd partial derivatives } \} \end{array}$$

In the quintic case and several other cases which have geometric models with sufficiently large symmetry group G, one has

$$\{ \text{ partial derivatives } \} \cap (\Omega_f^\circ)^G \subset \mathcal{F}_2.$$

Proof of vanishing of α_1

$$M = \Omega_f^\circ / \{ \text{ partial derivatives } \} \cong \oplus_{j=0}^3 R\, \theta^j (1/f)$$
$$M/\mathcal{F}_2 = R\, 1/f + R\, \theta(1/f)$$

$\rightsquigarrow$

$$\mathcal{C}_p(1/f) = \sum_{j=0}^{3} \lambda_j(t) \theta^j (1/f)^\sigma \quad \text{modulo partial derivatives}$$
$$= \mu_0(t) 1/f^\sigma + \mu_1(t)\theta(1/f)^\sigma \quad \text{mod } \mathcal{F}_2$$
$$\mu_0(0) = \lambda_0(0), \quad \mu_1(0) = \lambda_1(0)$$

For the expansion coefficients $\frac{1}{f(\mathbf{x})} = \sum a_{\mathbf{u}}(t)\mathbf{x}^{\mathbf{u}}$ this yields congruences

$$a_{p^{s+1}\mathbf{u}}(t) \equiv \mu_0(t) a_{p^s\mathbf{u}}(t^p) + \mu_1(t)(\theta a_{p^s\mathbf{u}})(t^p) \quad \text{mod } p^{2s}.$$

These explicit congruences allow us to check the vanishing of $\mu_1(0) = p\alpha_1$, which concludes the proof of integrality of instanton numbers.

Examples

In addition to the quintic case, we could prove p-integrality of instanton numbers for the following Calabi–Yau differential operators for all $p \geq 5$:

$$AESZ\#8 \qquad g(\mathbf{x}) = x_1 + x_2 + x_3 + x_4 + \frac{1}{x_1^2 x_2 x_3 x_4}$$

$$L = \theta^4 - 108^2 t^6 (\theta+1)(\theta+2)(\theta+4)(\theta+5)$$

$$AESZ\#15 \qquad g(\mathbf{x}) = x_1 + x_2 + x_3 + x_4 + \frac{1}{x_1 x_2} + \frac{1}{x_3 x_4}$$

$$L = \theta^4 - 3^3 t^3 (\theta+1)(\theta+2)(7\theta^2 + 21\theta + 18) + 18^3 t^6 (\theta+1)(\theta+2)(\theta+4)(\theta+5)$$

$$AESZ\#16 \qquad g(\mathbf{x}) = x_1 + \frac{1}{x_1} + x_2 + \frac{1}{x_2} + x_3 + \frac{1}{x_3} + x_4 + \frac{1}{x_4}$$

$$L = (1024t^4 - 80t^2 + 1)\theta^4 + 64(128t^4 - 5t^2)\theta^3 + 16(1472t^4 - 33t^2)\theta^2 + 32(896t^4 - 13t^2)\theta + 128(96t^4 - t^2)$$

Thank you!