Soliton theory is Abelian

Hiroshi Umemura
Graduate School of Mathematics
Nagoya University
Email umemura@math.nagoya-u.ac.jp

ACA 2011 Houston, June 28, 2011

§1 Integral

According to a dictionary,
the word *integral = whole, entire or complete.*

Do we have a hidden expectation that our knowledge is complete or our world is harmonious?

Example 1. The Pythagoreans:
The universe is beautifully dominated by integers.

They encountered, however, $\sqrt{2}$ in the length of the diagonal of square. $\sqrt{2}$ is outside their universe, which means that $\sqrt{2}$ is not integrable. So this fact is unpleasant for them.

Example 2. Plato says:
Only line and circle are perfect figures.

The problem of constructions of figures using only ruler and compass was born from this idea.

Unfortunately we can not construct in this way regular heptagon. Or regular heptagon is not integrable for the Greeks.

History of mathematics is the history of encounters with non-integrability.

(1) Introductions of negative numbers and 0,
and $\sqrt{-1}$ are radical ideas of enlarging our world.

(2) Impossibility of solving general quintic equation by extraction of radicals.

(3) Irreducibility of the Painlevé equations.

(4) Discovery that most of Hamiltonian systems are non-integrable.

§2 Integrable theories

On the other hand, there are perfect theories dealing with integrable phenomena.

(1) Theorem (Gauss). *We can construct the regular 17-gon by ruler and compass.*

(2) Examples of Hamiltonian systems integrable by theta functions.

§3 Our result

Theorem. Galois group of KP-hierarchy is Abelian.

Showing
Galois theoretically Soliton theory is integrable.

Background

Theory of Morales and Ramis says that if a Hamiltonian system is integrable in the sense of Liouville and Arnold, then the Lie algebra of Galois group of the variational equation along a solution is Abelian (Poincaré and Zyglin).

Malgrange noticed that if a Hamiltonian system is integrable by theta functions, then the Lie algebra of his general Galois groupoid is Abelian.

§4 KdV equation

J. S. Russel observed propagation of solitary waves in the canal 1834.
The KdV equation, which is the oldest soliton equation, is a mathematical model of Russel's observation.

$$4u_t = 12uu_x + u_{xxx}$$

proposed in 1895,

Later in the trial of solving the KdV by inverse scattering method, it was revealed that the KdV equation is written in the following Lax form

$$\frac{\partial L}{\partial t} = [B,\, L],$$

where

$$L = \partial^2 + 2u \text{ and } B = \partial^3 + 3u\partial + \frac{3}{2}u_x$$

are differential operators and

$$[B,\, L] := BL \,-\, LB,$$

$$u = u(x,\, t)$$

being a function of x and t and

$$\partial = \partial/\partial x.$$

§5 Spectral preserving deformation

General scheme is

Spectral preserving deformation of a Linear differential operator
$\Rightarrow$ Lax equation

We consider a system of differential equations

$$\begin{cases} L\psi = \lambda\psi, \\ \frac{\partial\psi}{\partial t} = B\psi, \end{cases}$$

where L and B are linear ordinary differential operators with respect to derivation ∂_x parametrized by t,
ψ is a function of x, t,
λ is a function of t,
the function $\psi = \psi(x, t)$ is an eigen function of the linear operator L with eigen value $\lambda(t)$.

We further assume that the eigen value λ is independent of t. So $\partial_t \lambda = 0$ and consequently λ is a constant function.

Therefore the differential system describes a deformation of the ordinary linear differential operator L parametrized by t with the eigen function $\psi(x, t)$ in such a way that the eigen value λ does not depend on the parameter t.

The Lax equation

$$\frac{\partial L}{\partial t} = [B, L]$$

arises from the compatibility condition

$$\partial_x \partial_t = \partial_t \partial_x$$

of the operators ∂_x and ∂_t.

KP-heirarchy is an infinite set of Lax equations

$$\frac{\partial}{\partial t_n} L = [B_n, L], \qquad n = 1, 2, 3, \cdots .$$

Sato solves KP-hierarchy by a flow on the infinite dimensional Grassmann variety $\mathbb{GM}(m, \infty)$.

