Continuity of the radius of convergence of p-adic differential equations on Berkovich analytic spaces

Francesco Baldassarri*   Lucia Di Vizio†

March 25, 2008

Contents

1 Introduction

The classical existence theorem of Cauchy [14, Chap.I] for local solutions of an analytic differential system at an ordinary point does not hold in general for differential equations on a smooth Berkovich analytic space X over a p-adic field k. We recall [3, 1.2.2] that to any point $\xi \in X$ one associates a completely valued extension field $\mathscr{H}(\xi)$ of k, called the *residue field at* ξ; the point ξ is k-rational if $\mathscr{H}(\xi) = k$. Any k-rational point ξ of X admits a neighborhood isomorphic to a polydisk centered at the origin O in an affine

---

1

(analytic) k-space, the isomorphism sending ξ to O. However, the neighborhoods of a non-rigid point are in general too coarse. So, a differential equation does in general have no solutions analytic in a full neighborhood of a non-rigid point $\xi \in X$, even if the point is not a singularity of the equation. In the very inspiring paper [1, §3] Y. André concentrates on differential equations which after pull-back to a finite étale covering admit a full set of multivalued analytic solutions. For such differential equations there is a notion of global monodromy group close to the one in the complex case. It would be interesting to pursue André's investigation into a description of integrable analytic connections locally for the étale topology of [4]. But this is not our approach here: we use the natural topology on Berkovich analytic spaces and regard an étale covering $f : Y \to X$ as producing a *highly non-trivial connection* $(f_*\mathcal{O}_Y, \nabla = f_*(d_{X/k}) : f_*\mathcal{O}_Y \to f_*\mathcal{O}_Y \otimes \Omega^1_X)$ on X. Moreover, the problem of the failure of Cauchy existence theorem would not be overcome in general by using some étale topology. On the other hand, it is possible and sometimes convenient to recover Cauchy's theorem at any given point $\xi \in X$, by performing the extension of scalars to $X \widehat{\otimes} \mathscr{H}(\xi)$, and passing to some canonical point ξ' of this space above ξ. This viewpoint has been systematically used by Dwork and Robba in their study of p-adic differential equations.

We actually assume that X comes with a local notion of distance, measured in terms of an embedding of X as an analytic domain in the generic fiber $\mathcal{X}_\eta$ of a smooth formal scheme $\mathcal{X}$ over k°. This does not mean that we privilege formal schemes over k° or $\tilde{k}$-schemes, over k-analytic spaces. The formal model of X is here a technical tool for expressing "local" radii of convergence of solutions of differential equations in the above sense. We will show in a subsequent paper that certain expressions in these local radii are in fact absolute invariants of a connection on an analytic space.

In practice, we consider all over this paper the following

Situation 1.1. The smooth formal scheme $\mathcal{X} = \mathrm{Spf}\, A$, is affine and étale over $\widehat{\mathbb{A}}^d_{k^\circ}$, the formal affine space over k°, with formal coordinates $\underline{x} = (x_1, \ldots, x_d)$. Then $\mathscr{A} := A \otimes k$, $X = \mathcal{M}(\mathscr{A})$, and U is an analytic domain in X.

We are given an integrable system of partial differential equations of the form

$$(1.1.1) \qquad \Sigma = \Sigma_{(\underline{x},\underline{G},U)} \quad : \quad \frac{\partial \vec{y}}{\partial x_i} = G_i\, \vec{y}\ , \ \forall\, i = 1, \ldots, d\,,$$

for $\vec{y}$ a column vector of unknown functions and G_i a $\mu \times \mu$ matrix of analytic functions on U. Notice that, for any k-rational point $\xi \in X$, the étale map $\underline{x} : X \to \mathbb{A}^d_k$ to the affine k-analytic space of dimension d, admits a unique local section $\sigma_\xi : D^d_k(\underline{x}(\xi), 1^-) \to X$, sending $\underline{x}(\xi)$ to ξ, where

$$(1.1.2) \qquad D^d_k(\underline{x}(\xi), 1^-) = \{\eta \in \mathbb{A}^d_k \,|\, |x_i(\eta) - x_i(\xi)| < 1\,, \text{ for } 1 = 1, \ldots, d\}\,.$$

The image of σ_ξ will be denoted $D_{\mathcal{X}}(\xi, 1^-)$, and called *the open disk of radius 1 centered at the k-rational point* $\xi \in X$. Similarly, we define open and closed disks $D_{\mathcal{X}}(\xi, r^\pm)$, of radius $r < 1$ centered at ξ. Notice that we use the term "disk" to refer to "polydisk with equal radii". The *diameter* $\delta_{\mathcal{X}}(\xi, U)$ of U at the k-rational point ξ, is the radius of the maximal open disk centered at ξ and contained in U, a notion obviously independent of the choice of the formal coordinates $\underline{x}$ on $\mathcal{X}$. Then $0 < \delta_{\mathcal{X}}(\xi, U) \le 1$ because, on the one hand, a k-rational point of U is necessarily an interior point of U in X; on the other hand, disks of radii ≥ 1 are not defined. Now, when $\xi \in U$ is a k-rational point of U, the definition of the *radius of convergence of the system (1.1.1) at ξ* is completely natural. It is the radius $r = R_{\mathcal{X}}(\xi, \Sigma) = R(\xi, \Sigma)$ of the maximal open disk $D_{\mathcal{X}}(\xi, r^-)$ contained in U, where a fundamental solution matrix Y of (1.1.1) at ξ converges. Notice that Y is a matrix with entries in $k[[x_1 - x_1(\xi), \ldots, x_d - x_d(\xi)]]$ and its convergence is really tested in $D^d_k(\underline{x}(\xi), 1^-)$.

If

$$(1.1.3) \qquad Y = \sum_{\underline{\alpha} \in \mathbb{N}^d} A_{\underline{\alpha}} (x_1 - x_1(\xi))^{\alpha_1} \cdots (x_d - x_d(\xi))^{\alpha_d} \ , \ \text{with } A_{\underline{\alpha}} \in M_\mu(k) \ ,$$

its radius of convergence is, as in the classical case,

$$(1.1.4) \qquad \widetilde{R}(\xi, \Sigma) = \liminf_{|\underline{\alpha}|_\infty \to \infty} \left| A_{\underline{\alpha}} \right|^{-1/|\underline{\alpha}|_\infty} \in \mathbb{R}_{\geq 0} \cup \{\infty\} \ ,$$

where $|\underline{\alpha}|_\infty = \alpha_1 + \cdots + \alpha_d$, and where the norm of a matrix is the maximum absolute value of its entries. Notice that the disk of radius $\widetilde{R}(\xi, \Sigma)$, centered at $\xi \in \mathbb{A}_k^d$, is not necessarily contained in U, as the example of the trivial connection $G_i = 0$, $\forall i$, on a small disk $U \subset \mathbb{A}_k^d$ shows. But we insist on defining

$$(1.1.5) \qquad R(\xi, \Sigma) = \min(\widetilde{R}(\xi, \Sigma), \delta_{\mathcal{X}}(\xi, U)) \ .$$

The reason is that the determinant of the matrix Y may vanish at a point $\zeta \in D(\underline{x}(\xi), \widetilde{R}(\xi, \Sigma)^-) \setminus D(\underline{x}(\xi), \delta_{\mathcal{X}}(\xi, U)^-)$, while this cannot be the case in $D(\underline{x}(\xi), R(\xi, \Sigma)^-)$, otherwise the differential system for the wronskian $w := \det Y$, namely

$$(1.1.6) \qquad \frac{\partial w}{\partial x_i} = (\operatorname{Tr} G_i) \, w \ , \ \forall \, i = 1, \ldots, d,$$

would have a singularity in U, which is not the case. Notice that $R(\xi, \Sigma)$ is then the maximum real number $r \leq 1$ such that the system Σ admits a solution matrix $Y \in GL(d, \mathcal{O}(D_{\mathcal{X}}(\xi, r^-)))$.

The advantage, and the intrinsic content, of this definition may be better appreciated if we consider the category $\mathbf{MIC}(U/k)$ of coherent $\mathcal{O}_U$-modules with integrable connection $(\mathcal{E}, \nabla)$,

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_{U/k} \ ,$$

and the object $(\mathcal{E} := \mathcal{O}_X^\mu, \nabla)$ associated to Σ. If $\underline{e} = (e_1, \ldots, e_\mu)$ denotes the canonical basis of global sections of $\mathcal{O}_X^\mu$, then, by convention,

$$(1.1.7) \qquad \nabla(\underline{e}) = - \sum_{i=1}^d (e_1 \otimes dx_i, \ldots, e_\mu \otimes dx_i) \, G_i \ ,$$

so that Σ is the differential system satisfied by the horizontal sections of $(\mathcal{E}, \nabla)$.

