Galois Theory for nonlinear q-difference equations

Guy Casale

Wuhan, 16-27/04/2012

In this course Malgrange pseudogroup for a rational q-difference equations will be defined and its elementary properties will be investigated. Such equations are

$$\begin{array}{rcl} y_1(qx) & = & R_1(x, y_1((x), \dots y_n(x)) \\ \vdots & & \\ y_n(qx) & = & R_n(x, y_1((x), \dots y_n(x)) \end{array}$$

with $\dfrac{\partial(R_1, \dots, R_n)}{\partial(y_1, \dots, y_n)} \not\equiv 0$.

For systems of linear q-difference equations, such object called the difference Galois group was defined around 1962 by Franke and Bialynicki-Birula following the definition given for differential equations by Picard and Vessiot. The important basic property of this group is that its solvability is equivalent to solvability of the equation by means of q-elementary functions. In 2001 B. Malgrange defined a generalization of the differential Glois group for nonlinear differential equations. This definition was adapted to the difference case by Granier.

In a first part one will define frame bundles and prolongations of the equation, its differential invariants and its Malgrange pseudogroup and prove the statement: *if a nonlinear q-difference equation is solvable by means of q-elementary functions then its Magrange pseudogroup is solvable.*

In a second part, one will investigate relations between (finite or infinitesimal) symmetries of a system of q-difference equations and its Malgrange pseudogroup. A good symmetry for the equation above is a h-difference equation

$$\begin{array}{rcl} y_1(hx) & = & S_1(x, y_1((x), \dots y_n(x)) \\ \vdots & & \\ y_n(hx) & = & S_n(x, y_1((x), \dots y_n(x)) \end{array}$$

such that formally $y(h(qx)) = y(q(hx))$ and constants for the q-difference equations are constants for the h-difference equation. Then one gets: *if Malgrage pseudogroup of the q-difference equation is solvable then Malgrage pseudogroup of the h-difference equation is solvable too.*