Theorem. Galois group of the foliation on the Grassmann variety is Abelian.

Soliton theory is integrable.

§6 Pseudo-differential operators

Let R be a commutative $\mathbb{Q}$-algebra and $\partial : R \to R$ a derivation so that

$$\partial(a+b) = \partial(a) + \partial(b) \text{ and } \partial(ab) = \partial(a)b + a\partial(b)$$

for any two elements $a, b \in R$.

An element $a \in R$ defines a linear operator

$$a : R \to R, \qquad u \mapsto au.$$

Commutation relation of the operators
$\partial : R \to R$ and $a : R \to R$ is given by

$$\partial a = \partial(a) + a\partial$$

Leibniz rule.

More generally, we know

$$\partial^n a = \sum_{i=0}^{n} \binom{n}{i} \partial^i(a)\partial^{n-i}$$

for a non-negative integer n.

$$R[\partial] := \{L = a_n\partial^n + a_{n-1}\partial^{n-1} + \cdots + a_0 \,|\, a_i \in R \text{ for } 0 \leq i \leq n, \text{ the order } n \text{ being dependent on } L\} \quad (1)$$

Non-commutative ring.

Now we introduce the formal inverse

$$\partial^{-1} \leftrightarrow \int$$

$$\partial a = \partial(a) + a\partial$$

Applying ∂^{-1} from left and right,

$$a\partial^{-1} = \partial^{-1}(\partial(a))\partial^{-1} + \partial^{-1}a$$

or

$$\partial^{-1}a = a\partial^{-1} - \partial^{-1}(\partial(a))\partial^{-1}.$$

We arrive at the commutation relation

$$\partial^{-1}a = \sum_{i=0}^{\infty} \binom{-1}{i} \partial^{i}(a)\partial^{n-i}$$

for every element $a \in R$.

General formula

$$\partial^{n}a = \sum_{i=0}^{\infty} \binom{n}{i} \partial^{i}(a)\partial^{n-i}$$

for every element $a \in R$ and for every integer n.

We consider the ring $R[[\partial^{-1}]][\partial]$ of pseudo-differential operators

$$\sum_{i=0}^{\infty} a_{n-i}\partial^{n-i} = a_n\partial^n + a_{n-1}\partial^{n-1} + \cdots ,$$

Lemma. Equivalent conditions. (1) A pseudo-differential operator

$$a_n\partial^n + a_{n-1}\partial^{n-1} + a_{n-2}\partial^{n-2} + \cdots \in R[[\partial^{-1}]][\partial]$$

with $a_n \neq 0$ is invertible in the ring $R[[\partial^{-1}]][\partial]$.
(2) The leading coefficient a_n is invertible in the ring R.

§7 KP-hierarchy

We consider an ordinary differential operator

$$W = \partial^m + w_1(x,t)\partial^{m-1} + w_2(x,t)\partial^{m-2} + \cdots + w_m(x,t)$$

of order m, ∂ being ∂_x as in the KdV.

So the differential operator $W(x,t)$ is parametrized by t.

After M. Sato, we introduce an infinite number of time, the deformation parameters. So we understand

$$t = (t_1, t_2, \cdots).$$

We set

$$L := W \partial W^{-1}.$$

W^{-1} means the formal inverse of the differential operator W. Therefore L is a pseudo-differential operator of order 1 so that

$$L = \partial + u_0(x, t) + u_{-1}(x, t)\partial^{-1} + \cdots .$$

Hence L^n is a pseudo-differential operator of order n for an integer $n \geq 1$. So we have a decomposition

$$L^n = L^n_+ + L^n_-$$

such that L^n_+ is the differential part of the pseudo-differential operator L^n so that we have

$$B_n := L^n_+ = \partial^n + a_1\partial^{n-1} + \cdots + a_n$$

and

$$L^n_- = b_{-1}\partial^{-1} + b_{-2}\partial^{-2} + \cdots .$$

KP-hierarchy is written in the following Lax form

$$\frac{\partial L}{\partial t_n} = [B_n, L], \text{for } n = 1, 2, \cdots .$$

The Lax equation describes the iso-spectral deformation of the pseudo-differential operator L.

$$\begin{cases} L\psi = \lambda\psi, \\ \frac{\partial\psi}{\partial t_n} = B_n\psi, \\ \frac{\partial\lambda}{\partial x} = \frac{\partial\lambda}{\partial t_n} = 0. \end{cases}$$

for $n = 1, 2, \cdots$.