The abelian sheaf $\mathcal{E}^\nabla = \operatorname{Ker}(\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_{U/k})$ for the G-topology of U, is not in general locally constant. If, on some analytic domain $V \subset U$, $\mathcal{E}^\nabla_{|V}$ is locally constant, then it is necessarily a local system of k-vector spaces of rank μ on V and the canonical monomorphism

$$(1.1.8) \qquad \mathcal{E}^\nabla \otimes_k \mathcal{O}_U \hookrightarrow \mathcal{E} \ ,$$

is in fact an *isomorphism*: this is the intrinsic content of our previous statement on the wronskian equation. Taking into account the fact that a locally constant sheaf of finite dimensional k-vector spaces on a disk is necessarily constant, we see that $D = D_{\mathcal{X}}(\xi, R(\xi, \Sigma)^-)$ is the maximal open disk centered at $\xi \in U$, and contained in U, where $(\mathcal{E}, \nabla)$ is isomorphic to the trivial connection $(\mathcal{O}_D, d_D)^\mu$. We may then give the

Definition 1.2 (Alternative form)**.** *Let $(\mathcal{E}, \nabla)$ be an object of $\mathbf{MIC}(U/k)$, with $\mathcal{E}$ locally free of rank μ for the G-topology. For any k-rational point $\xi \in U$, we define the radius of convergence $R_{\mathcal{X}}(\xi, (\mathcal{E}, \nabla))$ of $(\mathcal{E}, \nabla)$ at ξ as the maximal open disk D centered at ξ and contained in U, such that $(\mathcal{E}, \nabla)_{|D}$ is isomorphic to the trivial connection $(\mathcal{O}_D, d_D)^\mu$.*

3

Coming back to the explicit situation (1.1.1), there is a nice compact formula for the solution matrix $Y = Y_\xi$ of (1.1.1) at ξ, such that $Y_\xi(\xi) = I_\mu$. We write

$$(1.2.1) \qquad \underline{\alpha}! = \prod_i \alpha_i! \ , \ (\underline{x} - \underline{x}(\xi))^{\underline{\alpha}} = \prod_i (x_i - x_i(\xi))^{\alpha_i} \ , \ \partial^{\underline{\alpha}} = \prod_i \frac{\partial^{\alpha_i}}{\partial x_i^{\alpha_i}} \ , \ \partial^{[\underline{\alpha}]} = \frac{1}{\underline{\alpha}!} \partial^{\underline{\alpha}} \ .$$

By iteration of the system (1.1.1) we obtain, for any $\underline{\alpha} \in \mathbb{N}^d$, the equations

$$(1.2.2) \qquad \partial^{[\underline{\alpha}]} \vec{y} = G_{[\underline{\alpha}]} \vec{y} \qquad (\text{resp. } \partial^{\underline{\alpha}} \vec{y} = G_{\underline{\alpha}} \vec{y}) \ ,$$

with $G_{[\underline{\alpha}]}$ and $G_{\underline{\alpha}} = \underline{\alpha}! G_{[\underline{\alpha}]}$, $\mu \times \mu$ matrices of functions analytic in U. In particular, $G_{\underline{0}} = I_\mu$ and G_i is now written $G_{[\underline{1}_i]} = G_{\underline{1}_i}$, where $\underline{1}_i = (0, \ldots, 0, 1, 0, \ldots, 0)$, with 1 only at the i-th place.

The $G_{\underline{\alpha}}$ satisfy the recursion relations

$$(1.2.3) \qquad G_{\underline{\alpha}+\underline{1}_i} = \frac{\partial}{\partial x_i}(G_{\underline{\alpha}}) + G_{\underline{\alpha}} G_{\underline{1}_i} \ .$$

The Taylor series of the fundamental solution matrix Y_ξ of (1.1.1) at $\xi \in U$ is

$$(1.2.4) \qquad Y_\xi = \sum_{\underline{\alpha} \in \mathbb{N}^d} G_{[\underline{\alpha}]}(\xi)(\underline{x} - \underline{x}(\xi))^{\underline{\alpha}} \in GL(\mu, \mathscr{H}(\xi)[[\underline{x} - \underline{x}(\xi)]]) \ ,$$

(for the k-rational point ξ, $\mathscr{H}(\xi) = k$, of course) with radius of convergence

$$(1.2.5) \qquad \widetilde{R}(\xi, \Sigma) = \liminf_{|\underline{\alpha}|_\infty \to \infty} \left|G_{[\underline{\alpha}]}(\xi)\right|^{-1/|\underline{\alpha}|_\infty} \in \mathbb{R}_{\geq 0} \cup \{\infty\} \ .$$

We now extend the previous definitions to all points $\xi \in U$. The function $\xi \mapsto \widetilde{R}(\xi, \Sigma)$ will be defined in general by formula (1.2.5). This amounts to the following consideration on Berkovich analytic spaces. As explained in [4, 1.4], we may consider the ground field extension of U to $\mathscr{H}(\xi)$, $U_{\mathscr{H}(\xi)} = U \widehat{\otimes}_k \mathscr{H}(\xi)$. It is a $\mathscr{H}(\xi)$-analytic space equipped with a canonical compact projection map $\psi_\xi : U_{\mathscr{H}(\xi)} \to U$, and there is a canonical $\mathscr{H}(\xi)$-rational point ξ' above ξ. The system (1.1.1) may be viewed, with no change in notation, on $U_{\mathscr{H}(\xi)} \to \mathbb{A}^d_{\mathscr{H}(\xi)}$, where the field of constants is now $\mathscr{H}(\xi)$, and formula (1.2.5) represents the radius of convergence of the fundamental solution matrix $Y_{\xi'}$ of (1.1.1), viewed on $U_{\mathscr{H}(\xi)}$ at ξ'. We then define, for general $\xi \in U$, $D_{\mathcal{X}}(\xi, r^\pm) := D_{\mathcal{X}_{\mathscr{H}(\xi)^\circ}}(\xi', r^\pm)$, where $\mathcal{X}_{\mathscr{H}(\xi)^\circ} = \mathcal{X} \times_{k^\circ} \text{Spf} \mathscr{H}(\xi)^\circ$, $\delta_{\mathcal{X}}(\xi, U) := \delta_{\mathcal{X}_{\mathscr{H}(\xi)^\circ}}(\xi', U_{\mathscr{H}(\xi)})$, and $R(\xi, \Sigma) = \min(\widetilde{R}(\xi, \Sigma), \delta_{\mathcal{X}}(\xi, U))$ (resp. $R_{\mathcal{X}}(\xi, (\mathcal{E}, \nabla)) = R_{\mathcal{X}_{\mathscr{H}(\xi)^\circ}}(\xi', \psi_\xi^*(\mathcal{E}, \nabla)))$. We abusively call $D_{\mathcal{X}}(\xi, r^\pm)$ the *open* (resp. *closed*) disk of radius r centered at $\xi \in X$.

In the situation (1.1), under the further condition that *U is a Laurent domain in X*, we prove that the function $\xi \mapsto R(\xi, \Sigma)$ is upper semicontinuous on U, for its natural Berkovich topology. A preliminary fact, and this is where we need U to be a Laurent domain in X, is that the function $\xi \mapsto \delta_{\mathcal{X}}(\xi, U)$ is upper semicontinuous on U. Moreover, if U is the inverse image of a Laurent domain in $D_k^d(0, 1^+)$, the function $\xi \mapsto \delta_{\mathcal{X}}(\xi, U)$ is continuous. If $U = X$ then $\xi \mapsto R(\xi, \Sigma)$ is continuous at the maximal point η_X of X. If $\dim X = 1$ and $\mathcal{X} = \widehat{\mathbb{A}}^1_{k^\circ}$, we prove directly that $\xi \mapsto R(\xi, \Sigma)$ is continuous on U. Combining the last two results, we deduce that $\xi \mapsto R(\xi, \Sigma)$ is continuous if $\dim X = 1$ and U is any affinoid neighborhood of η_X.

Surprisingly enough, the simple statement above seems to be new even in the case when U is the closed unit disk $D_k(0, 1^+)$ of dimension 1, a case extensively discussed in the literature (*cf.* [11] and [9] for reference). In the case of an ordinary differential system Σ as (1.1.1) on an annulus

$$U = C(r_1, r_2) = \{\xi : r_1 < |x(\xi)| < r_2\} \subset \mathbb{A}^1_k$$

with $0 < r_1$, a simple convexity argument due to Christol and Dwork [5] shows that the function $\widetilde{R}$ is continuous when restricted to the segment of points $(r_1, r_2) \to C(r_1, r_2)$, $r \mapsto t_r$, where $t_r = t_{0,r}$ is the "generic point at distance r from 0", *i.e.* the point at the boundary of the disk $D_k(0, r^-)$. They actually consider, precisely as we do, the more invariant function

(1.2.6)
$$R: \quad (r_1, r_2) \quad \longrightarrow \quad \mathbb{R}_{\geq 0} \cup \{\infty\}$$
$$r \quad \longmapsto \quad R(t_r, \Sigma) := \min(r, \widetilde{R}(t_r, \Sigma)) \quad .$$

It is easy to show [5, 2.3] that the function $\log r \mapsto \log \widetilde{R}(t_r, \Sigma)$ is concave (*i.e.* $\cap$-shaped), hence continuous, in (r_1, r_2). The function $r \mapsto \widetilde{R}(t_r, \Sigma)$ is therefore continuous on (r_1, r_2). In this situation, the system is said to be *solvable at r_2* if the $\lim_{r \to r_2^-} R(t_r, \Sigma)$, which certainly exists, is $= r_2$ (and similarly for r_1). Systems solvable at r_2 (resp. r_1) are only understood on $C(r_2 - \varepsilon, r_2)$ (resp. $C(r_1, r_1 + \varepsilon)$), for small values of $\varepsilon > 0$, by the theory of *factorization according to the slopes* due to Christol and Mebkhout [7] [8]. In the special case of a *Robba system* [6, 3.1], *i.e.* of a system Σ on $C(r_1, r_2)$, such that $R(t_r, \Sigma) = r$ for every $r \in (r_1, r_2)$, it follows from Dwork transfer theorems [11, IV.5.2], that $R(\xi, \Sigma) = |x(\xi)|$, for every $\xi \in C(r_1, r_2)$.[1] This simplest case is of high interest, even (or maybe *especially*) when its features depend on the existence of a strong Frobenius structure. A notion of *exponents* is then available, and under an arithmetic condition on them (automatic in case of a strong Frobenius structure) the system admits a Fuchs-type decomposition over $C(r_1, r_2)$ [6] [10]. *Our paper deals with the deviation of a system from being of Robba type.*

We prove a far-reaching generalization of the Dwork-Robba theorem [11, IV.3.1] on effective bounds for the growth of local solutions (theorem (4.7) and its corollaries). The only difference from the version published by Gachet [13] is the formulation on Berkovich analytic spaces, which is however crucial in our proof of the upper semicontinuity of the radius of convergence (*cf.* §4.3).

It turns out that Berkovich analytic spaces represent an ideal framework for the study of p-adic differential equations. They contain the *generic points* in the sense of Dwork and Robba, as honest points. This gives great flexibility to the "rigid" geometry setting and permits in the end to generalize classical one-dimensional results of Dwork, Robba and Christol to analytic spaces.

2 Generalities and notation

We refer to our assumptions (1.1). For any subset $S \subset X$, the notation $|| \ ||_S$ will refer to the supnorm on S. For example, for an analytic domain $V \subset X$, let $\mathscr{A}_V^+$, denote the k-Banach algebra of *bounded* analytic functions on V, equipped with k-Banach norm $|| \ ||_V$. If $V = \mathcal{M}(\mathscr{A}_V)$ is affinoid, then $\mathscr{A}_V^+ = \mathscr{A}_V$, and the k-affinoid algebra $\mathscr{A}_V$ will be viewed as a k-Banach algebra via $|| \ ||_V$. We will denote by $\mathcal{L}_k(\mathscr{A}_V^+)$ the k-vector space of $|| \ ||_V$-bounded k-linear endomorphisms of $\mathscr{A}_V^+$, equipped with the corresponding operator norm $| \ |_V := || \ ||_{V,\mathrm{op}}$. Since $\forall \varphi, \psi \in \mathcal{L}_k(\mathscr{A}_V^+)$, $||\psi \circ \varphi||_{V,\mathrm{op}} \leq ||\psi||_{V,\mathrm{op}} ||\varphi||_{V,\mathrm{op}}$, $\mathcal{L}_k(\mathscr{A}_V^+) = (\mathcal{L}_k(\mathscr{A}_V^+), | \ |_V)$ is a k-Banach algebra.

---

[1] If for two values R_1 and R_2, with $r_1 < R_1 < R_2 < r_2$, $R(t_{R_i}, \Sigma) = R_i$, $i = 1, 2$, then $R(t_r, \Sigma) = r$, for all $r \in [R_1, R_2]$ [11, Cor. in App. I].

For a matrix $G = (g_{ij})$ of elements in a k-Banach algebra $(\mathscr{B}, || \; ||)$, we will set

$$||G|| := \sup_{i,j} ||g_{ij}|| \; .$$

Then $||G \cdot H|| \leq ||G|| \cdot ||H||$, whenever multiplication of matrices makes sense, and the k-algebra $\mathcal{M}_{n \times n}(\mathscr{B})$ of $n \times n$-matrices with entries in $\mathscr{B}$, equipped with the norm $|| \; ||$, is a Banach k-algebra.

2.1 Ground extension functor and continuity

We need a definition extracted from [4, 1.4]. Let L be any complete valued field extension of k; the *ground extension functor* associates to any k-analytic space Y an L-analytic space $Y_L = Y \widehat{\otimes}_k L$ equipped with a canonical projection $\psi_{Y,L} = \psi_{Y,L/k} : Y_L \to Y$. In the case of a k-affinoid space $Y = \mathscr{M}(\mathscr{A})$, $Y_L = \mathscr{M}(\mathscr{A}_L)$, where $\mathscr{A}_L$ is the L-affinoid algebra $\mathscr{A} \widehat{\otimes}_k L$, and the map $\psi_{Y,L}$ corresponds to the inclusion $\mathscr{A} \to \mathscr{A}_L$, $a \mapsto a \otimes 1$. By construction, the map $\psi_{Y,L}$ is compact, *i.e.* for any compact subset C of Y, $\psi_{Y,L}^{-1}(C)$ is compact. We will be dealing with a family $\mathscr{F}$ of functions defined on the analytic spaces over k in a class $\mathscr{S} = \bigcup_L \mathscr{S}_L$, where $\mathscr{S}_L$ is a class of L-analytic spaces and L varies over completely valued field extensionof k. We assume that $\mathscr{F} = \bigcup_L \mathscr{F}_L$, $\mathscr{F}_L = \{\varphi_Y : Y \to S\}_{Y \in \mathscr{S}_L}$, all functions taking values in a fixed topological space S. We will assume that the class $\mathscr{S}$ is stable by ground extensions, and that the family $\mathscr{F}$ is *compatible with base change*, in the sense that if $Y \in \mathscr{S}_L$, $\varphi_Y \in \mathscr{F}_L$, and L'/L is a completely valued extension, $Y_{L'} \in \mathscr{S}_{L'}$, and $\varphi_{Y_{L'}} = \varphi_Y \circ \psi_{Y,L'/L} \in \mathscr{F}_{L'}$. The following general lemma shows that, to prove continuity of the functions in $\mathscr{F}_k$, no loss of generality is involved in assuming that the base field k is maximally complete and algebraically closed.

Lemma 2.1. *Let Y be any k-analytic space, L be a complete valued field extension of k and $Y_L = Y \widehat{\otimes}_k L$ be the extension of Y over L. Then the natural topology of Y is the quotient topology of the projection map $\psi_L = \psi_{Y,L} : Y_L \to Y$.*

Proof. We first prove that the map ψ_L is closed. Let C be a closed subset of Y_L. Let y be a point of $Y \setminus \psi_L(C)$, and let D_2 be a compact neighborhood of y in Y. Then $D_1 = \psi_L^{-1}(D_2)$ is a compact subset of Y_L. The intersection $C \cap D_1$ is then compact; its image $\psi_L(C \cap D_1)$ is then closed, so that $D_2 \setminus \psi_L(C \cap D_1)$ is a neighborhood of y in Y not intersecting $\psi_L(C)$. The conclusion follows from [12, 2.4]. $\qquad \square$

It follows from the previous lemma that a function on Y is continuous if and only if its lift to Y_L is continuous. In particular,

Corollary 2.2. *The functions in $\mathscr{F}_k$ are continuous if there exists a completely valued extension field L/k such that all functions in $\mathscr{F}_L$ are continuous.*

This will allow us to assume in certain cases, without loss of generality, that the ground field k is maximally complete and algebraically closed.

We recall here for completeness that a function $\varphi : T \to \mathbb{R}$, where T is any topological space is *upper semicontinuous* or *USC* (resp. *lower semicontinuous* or *LSC*) if $\forall t_0 \in T$ and $\varepsilon > 0$, there exists a neighborhood $U_{t_0, \varepsilon}$ of t_0 in T such that

$$\varphi(t) < \varphi(t_0) + \varepsilon \;\; (\text{resp. } \; \varphi(t) > \varphi(t_0) - \varepsilon \;)$$

$\forall t \in U_{t_0, \varepsilon}$. If $\forall \alpha \in I$, φ_α is USC (resp. LSC), then

$$\varphi = \inf_{\alpha \in I} \varphi_\alpha \;\; (\text{resp. } \; \varphi = \sup_{\alpha \in I} \varphi_\alpha \;)$$

is USC (resp. LSC).