§8 Grassmann variety

Let m be a positive integer chosen once for all.
$\mathbb{N} = \{0, 1, 2, \cdots\}$.
We consider the $\mathbb{C}$-vector space $V = \mathbb{C}^{\mathbb{N}}$ of column vectors so that

$$V = \{[\xi_0\, \xi_1\, \xi_2 \cdots]^t \mid \xi_i \in \mathbb{C} \text{ for every } i \in \mathbb{N}\}$$

We identify the vector space V with $\mathbb{C}[[x]]$.

$$\xi = [\xi_0\, \xi_1\, \xi_2 \cdots]^t \in V$$

$\leftrightarrow$

$$f(x) = \xi_0 + \xi_1 x + \frac{1}{2!}\xi_2 x^2 + \frac{1}{3!}\xi_3 x^3 + \cdots$$

We denote by $\mathbb{GM}(m,\infty)$ the Grassmann variety of all the m-dimensional linear sub-spaces of the vector space $V=\mathbb{C}^{\mathbb{N}}$.

$$\begin{aligned}\mathbb{GM}(m,\infty) &= \{m\text{-dimensional subspace } \subset V\}.\\ &= \{m\text{-dimensional subspace } \subset \mathbb{C}[[x]]\}.\end{aligned}$$

So to a point P of the Grassmann variety $\mathbb{GM}(m,\infty)$,
there corresponds
a sub-space Z_P of dimension m, of the vector space V.

A frame of the point P is an $\mathbb{N}\times m$-matrix

$$\Xi = [\xi_i^{(j)}]_{0\le i<\infty;\,1\le j\le m}$$

such that the m-column vectors $\Xi^{(j)} = [\xi_i^{(j)}]_{0\le i<\infty} \in V$ of the matrix Ξ span the vector space Z_P. Therefore

$$Z_P = \bigoplus_{j=1}^{m} \mathbb{C}\Xi^{(j)}.$$

The point $P \in \mathbb{GM}(m,\infty)$ determines a $\mathbb{C}$-vector subspace

$$< f^{(1)}(x), f^{(2)}(x), \cdots, f^{(m)}(x) > \subset \mathbb{C}[[x]],$$

where

$$f^{(j)}(x) = \sum_{i=0}^{\infty} \frac{1}{i!} \xi_i^{(j)} x^i$$

for $1 \leq j \leq m$.

Notation

$$f^{(j)}(x) = [1, x, \frac{1}{2!}x^2, \cdots][\xi_i^{(j)}]$$

$$f_i^{(j)}(x) := \frac{d^i}{dx^i} f^{(j)(x)} = \sum_{l=0}^{\infty} \frac{1}{l!} \xi_{l+i}^{(j)} x^l.$$

We introduce $\mathbb{N} \times \mathbb{N}$-matrix

$$\Lambda = [\lambda_{ij}]_{(i,j) \in \mathbb{N} \times \mathbb{N}}$$

such that

$$\lambda_{ij} = \begin{cases} 1, & \text{if } j = i + 1, \\ 0, & \text{otherwise} \end{cases}$$

so that the $\mathbb{N} \times \mathbb{N}$ matrix

$$\Lambda = \begin{bmatrix} 0 & 1 & 0 & . & . \\ . & 0 & 1 & 0 & . \\ . & . & 0 & 1 & 0 \\ . & . & . & . & . \end{bmatrix}.$$

Then we have

$$\exp(x\Lambda) = I + x\Lambda + \frac{1}{2!}x^2\Lambda^2 + \cdots .$$

So

$$\exp(x\Lambda) = \begin{bmatrix} 1 & x & x^2/2! & x^3/3! & \cdots \\ 0 & 1 & x & x^2/2! & \cdots \\ 0 & 0 & 1 & x & \cdots \\ . & . & . & . & \cdots \end{bmatrix}.$$