3 (Semi-) continuity of formal invariants

3.1 Diameter

We work here under the assumptions of (1.1). We recall that we have defined open (resp. closed) disks $D_{\mathcal{X}}(\xi, r^{\pm})$ centered at $\xi \in X$ of radius $r \in (0,1]$ (resp. $r \in (0,1)$). We say that a disk $D_{\mathcal{X}}(\xi, r^{\pm})$ is k-rational if its center ξ may be chosen in $X(k)$.

Proposition 3.1. *For any $\xi \in U$, the diameter $\delta_{\mathcal{X}}(\xi, U) > 0$.*

Proof. We follow the notation of [3, 2.5]. We may assume that ξ is a k-rational point of U, and that $U = \mathcal{M}(\mathscr{A}_U)$ is an affinoid contained in a disk $D = D_{\mathcal{X}}(\xi, r^+)$, with $r \in |k^{\times}|$. Then D is isomorphic as a k-analytic space to $\mathcal{M}(k\{r^{-1}X\})$, with $k\{r^{-1}X\} = k\{r^{-1}X_1, \ldots, r^{-1}X_d\}$, so we regard U as an affinoid in $D_k^d(0, r^+)$. Let $\chi_{\xi} : \mathscr{A}_U \to k$ (resp. $\chi'_{\xi} : k\{r^{-1}X\} \to k$) be the bounded character corresponding to $\xi \in U$ (resp. $\xi \in D$). Notice that χ_{ξ} may be viewed as a bounded $k\{r^{-1}X\}$-homomorphism $\mathscr{A}_U \to k$. The reduced character $\widetilde{\chi}_{\xi} : \widetilde{\mathscr{A}_U} \to \widetilde{k}$ obviously satisfies condition (d) of [3, 2.5.2] *i.e.* the ring $\widetilde{\chi}_{\xi}(\widetilde{\mathscr{A}_U}) = \widetilde{k}$ is integral over $\widetilde{\chi}'_{\xi}(\widetilde{k\{r^{-1}X\}}) = \widetilde{k}$, hence χ_{ξ} is inner with respect to $k\{r^{-1}X\}$. In that case, it is known that ξ lies in the topological interior of U in D [3, 2.5.13]. So, U hence contains a non trivial disk $D_{\mathcal{X}}(\xi, \varepsilon^-)$ centered at ξ and $\delta_{\mathcal{X}}(\xi, U) \geq \varepsilon > 0$. $\qquad\square$

We recall that a *Laurent (affinoid) domain* in any k-affinoid space $Y = \mathcal{M}(\mathscr{B})$ is a domain of the form

$$(3.1.1) \qquad Y(r^{-1}f, sg^{-1}) = \{x \in Y \mid |f_i(x)| \leq r_i, |g_j(x)| \geq s_j, 1 \leq i \leq n, 1 \leq j \leq m\}$$

where $f_i, g_j \in \mathscr{B}$, and r_i, s_j are positive real numbers.

3.2 Trivial estimate

We will assume here that the entries of the matrices G_i in (1.1.1) are *bounded* analytic functions on U, *i.e.* elements of the k-Banach algebra $\mathscr{A}_U^+$. For any analytic domain $V \subset X = \mathcal{X}_{\eta}$, the derivations $\frac{\partial}{\partial x_i}$, for $i = 1, \ldots, d$, are bounded k-linear operators on $\mathscr{A}_V^+$. Let $|\frac{\partial}{\partial x_i}|_V$ denote the operator norm of $\frac{\partial}{\partial x_i}$ on the k-Banach algebra $\mathcal{L}_k(\mathscr{A}_V^+)$. Then,

Proposition 3.2. *For any $\xi \in U$ we have:*

$$(3.2.1) \qquad \widetilde{R}(\xi, \Sigma) \geq \frac{|p|^{\frac{1}{p-1}}}{\max_{i=1,\ldots,d}\left(|\frac{\partial}{\partial x_i}|_U, \|G_i\|_U\right)} > 0.$$

Proof. It follows from (1.2.3) that for any $\underline{\alpha} \in \mathbb{N}^d$, with $\alpha_i > 0$, we have

$$\|G_{\underline{\alpha}}\|_U \leq \sup\left(\left\|\frac{\partial G_{\underline{\alpha}-\underline{1}_i}}{\partial x_i}\right\|_U, \|G_{\underline{\alpha}-\underline{1}_i}G_i\|_U\right)$$

$$\leq \|G_{\underline{\alpha}-\underline{1}_i}\|_U \sup\left(|\frac{\partial}{\partial x_i}|_U, \|G_i\|_U\right).$$

Recursively we obtain

$$\|G_{\underline{\alpha}}\|_U \leq \sup_{i=1,\ldots,d}\left(|\frac{\partial}{\partial x_i}|_U, \|G_i\|_U\right)^{|\alpha|_{\infty}},$$

hence

$$||G_{[\underline{\alpha}]}||_U \leq \sup_{i=1,\ldots,d} \left(|\frac{\partial}{\partial x_i}|_U, ||G_i||_U\right)^{|\underline{\alpha}|_\infty} /||\underline{\alpha}|_\infty!|_p .$$

Finally

$$||G_{[\underline{\alpha}]}||_U^{1/|\underline{\alpha}|_\infty} \leq \sup_{i=1,\ldots,d} \left(|\frac{\partial}{\partial x_i}|_U, ||G_i||_U\right) /|p|^{\frac{|\underline{\alpha}|_\infty - S_p(|\underline{\alpha}|_\infty)}{(p-1)|\underline{\alpha}|_\infty}} ,$$

where $S_p(n) \leq \log_p n$ is the sum of p-adic digits of the natural number n, from which we deduce the formula in the statement. $\qquad\square$

3.3 (Upper semi-)continuity of $\xi \mapsto \delta_{\mathcal{X}}(\xi, U)$ for U a Laurent domain in X

Let $U = X(r^{-1}f, sg^{-1})$ be a Laurent domain in X (3.1.1), so that $f_i, g_j \in \mathscr{A}$ and r_i, s_j are positive real numbers. We will say that $U = X(r^{-1}f, sg^{-1})$ is a *special* Laurent domain in X, if $f_i, g_j \in k\{x_1, \ldots, x_d\}$. Since

$$X(r^{-1}f, sg^{-1}) = \left(\underset{i}{\cap} X(r_i^{-1}f_i)\right) \cap \left(\underset{j}{\cap} X(s_j g_j^{-1})\right)$$

we actually have

$$\delta_{\mathcal{X}}(\xi, X(r^{-1}f, sg^{-1})) = \min_{i,j}\left(\delta_{\mathcal{X}}\left(\xi, X(r_i^{-1}f_i)\right), \delta_{\mathcal{X}}\left(\xi, X(s_j g_j^{-1})\right)\right).$$

Proposition 3.3. (*cf.* [2]) *Let $f, g \in \mathscr{A}$, and let $U = X(r^{-1}f)$, with $r > 0$ (resp. $U = X(sg^{-1})$, with $s > 0$), and let $\xi \in U$. Then:*

(3.3.1) $\qquad \delta_{\mathcal{X}}(\xi, U) = \min(1, \inf_{1 \leq |\underline{\alpha}|_\infty, f^{[\underline{\alpha}]}(\xi) \neq 0}\{r^{1/|\underline{\alpha}|_\infty}|f^{[\underline{\alpha}]}(\xi)|^{-1/|\underline{\alpha}|_\infty}\}) .$

(resp.

(3.3.2) $\qquad \delta_{\mathcal{X}}(\xi, U) = \min(1, \inf_{1 \leq |\underline{\alpha}|_\infty, g^{[\underline{\alpha}]}(\xi) \neq 0}\{|g(\xi)|^{1/|\underline{\alpha}|_\infty}|g^{[\underline{\alpha}]}(\xi)|^{-1/|\underline{\alpha}|_\infty}\}) .)$

In particular, for any Laurent domain $U \subset X$, the function $\xi \mapsto \delta_{\mathcal{X}}(\xi, U)$ is upper semicontinuous on U. If $f, g \in k\{x_1, \ldots, x_d\}$, the infima (3.3.1), (3.3.2) are realized on a finite set of $\underline{\alpha} \in \mathbb{N}^d$, depending only upon U. In particular, if U is a special Laurent domain in X, the function $\xi \mapsto \delta_{\mathcal{X}}(\xi, U)$ is continuous on U.

Proof. We consider the case of $U = X(r^{-1}f)$, for $r \in (0, 1)$ first. We extend the base field to $\mathscr{H}(\xi)$, so that the canonical point ξ' over ξ has a neighborhood which is a disk centered at ξ'. We set $\underline{\xi} = (\xi_1, \ldots, \xi_d) = (x_1(\xi), \ldots, x_d(\xi))$. The Taylor expansion at ξ, $g \mapsto \sum_{\underline{\alpha}} g^{[\underline{\alpha}]}(\xi)(\underline{X} - \underline{\xi})^{\underline{\alpha}}$, produces an isometric embedding

(3.3.3) $\qquad T_{\xi, \underline{x}} : \mathscr{A} \to k \otimes k^\circ\{x_1 - \xi_1, \ldots, x_d - \xi_d\} ,$

of $\mathscr{A}$, equipped with the supnorm on X, into the ring of bounded analytic functions on $D_k^d(\underline{\xi}, 1^-)$, with the natural norm. The diameter $\delta_{\mathcal{X}}(\xi, U)$ is then characterized as follows

$$\delta_{\mathcal{X}}(\xi, U) = \sup\left\{\varepsilon \in (0, 1) \ : \ |f(x)| \leq r \ \forall x \in D_{\mathscr{H}(\xi)}^d((\xi_1, \ldots, \xi_d), \varepsilon^+)\right\} .$$

Since

(3.3.4) $\qquad \sup_{x \in D_{\mathscr{H}(\xi)}^d((\xi_1, \ldots, \xi_d), \varepsilon^+)} |f(x)| = \sup_{\underline{\alpha} \in \mathbb{N}^d} |f^{[\underline{\alpha}]}(\xi)|\varepsilon^{|\underline{\alpha}|_\infty} \leq r$

8

we deduce that

$$(3.3.5) \qquad \delta_{\mathcal{X}}(\xi, U) = \min(1, \inf_{1 \leq |\underline{\alpha}|_\infty,\, f^{[\underline{\alpha}]}(\xi) \neq 0} \{ r^{1/|\underline{\alpha}|_\infty} |f^{[\underline{\alpha}]}(\xi)|^{-1/|\underline{\alpha}|_\infty} \}) \,,$$

and hence that $\xi \mapsto \delta_{\mathcal{X}}(\xi, U)$ is an upper semi-continuous function of $\xi \in U$.

If we now assume that $f \in k\{x_1, \ldots, x_d\}$, then $\lim_{|\underline{\alpha}|_\infty \to \infty} ||f^{[\underline{\alpha}]}||_X = 0$. Then there exists a natural number N such that $||f^{[\underline{\alpha}]}||_X < r$, $\forall x \in X$, as soon as $|\underline{\alpha}|_\infty \geq N$. The infimum in (3.3.5) is then really a minimum on the finite set $|\underline{\alpha}|_\infty < N$. The function $\xi \mapsto \delta_{\mathcal{X}}(\xi, U)$ is continuous in this case.

We now consider the case of $U = X(sg^{-1})$, $g \in \mathscr{A}$. As in the previous case, we extend our spaces to $\mathscr{H}(\xi)$, so that we have the canonical point ξ' over ξ with $|g(\xi')| = |g(\xi)| \geq s$. Suppose that there exists $\omega \in D^d_{\mathscr{H}(\xi)}((\xi_1, \ldots, \xi_d), \varepsilon^+)$, for some $\varepsilon \in (0, 1)$, such that $|g(\omega)| < |g(\xi)|$. We deduce from Corollary 5.6 in the appendix, that g has a zero in the disk $D^d_{\mathscr{H}(\xi)}((\xi_1, \ldots, \xi_d), \varepsilon^+)$ so that $\varepsilon > \delta_{\mathcal{X}}(\xi, U)$.

In other words, we have proven that $\delta_{\mathcal{X}}(\xi, U)$ is precisely the minimum distance of a zero of g from ξ'. We use Robba's theory of Newton polygons (*cf.* corollary 5.6 in the appendix) to obtain an explicit formula. The conclusion is that

$$\delta_{\mathcal{X}}(\xi, U) = \min(1, \inf_{1 \leq |\underline{\alpha}|_\infty} \{ |g(\xi)|^{1/|\underline{\alpha}|_\infty} |g^{[\underline{\alpha}]}(\xi)|^{-1/|\underline{\alpha}|_\infty} \}) \,.$$

As in the previous case, the infimum is really a minimum, and if $g \in k\{x_1, \ldots, x_d\}$ it is a minimum in a finite set of $\underline{\alpha}$'s. We conclude as in the previous case. $\qquad \square$

4 The Dwork-Robba theorem and the upper semicontinuity of $\xi \mapsto R(\xi, \Sigma)$

4.1 The global growth estimate

We set ourselves in the situation of (1.1.1). We will need the following estimate, a corollary of the generalized form of the theorem of Dwork and Robba [11, Chap. IV, Thm. 3.1] given below.

Theorem 4.1. (Growth estimate) *Assume the entries of the matrices G_i in (1.1.1) are bounded analytic functions on the analytic domain U. For any $\xi \in U$ let $R(\xi) = R(\xi, \Sigma)$ be the radius of convergence of Σ at ξ. Let, for any $\underline{\beta} \in \mathbb{N}^d$, $C_{\underline{\beta}} = C_{\underline{\beta}}(\Sigma, U)$ be the constant*

$$(4.1.1) \qquad\qquad C_{\underline{\beta}} = \left|\left| G_{\underline{\beta}} \right|\right|_U = \sup_{\zeta \in U} \left| \underline{\beta}!\, G_{[\underline{\beta}]}(\zeta) \right| \,,$$

and $C = C(\Sigma, U)$ be

$$(4.1.2) \qquad\qquad C = \max_{|\underline{\beta}|_\infty \lneqq \mu} C_{\underline{\beta}} \,.$$

For any $\underline{\alpha} \in \mathbb{N}^d$ we have the following growth estimate on the coefficients of Y_ξ
(4.1.3)

$$|G_{[\underline{\alpha}]}(\xi)| \leq \left(\sum_{|\underline{\beta}|_\infty \lneqq \mu} C_{\underline{\beta}} R(\xi)^{|\underline{\beta}|_\infty} \right) \{ |\underline{\alpha}|_\infty, (\mu-1) \}_p R(\xi)^{-|\underline{\alpha}|_\infty} \leq C \{ |\underline{\alpha}|_\infty, (\mu-1) \}_p R(\xi)^{-|\underline{\alpha}|_\infty} \,,$$

where

$$\{ s, n \}_p = \sup_{1 \leq \lambda_1 \lneqq \lambda_2 \lneqq \cdots \lneqq \lambda_n \leq s} \left(\frac{1}{|\lambda_1 \cdots \lambda_n|_p} \right) \,.$$

9

Remark 4.2. $\{s, n\}_p \leq s^n$.

Remark 4.3. In practice, the estimate (4.1.3) is used in the form

$$(4.3.1) \qquad |G_{[\underline{\alpha}]}(\xi)| \leq C|\underline{\alpha}|_\infty^{\mu-1} R(\xi)^{-|\underline{\alpha}|_\infty} .$$

Corollary 4.4. *For any $\varepsilon > 0$, there exists $s_\varepsilon \in \mathbb{N}$, such that for every $\underline{\alpha} \in \mathbb{N}^d$, with $|\underline{\alpha}|_\infty \geq s_\varepsilon$ and every $\xi \in U$*

$$(4.4.1) \qquad |G_{[\underline{\alpha}]}(\xi)|^{1/|\underline{\alpha}|_\infty} \leq (1 + \varepsilon)/R(\xi) .$$

We mention a variation of (4.1), which is often useful. Let $\mathrm{cl}_X(U)$ be the closure of U in X, and let $\mathscr{A}_{(U)}$ be the localization of the algebra $\mathscr{A}$ with respect to the elements which do not vanish on $\mathrm{cl}_X(U)$. Let $\mathscr{H}(U) \subset \mathscr{A}_U^+$ denote the completion of $\mathscr{A}_{(U)}$ in the supnorm $||\ ||_U$. The elements of $\mathscr{H}(U)$ will be called *analytic elements on U*; they define continuous real valued functions on $\mathrm{cl}_X(U)$. Namely, if $h \in \mathscr{H}(U)$ is the uniform limit $h = \lim_i R_i/S_i$, where $R_i, S_i \in \mathscr{A}$, and S_i does not vanish on $\mathrm{cl}_X(U)$, we may define for any limit point $\xi \in \mathrm{cl}_X(U)$, $\xi = \lim_{j \to \infty} \eta_j$, $\eta_j \in U$, $|h(\xi)| = \lim_i R_i(\eta_j)/S_i(\eta_j)$. Assume the entries of the matrices G_i in (1.1.1) are analytic elements on the analytic domain U, and that the function $\xi \mapsto \delta_{\mathcal{X}}(\xi, U)$ admits a continuous extension on $\mathrm{cl}_X(U)$. Then, $|G_{[\underline{\alpha}]}(\xi)|$ exists $\forall \underline{\alpha}$ and $\forall \xi \in \mathrm{cl}_X(U)$, and $\widetilde{R}(\xi, \Sigma)$ is defined by formula (1.1.4) $\forall \xi \in \mathrm{cl}_X(U)$. Let us define $R(\xi, \Sigma)$, $\forall \xi \in \mathrm{cl}_X(U)$, by formula (1.1.5).