Therefore

$$\mathcal{H}(x) := \exp(x\Lambda)\Xi = \begin{bmatrix} f^{(1)} & f^{(2)} & \dots & f^{(m)} \\ \partial f^{(1)} & \partial f^{(2)} & \dots & \partial f^{(m)} \\ \partial^2 f^{(1)} & \partial^2 f^{(2)} & \dots & \partial^2 f^{(m)} \\ . & . & \dots & . \end{bmatrix}$$

$$= \begin{bmatrix} f^{(1)} & f^{(2)} & \dots & f^{(m)} \\ f_1^{(1)} & f_1^{(2)} & \dots & f_1^{(m)} \\ f_2^{(1)} & f_2^{(2)} & \dots & f_2^{(m)} \\ . & . & \dots & . \end{bmatrix}.$$

§9 Grassmann variety $\mathbb{GM}(m,\infty)$ and pseudo-differential operator

Consider a pseudo-differential operator

$$W = 1 + w_1(x)\partial^{-1} + w_2(x)\partial^{-2} + \cdots + w_m(x)\partial^{-m}$$

with coefficients $w_l(x) \in \mathbb{C}[[x]]$ for $1 \leq l \leq m$.

So

$$W\partial^m = \partial^m + w_1(x)\partial^{m-1} + w_2(x)\partial^{m-2} + \cdots + w_m(x) \in \mathbb{C}[[x]][\partial].$$

is a linear ordinary differential operator with coefficients in $\mathbb{C}[[x]]$.
We consider a linear ordinary differential equation

$$W\partial^m f = 0$$

of order m that is equivalent to

$$\partial^m f + w_1(x)\partial^{m-1} f + w_2(x)\partial^{m-2} f + \cdots + w_m(x) f = 0. \qquad (2)$$

Proposition. The linear differential equation (2) determines a $\mathbb{C}$-vector subspace $Sol \subset \mathbb{C}[[x]]$ of dimension m.
Conversely the sub-space $Sol \subset \mathbb{C}[[x]]$ determines the differential equation (2) and hence the operator W.

Proof of the second assertion.
Given m-linearly independent formal power series solutions

$$f^{(j)}(x) = \sum_{i=0}^{\infty} \frac{1}{i!}\xi_i^{(j)} x^i \in \mathbb{C}[[x]]$$

of (2) for for $1 \leq j \leq m$.

$$\begin{cases} f^{(1)}_{m-1}w_1(x) + f^{(1)}_{m-2}w_2(x) + \cdots + f^{(1)}_0 w_m(x) = & -f^{(1)}_m, \\ f^{(2)}_{m-1}w_1(x) + f^{(2)}_{m-2}w_2(x) + \cdots + f^{(2)}_0 w_m(x) = & -f^{(2)}_m, \\ \cdots & \\ f^{(m)}_{m-1}w_1(x) + f^{(m)}_{m-2}w_2(x) + \cdots + f^{(m)}_0 w_m(x) = & -f^{(m)}_m, \end{cases}$$

where $f_i^{(j)}(x) := d^i f^{(j)}/dx^i$.

Since the Wronskian

$$Wr := \begin{vmatrix} f_{m-1}^{(1)} & f_{m-2}^{(1)} & \cdots & f_0^{(1)} \\ f_{m-1}^{(2)} & f_{m-2}^{(2)} & \cdots & f_0^{(2)} \\ . & . & \cdots & . \\ f_{m-1}^{(m)} & f_{m-2}^{(m)} & \cdots & f_0^{(m)} \end{vmatrix} \neq 0$$

because she solutions are linearly independent. the Wronskian Wr that is a formal power series is invertible in $\mathbb{C}[[x]][x^{-1}$. It follows from by the formula of Cramer that we can solve the $w_l(x)$'s in the formal Laurent series ring $\mathbb{C}[[x]][x^{-1}]$;

$$w_l(x) = \frac{A_l}{Wr} \in \mathbb{C}[[x]],$$

where the numerator

$$A_l := \begin{vmatrix} f_{m-1}^{(1)} & \cdots & -f_m^{(1)} & \cdots & f_0^{(1)} \\ f_{m-1}^{(2)} & \cdots & -f_m^{(2)} & \cdots & f_0^{(2)} \\ . & \cdots & . & \cdots & . \\ f_{m-1}^{(m)} & \cdots & -f_m^{(m)} & \cdots & f_0^{(m)} \end{vmatrix}$$

obtained by replacing the l-th column of the Wronskian by the column vector

$$[-f_m^{(1)} - f_m^{(2)} \cdots - f_m^{(m)}]^t.$$

Corollary to the proof.