Then we have

Theorem 4.5. *Assume the entries of the matrices G_i in (1.1.1) are analytic elements on the analytic domain U, and that the function $\xi \mapsto \delta_{\mathcal{X}}(\xi, U)$ admits a continuous extension on $\mathrm{cl}_X(U)$. Let, for any $\underline{\beta} \in \mathbb{N}^d$, $C_{\underline{\beta}} = C_{\underline{\beta}}(\Sigma, U)$ and $C = C(\Sigma, U)$ be the constants defined in (4.1). For any $\underline{\alpha} \in \mathbb{N}^d$ we have again the growth estimate (4.1.3).*

Corollary 4.6. *Remark (4.3.1) and corollary (4.4) hold under the assumptions of theorem (4.5).*

4.2 The generalized Dwork-Robba theorem

The following discussion, due to Dwork, has been previously made available by Gachet [13]. We rediscuss it here in the framework of Berkovich spaces. We set ourselves in a slightly more general situation, namely we assume that the matrices G_i of the system Σ in (1.1.1), are meromorphic functions in a polydisk $D(a, r^-)$, for $a = (a_1, \ldots, a_d) \in k^d$, $r = (r_1, \ldots, r_d) \in \mathbb{R}_{>0}^d$

$$D(a, r^-) = D_k^d(a, r^-) = \{\xi \in \mathbb{A}_k^d \,|\, |x_i(\xi) - a_i| < r_i \,,\, \forall i = 1, \ldots, d\ \} .$$

The *field $\mathcal{M}(D(a, r^-))$ of meromorphic functions on $D(a, r^-)$* is defined as the quotient field of the integral domain $\mathcal{O}(D(a, r^-))$. For any $\rho = (\rho_1, \ldots, \rho_d)$, $0 < \rho_i < r_i$, the maximal point $t_{a,\rho}$ of $D(a, \rho^+)$ belongs to $D(a, r^-)$ and defines a multiplicative map $\mathcal{M}(D(a, r^-)) \to \mathscr{H}(t_{a,\rho})$, $f \mapsto f(t_{a,\rho})$. For $f \in \mathcal{M}(D(a, r^-))$, the function $\rho \mapsto f(t_{a,\rho})$ is continuous, as shown in the appendix, but not necessarily bounded for $0 < \rho_i < r_i$. We define the *boundary seminorm $||\ \ ||_{a,r}$* on $\mathcal{O}(D(a, r^-))$ as

$$(4.6.1) \qquad ||f||_{a,r} = \limsup_{\rho \to r} |f(t_{a,\rho})| \in \mathbb{R}_{\geq 0} \cup \{\infty\} \,,\, f \in \mathcal{M}(D(a, r^-)) .$$

It is clear that

$$||f + g||_{a,r} \leq \sup(||f||_{a,r}, ||g||_{a,r}) ,$$

10

for all $f, g \in \mathcal{M}(D(a, r^-))$, and that

$$\|f\,g\|_{a,r} \leq \|f\|_{a,r}\,\|g\|_{a,r}\,,$$

whenever the right side is defined (the only case excluded is $\|f\|_{a,r} = 0$, $\|g\|_{a,r} = \infty$). Notice that, in one variable X, $\|1/\log(1-X)\|_{0,1} = 0$. If $f = \sum_{\underline{\alpha} \in \mathbb{Z}^d} a_{\underline{\alpha}} (\underline{X} - \underline{a})^{\underline{\alpha}}$, then

$$(4.6.2) \qquad \|f\|_{a,r} = \sup_{\underline{\alpha}} |a_{\underline{\alpha}}|\,.$$

We have the following generalization of the theorem of Dwork and Robba [11, Chap. IV, Thm. 3.1].

Theorem 4.7. *Suppose that the system (1.1.1) has meromorphic coefficients on $D(a, r^-) \subset \mathbb{A}_k^d$, $a = (a_1, \ldots, a_d) \in k^d$, and that it admits a solution matrix $Y \in GL(\mu, \mathcal{M}(D(a, R^-)))$, meromorphic in $D(a, R^-) = D(a, (R_1, \ldots, R_d)^-) \subset D(a, r^-)$. Then, for any $\underline{\alpha} \in \mathbb{N}^d$ we have the following estimate*

$$(4.7.1) \qquad \left\|G_{[\underline{\alpha}]}\right\|_{a,R} \leq C\{|\underline{\alpha}|_\infty, (\mu - 1)\}_p \underline{R}^{-\underline{\alpha}}\,,$$

where $\underline{R}^{-\underline{\alpha}} = R_1^{-\alpha_1} \cdots R_d^{-\alpha_d}$,

$$C = \max_{|\underline{\beta}|_\infty \lneqq \mu} \left(\underline{R}^{\underline{\beta}} \left\|\underline{\beta}!\, G_{[\underline{\beta}]}\right\|_{a,R} \right)\,,$$

$\| \; \|_{a,R}$ *denotes the boundary seminorm on $\mathcal{M}(D_k^d(a, (R_1, \ldots, R_d)^-))$, and*

$$\{s, n\}_p = \sup_{1 \leq \lambda_1 \lneqq \lambda_2 \lneqq \cdots \lneqq \lambda_n \leq s} \left(\frac{1}{|\lambda_1 \cdots \lambda_n|_p} \right)\,.$$

Proof. We may assume that $a = 0$, which will simplify notations. Let us consider the completion $\mathscr{K}_{\underline{b},R}$ of the field $k(\underline{b}) = k(b_1, \ldots, b_d)$ (of rational functions in the variables $b_1, \ldots, b_d$) with respect to the absolute value $|\;|_{\underline{b},R} : f \mapsto |f(t_{0,R})|$, with respect to the variables $\underline{b}$, so that $|b_i|_{\underline{b},R} = R_i$, for any i, and $|c|_{\underline{b},R} = |c|$, for any $c \in k$. We have an injective map of k-algebras

$$(4.7.2) \qquad \begin{array}{ccc} \mathcal{M}(D_k^d(0, (R_1, \ldots, R_d)^-)) & \longrightarrow & \mathcal{M}(D_{\mathscr{K}_{\underline{b},R}}(0, 1^-)) \\[4pt] f(X_1, \ldots, X_d) & \longmapsto & f(b_1 Z, \ldots, b_d Z) \end{array}\,.$$

For any $\underline{\alpha} \in \mathbb{N}^d$, we will shorten $\underline{\alpha}!\, G_{[\underline{\alpha}]}$ into $G_{\underline{\alpha}}$, so that (1.2.2) becomes

$$(4.7.3) \qquad \partial^{\underline{\alpha}} \vec{y} = G_{\underline{\alpha}} \vec{y}\,.$$

We denote by $\widetilde{G}_{\underline{\alpha}}(Z\underline{b})$ the image of $G_{\underline{\alpha}}$, via the injective morphism (4.7.2), and define for any $l \in \mathbb{N}$

$$(4.7.4) \qquad \mathcal{H}_{[l]}(Z) = \frac{1}{l!}\mathcal{H}_l(Z) = \frac{1}{l!} \left(\sum_{|\underline{\alpha}|_\infty = l} \widetilde{G}_{\underline{\alpha}}(Z\underline{b})\underline{b}^{\underline{\alpha}} \right)\,,$$

We reduce to the case of dimension 1 *via* a generic line argument:

Lemma 4.8. *Consider the system of ordinary differential equations*

$$(4.8.1) \qquad \frac{d}{dZ}\vec{y} = \mathcal{H}_1(Z)\vec{y}\,,$$

11

where $\mathcal{H}_1(Z)$ is the matrix of meromorphic functions on $D_{\mathscr{K}_{\underline{b},R}}(0,1^-)$ appearing in (4.7.4). Then, in the notation (4.7.4)

(4.8.2)
$$\left(\frac{d}{dZ}\right)^l \vec{y} = \mathcal{H}_l(Z)\vec{y},$$

Proof. It is enough to prove that the matrices $\mathcal{H}_l(Z)$ verify the recursive relations induced by the Leibnitz formula, namely:

$$
\begin{aligned}
\tfrac{d}{dZ}\mathcal{H}_l(Z) + \mathcal{H}_l(Z)\mathcal{H}_1(Z) &= \sum_{\substack{|\underline{\alpha}|_\infty=l \\ i=1,\dots,d}} \left(\left(\frac{\partial}{\partial X_i}\widetilde{G}_{\underline{\alpha}}\right)(Z\underline{b}) + \widetilde{G}_{\underline{\alpha}}(Z\underline{b})\widetilde{G}_{\underline{1}_i}(Z\underline{b})\right)\underline{b}^{\alpha+\underline{1}_i} \\
&= \sum_{|\underline{\alpha}|_\infty=l+1} \widetilde{G}_{\underline{\alpha}}(Z\underline{b})\underline{b}^{\alpha} = \mathcal{H}_{l+1}(Z).
\end{aligned}
$$

$\square$

We can now conclude the proof of the theorem. We denote by $||\ \ ||_{\underline{b},R}$ the boundary seminorm on $\mathcal{M}(D_{\mathscr{K}_{\underline{b},R}}(0,1^-))$, defined at the beginning of this section, relative to the complete field $\mathscr{K}_{\underline{b},R}$. We on the other hand keep denoting $||\ \ ||_{0,R}$ the boundary seminorm on $\mathcal{M}(D_k^d(0,(R_1,\dots,R_d)^-))$. We have

$$||\mathcal{H}_{[l]}||_{\underline{b},R} = \sup_{|\underline{\beta}|_\infty=l} \left|\frac{1}{l!}G_{\underline{\beta}}\right|_{0,R} \underline{R}^{\underline{\beta}}.$$

The classical theorem of Dwork-Robba in the one variable case (*cf.* [DGS, IV.3.2]) implies that for any $l=|\underline{\alpha}|_\infty$ we obtain the estimate

$$
\begin{aligned}
||G_{[\underline{\alpha}]}||_{0,R}\underline{R}^{\underline{\alpha}} &\leq ||\mathcal{H}_{[l]}||_{\underline{b},R} \\
&\leq \{l,\mu-1\}_p \sup_{j\leq\mu-1} ||\mathcal{H}_j||_{\underline{b},R} \\
&\leq \{|\underline{\alpha}|_\infty,\mu-1\}_p \sup_{|\underline{\beta}|_\infty\leq\mu-1} (\underline{R}^{\underline{\beta}}||G_{\underline{\beta}}||_{0,R}) \\
&\leq C\{|\underline{\alpha}|_\infty,\mu-1\}_p.
\end{aligned}
$$

This ends the proof. $\square$

Corollary 4.9. *Suppose the matrices G_i are holomorphic and bounded in $D(a,R^-)$. Let*

$$C = \max_{|\underline{\beta}|_\infty\lneqq\mu} \left(\underline{R}^{\underline{\beta}}\left|\left|G_{\underline{\beta}}\right|\right|_{D(a,R^-)}\right).$$

Then, for any $\underline{\alpha}\in\mathbb{N}^d$ we have the following estimate

(4.9.1)
$$|G_{[\underline{\alpha}]}(a)| \leq C\{|\underline{\alpha}|_\infty,(\mu-1)\}_p\underline{R}^{-\underline{\alpha}}.$$

Corollary 4.10. *Suppose the matrices G_i are holomorphic and bounded in $D(a,r^-)$. Let $\xi\in D(a,r^-)$, let $\xi'\in D_{\mathscr{H}(\xi)}^d(0,r^-)$ be the canonical point above ξ and let us assume that the fundamental solution matrix (1.2.4) of (1.1.1) at ξ' converges in the polydisk $D_{\mathscr{H}(\xi)}(\xi',(R_1,\dots,R_d)^-)\subseteq D_{\mathscr{H}(\xi)}(a,r^-)$. Let*

$$C = \max_{|\underline{\beta}|_\infty\lneqq\mu} \left(\underline{R}^{\underline{\beta}}\left|\left|G_{\underline{\beta}}\right|\right|_{D(a,r^-)}\right).$$

Then, for any $\underline{\alpha} \in \mathbb{N}^d$ we have the following estimate

$$(4.10.1) \qquad |G_{[\underline{\alpha}]}(\xi)| \leq C\{|\underline{\alpha}|_\infty, (\mu-1)\}_p \underline{R}^{-\underline{\alpha}} .$$

The proof of (4.1) now follows directly. We consider $\xi \in U$, and the canonical point $\xi' \in U_{\mathscr{H}(\xi)}$ above it; let $R = R(\xi, \Sigma) \leq 1$. The disk $D_{\mathcal{X}}(\xi, R^-) \subset U_{\mathscr{H}(\xi)}$ is isomorphic via the coordinate functions to $D_{\mathscr{H}(\xi)}(\underline{x}(\xi), (R, \ldots, R)^-)$. We apply (4.10) to the restriction of Σ to $D_{\mathcal{X}}(\xi, R^-)$, taking $r = R$.

4.3 Upper semicontinuity of $\xi \mapsto R(\xi, \Sigma)$

We are now back to the assumptions in (1.1) and (4.1), so in particular the matrices G_i are supposed to be *bounded* on U, and let us further insist that the function $\xi \mapsto \delta_{\mathcal{X}}(\xi, U)$ be USC on U. For example, by (3.3), this happens when U be a Laurent domain in X. For $s = 1, 2, \ldots$ and for $\xi \in U$, let

$(4.10.2)$
$$\varphi_s(\xi) = \min(\delta_{\mathcal{X}}(\xi, U), \inf_{|\underline{\alpha}|_\infty \geq s} |G_{[\underline{\alpha}]}(\xi)|^{-1/|\underline{\alpha}|_\infty}) = \inf_{|\underline{\alpha}|_\infty \geq s} \min(\delta_{\mathcal{X}}(\xi, U), |G_{[\underline{\alpha}]}(\xi)|^{-1/|\underline{\alpha}|_\infty}) .$$

So, $\eta \mapsto \varphi_s(\xi)$ is USC on U, and

$$(4.10.3) \qquad R_{\mathcal{X}}(\xi, \Sigma)) = \lim_{s \to \infty} \varphi_s(\xi) ,$$

where $R_{\mathcal{X}}(\xi, \Sigma))$ is the function introduced in (1.1.5). The corollary (4.4) of the Dwork-Robba theorem says that, $\forall \varepsilon > 0$, $\exists s_\varepsilon$ such that $\forall \underline{\alpha}$ with $|\underline{\alpha}|_\infty \geq s_\varepsilon$

$$(4.10.4) \qquad |G_{[\underline{\alpha}]}(\xi)|^{1/|\underline{\alpha}|_\infty} \leq (1+\varepsilon)/R_{\mathcal{X}}(\xi, \Sigma) , \quad \forall \xi \in U .$$

So,

$$(4.10.5) \qquad |G_{[\underline{\alpha}]}(\xi)|^{-1/|\underline{\alpha}|_\infty} \geq \frac{R_{\mathcal{X}}(\xi, \Sigma)}{1+\varepsilon} , \quad \forall \xi \in U .$$

Hence

$$(4.10.6) \qquad \begin{array}{cccc} \forall \varepsilon > 0 & \exists s_\varepsilon & \text{such that} & \forall s \geq s_\varepsilon \\[2mm] \varphi_s(\xi) \leq & R_{\mathcal{X}}(\xi, \Sigma) & \leq (1+\varepsilon)\varphi_s(\xi) & \forall \xi \in U , \end{array}$$

because the sequence $s \mapsto \varphi_s$ is an increasing sequence of functions on U. Then, $\forall \varepsilon > 0$, $\exists s_\varepsilon$ such that

$$(4.10.7) \qquad 0 \leq R_{\mathcal{X}}(\xi, \Sigma) - \varphi_s(\xi) \leq \varepsilon \; \forall \xi \in U , \quad \forall s \geq s_\varepsilon .$$

Then $\xi \mapsto R_{\mathcal{X}}(\xi, \Sigma)$ is a uniform limit of USC functions, and is therefore USC. We then state

Theorem 4.11. *Assume the matrices G_i are bounded analytic functions on the analytic domain U, and suppose the function $\xi \mapsto \delta_{\mathcal{X}}(\xi, U)$ is USC on U. Then $\xi \mapsto R_{\mathcal{X}}(\xi, \Sigma)$ is USC on U.*

Similarly we have

Theorem 4.12. *Assume the matrices G_i are analytic elements on the analytic domain U, and suppose the function $\xi \mapsto \delta_{\mathcal{X}}(\xi, U)$ admits a continuous extension to $\mathrm{cl}_X(U)$. Let us define $R_{\mathcal{X}}(\xi, \Sigma)$ on $\mathrm{cl}_X(U)$ as in theorem (4.5). Then $\xi \mapsto R_{\mathcal{X}}(\xi, \Sigma)$ is USC on $\mathrm{cl}_X(U)$.*