Given a point $P \in \mathbb{GM}(m, \infty)$

$\Downarrow$

A subspace of dimension $m \subset \mathbb{C}[[x]]$

$\Downarrow$ Formula of Cramer

Differential operator

$$\partial^m + w_1(x)\partial^{m-1} + w_2(x)\partial^{m-2} + \cdots + w_m(x)$$

$\Downarrow$

Pseudo-differential operator

$$W = 1 + w_1(x)\partial^{-1} + w_2(x)\partial^{-2} + \cdots + w_m(x)\partial^{-m} \in \mathbb{S}_{)}(m)$$

with coefficients in $\mathbb{C}[[x]][x^{-1}]$.

Remarkable picture.

$$\mathbb{GM}(m,\infty) \to R[[\partial^{-1}]][\partial],$$

R being $\mathbb{C}[[x]][x^{-1}]$.

To have a clearer picture, we set

$$U_0 := \{P \in \mathbb{GM}(m,\infty) \mid \text{ There exists a frame } \Xi \text{ of the point } P \text{ such that } \det [\xi_i^{(j)}]_{0\le i\le m-1,\, 1\le j\le m} \ne 0. \}$$

So U_0 is a Zariski open sub-set of the Grassmann variety.

On the Zariski open set $U_0 \subset \mathbb{GM}(m,\infty)$

$$U_0 \to \mathbb{S}_0(m) := \{W = 1 + w_1(x)\partial^{-1} + w_2(x)\partial^{-2} + \cdots \in \mathbb{C}[[x]][[\partial^{-1}]][\partial] \mid w_l = 0 \text{ for } l > m \}$$

§10 Dynamical system on $\mathbb{GM}(m,\infty)$

Sato introduces a dynamical system on the Grassmann variery $\mathbb{GM}(m,\infty)$

$\Downarrow$

Dynamical system on

$\mathbb{S}_0(m) = \{1 + w_1(x)\partial^{-1} + w_2(x)\partial^{-2} = \cdots + w_m(x)\partial^{-m} \in]\mathbb{C}[[x]][[\partial]][\partial^{-1}\}$

$\Downarrow$

Pseudo-differential operator parametrized by $t = (t_1, t_2, t_3, \cdots)$

$W := W(t_1, t_2, t_3, \cdots) = 1 + w_1(t,x)\partial^{-1} + w_2(t,x)\partial^{-2} + \cdots w_m(t,x)\partial^{-m}$

$\Downarrow$

Pseudo-differential operator parametrized by $t = (t_1, t_2, t_3, \cdots)$

$L := W\partial W^{-1}$

$\Downarrow$

Lax equation is solved for all $n = 1, 2, 3, \cdots$

$$\frac{\partial L}{\partial t_n} = [B_n, L].$$

Definition of the dynamical system on $\mathbb{GM}(m,\infty)$

Let $\Xi = [\xi_i^{(j)}] \in M_{\mathbb{N},m}(\mathbb{C})$

Dynamical system on the frames

$\exp(t_1\Lambda + t_2\Lambda^2 + t_3\Lambda^3 + \cdots)\Xi \in M_{\mathbb{N},m}(\mathbb{C})$

$\Downarrow$

Dynamical system on $\mathbb{C}[[x]]$

$$\exp(x\Lambda)\exp(t_1\Lambda + t_2\Lambda^2 + t_3\Lambda^3 + \cdots)\Xi$$
$$= \exp((x+t_1)\Lambda + t_2\Lambda^2 + t_3\Lambda^3 + \cdots)\Xi \in M_{\mathbb{N},m}(\mathbb{C}[[x]]) \quad (3)$$

$$[f_i^{(j)}(t,x)]_{0\le i<\infty, 1\le j\le m} := \exp((x+t_1)\Lambda + t_2\Lambda^2 + t_3\Lambda^3 + \cdots)\Xi \in M_{\mathbb{N},m}(\mathbb{C}[[t,x]])$$

(1) $$\frac{\partial^l}{\partial x^l} f_0^{(j)}(t,x) = f_l^{(j)}(t,x), \tag{4}$$

(2) $$\frac{\partial^l}{\partial x^l} f_i^{(j)}(t,x) = \frac{\partial}{\partial t_l} f_i^{(j)}(t,x) = f_{i+l}^{(j)}(t,x) \tag{5}$$

$$f_i^{(j)}(t,x) = \sum_{k=0}^{\infty} \xi_{i+k}^{(j)} p_k(t,x),$$

where

$$p_n(x+t_1,\, t_2,\ t_3, \cdots) = \sum \frac{x^{\nu_0} t_1^{\nu_1} t_2^{\nu_2} \cdots}{\nu_0! \nu_1! \nu_2! \cdots} \in \mathbb{Q}[x, t_2, t_2, \cdots],$$

where the summation extends all the positive integers

$$\nu_0,\ \nu_1,\ \nu_2,\ \cdots$$

such that

$$\nu_0 + \nu_1 + 2\nu_2 + 3\nu_3 + \cdots = n.$$

Schur polynomial.

§11 Galois group of the KP-hierarchy

Take $\Xi = [\xi_i^{(j)}] \in M_{\infty,m}(\mathbb{C})$ such that the $\xi_i^{(j)}$'s are algebraically independent over $\mathbb{C}$.

Then the $f_i^{(j)}(t,x)$'s are algebraically independent over $\mathbb{C}$.
Let L be the differential field generated by $w_k(t,x)$'s over $\mathbb{C}$ with derivations $\partial/\partial x, \partial/\partial t_1, \partial/\partial t_2, \cdots$.

Theorem. Galois group Inf-gal$(L/\mathbb{C})$ is Abelian.

Let L' be the differential field generated by the $f_i^{(j)(t,x)}$ with derivations $\partial/\partial x, \partial/\partial t_1, \partial/\partial t_2, \cdots$.
So $L'/L/\mathbb{C}$. Theorem follows from

Theorem. Inf-gal$(L'/\mathbb{C})$ is Abelian.

$$L'/\mathbb{C} = \mathbb{C} < f_i^{(j)}(t,x) >_{i \in \mathbb{N}, 1 \leq j \leq m} /\mathbb{C}$$

The $f_i^{(j)}(t,x)$'s are transcendental over $\mathbb{C}$ satisfying the lineair partial differential equations!!

$$(1) \quad \frac{\partial^l}{\partial x^l} f_0^{(j)}(t,x) = f_l^{(j)}(t,x), \tag{6}$$

$$(2) \quad \frac{\partial^l}{\partial x^l} f_i^{(j)}(t,x) = \frac{\partial}{\partial t_l} f_i^{(j)}(t,x) = f_{i+l}^{(j)}(t,x) \tag{7}$$

They are the generic solution of the linear partial differential equations (1) and (2).

§11 Galois theory for differential equations

Galois tried applied his *ambiguity thoery to analysis.*

S. Lie, E. Picard, J. Drach, E. Vessiot, E. Kolchin, . . .

Infinite dimensional theory and finite dimensional theory

Given an algebraic vector field X on an algebraic vriety V over $\mathbb{C}$.

Assume there exists an operation (G, V) of algebraic group G on the algebraic variety V over $\mathbb{C}$ such that

$$X \in \mathbf{Lie}\, G \subset \mathbf{H}^0(V, \Theta_V).$$

Intuitively, Galois group of the vector field X on V is the smallest algebraic sub-group H of G such that

$$X \in \mathbf{Lie}\, H \subset \mathbf{Lie}\, G \subset \mathrm{H}^0(V, \Theta_V).$$

It is not canonical? What do we do, when there is no algebraic group action?

Replace the group action by the jet space $J(V \times V)$ that is an algebraic groupoid. Canonical choice!