13

4.4 Continuity of $\xi \mapsto R(\xi, \Sigma)$ at maximal points (Dwork's transfer theorem)

An immediate consequence of formula (4.3.1) is the following. Let U be an affinoid domain in X, and let $\Gamma(U) = \{\eta_1, \ldots, \eta_N\}$ be the Shilov boundary of U. Then, for the constant $C = C(\Sigma, U)$ of (4.1),

$$(4.12.1) \qquad ||G_{[\underline{\alpha}]}||_U \leq C|\underline{\alpha}|_\infty^{\mu-1} \left(\min_{i=1,\ldots,N} R(\eta_i, \Sigma) \right)^{-|\underline{\alpha}|_\infty} .$$

This shows that, for any $\xi \in U$,

$$(4.12.2) \qquad \widetilde{R}(\xi, \Sigma) \geq \min_{i=1,\ldots,N} R(\eta_i, \Sigma) .$$

Proposition 4.13. *Let us assume that U is a Laurent domain in X with a unique maximal point η_U, and that the function $\xi \mapsto \delta_{\mathcal{X}}(\xi, U)$ be continuous at η_U. Then, $\xi \mapsto R(\xi, \Sigma)$ is continuous at η_U.*

Proof. USC of $\xi \mapsto R(\xi, \Sigma)$, shows that

$$\lim_{\xi \to \eta_U} R(\xi, \Sigma) \leq R(\eta_U, \Sigma) .$$

Since

$$\lim_{\xi \to \eta_U} \delta_{\mathcal{X}}(\xi, U) = \delta_{\mathcal{X}}(\eta_U, U) ,$$

we conclude by (4.12.2) that

$$\lim_{\xi \to \eta_U} R(\xi, \Sigma) = R(\eta_U, \Sigma) .$$

$\square$

Corollary 4.14. *If $U = X$, $\xi \mapsto R(\xi, \Sigma)$ is continuous at η_X.*

Proof. $\delta_{\mathcal{X}}(\xi, X) = 1$, $\forall \xi \in X$. $\square$

5 The one-dimensional case

5.1 The theorem of Christol-Dwork revisited

Christol and Dwork (*cf.* [5]) consider a differential system

$$(5.0.1) \qquad \Sigma = \Sigma_{x,G,U} \;\; : \;\; \frac{d}{dx}\vec{y} = G\,\vec{y}$$

with G a $\mu \times \mu$ matrix of analytic elements on the annulus

$$C(r_1, r_2) = \{\xi : r_1 < |x(\xi)| < r_2\} \subset D_k(0, 1^+) .$$

So, the entries of G are elements of the k-Banach algebra $\mathscr{H}(r_1, r_2)$ of uniform limits on $C(r_1, r_2)$ of rational functions in $k(x)$, having no pole in $C(r_1, r_2)$. Here

$$\mathrm{cl}_X(U) = C^+(r_1, r_2) = C(r_1, r_2) \cup \{t_{0,r_1}, t_{0,r_2}\} .$$

Christol and Dwork consider the function radius of convergence of (5.0.1), restricted to a segment of points in $C^+(r_1, r_2)$, namely

$$R : \quad [r_1, r_2] \quad \longrightarrow \qquad \mathbb{R}_{\geq 0}$$
$$r \quad \longmapsto \quad R(r) := R(t_r, \Sigma) = \min(r, \widetilde{R}(t_r, \Sigma)) \quad ,$$

where $t_r = t_{0,r}$ is the point at the boundary of $D(0, r^-)$. This coincides with our definitions (4.5) taking into account the fact that the function $\xi \mapsto |x(\xi)|$ extends continuously $\xi \mapsto \delta_{\mathcal{X}}(\xi, C(r_1, r_2))$ to $C^+(r_1, r_2)$.

So the problem is to describe

$$r \longmapsto \widetilde{R}(r, \Sigma) = \liminf_{s \to \infty} \left| G_{[s]}(t_r) \right|^{-1/s}$$

on $[r_1, r_2]$. They use the well-known fact that, for any $f \in \mathscr{H}(r_1, r_2)$, the function $\rho \mapsto \log |f(t_{e^\rho})|$ is convex and continuous on the interval $[r_1, r_2]$. It is an elementary fact that, if $\forall i \in \mathbb{N}, \varphi_i : [r_1, r_2] \to \mathbb{R}$ is a convex (resp. concave) function, then

$$\varphi = \limsup_{i \to \infty} \varphi_i \quad (\text{resp.} \quad \varphi = \liminf_{i \to \infty} \varphi_i)$$

is convex (resp. concave). They conclude that the function $\rho \mapsto \log \widetilde{R}(e^\rho)$ is concave (*i.e.* $\cap$-shaped) in $[\log r_1, \log r_2]$. So the function $\widetilde{R}$ is continuous in $(\log r_1, \log r_2)$ and LSC at $\log r_1$ and $\log r_2$. Then the same is true for the function R. But we have proven in section (4.3), that the function R is USC in U, so, in the present case, R is continuous. The conclusion is that:

Theorem 5.1 (Christol-Dwork). *Let* $\mathcal{X} = \widehat{\mathbb{A}}^1_{k^\circ}$, $U = C(r_1, r_2)$, *and assume the entries of* G *are analytic elements on* $C(r_1, r_2)$. *Then the function*

$$[r_1, r_2] \quad \longrightarrow \quad \mathbb{R}_{>0}$$

$$r \quad \longmapsto \quad R(r) = R(t_{0,r}, \Sigma)$$

is continuous.

Remark 5.2. We do not claim that the function $r \mapsto \widetilde{R}(r, \Sigma)$ is continuous at r_1, r_2.

5.2 Continuity of $\xi \mapsto R_{\mathcal{X}}(\xi, \Sigma)$ on an affinoid $U \subset D(0, 1^+)$ of dimension 1

In this section we consider a system $\Sigma = \Sigma_{x,G,U}$ of the form (5.0.1) on an affinoid domain U of $D(0, 1^+)$. So, this is the case of system (1.1.1) under the assumptions of (1.1), in dimension one, and with the further condition that U is affinoid and $\mathcal{X} = \widehat{\mathbb{A}}^1_{k^\circ}$.

We prove continuity of $\xi \mapsto R(\xi, \Sigma)$. Since our definitions of diameter and radius of convergence of a system are invariant by base-field extension, we may apply the discussion of (2.1), and assume that the field k is maximally complete and algebraically closed. An affinoid U of $D(0, 1^+)$ is of the form

(5.2.1) $$U = D(0, 1^+) \smallsetminus \cup_{i \in I} D(a_i, r_i^-) \subset X ,$$

where I is a finite set and a_i is a k-rational point of $D(0, 1^+)$. We are left to prove LSC continuity of $\xi \mapsto R(\xi, \Sigma)$ for this system.

Notice that, because k is maximally complete and algebraically closed, the points of $D(0, 1^+)$ are either k-rational points or of the form $t_{a,r} =$ the boundary point of a disk $D(a, r^-)$, centered at a k-rational point a and of radius $r \in (0, 1]$.

Theorem 5.3. *The function* $\xi \mapsto R(\xi, \Sigma)$ *is continuous on* U.

Proof. We will have to restrict the system Σ to various affinoid subdomains V of U. We then write $R(\xi, V)$ for $R(\xi, \Sigma)$, when Σ is restricted to the affinoid $V \subset U$. Let $\xi \in U$ be a k-rational point. Then the function $\eta \mapsto R(\eta, U)$, which expresses the radius of the maximal

15

open disk centered at η and contained in U on which Σ is the trivial differential system, is clearly constant in the neighborhood $D_k(\xi, R(\xi, U)^-)$ of ξ. This neighborhood is non empty since $R(\xi, U) = \min(\widetilde{R}(\xi, \Sigma), \delta_{\mathcal{X}}(\xi, U))$, and we compute:

$$\text{(5.3.1)} \qquad\qquad \delta_{\mathcal{X}}(\xi, U) = \min_{i \in I} |x(\xi) - a_i| > 0 \,,$$

and, by (3.2),

$$\text{(5.3.2)} \qquad\qquad \widetilde{R}(\xi, \Sigma) \geq \min(1, \frac{|p|^{\frac{1}{p-1}}}{\max\left(|\frac{d}{dx}|_U, \|G\|_U\right)}) > 0 \,.$$

We are left to prove continuity (in fact just LSC) of $\xi \mapsto R(\xi, U)$ at a point $\xi \in U$ of the form $\xi = t_{a,r} \in U$. Notice that we may (and will) assume $R = R(t_{a,r}, U) \leq r$, otherwise on the disk $D(a, R^-)$, which is an open neighborhood of $t_{a,r}$, the function $\xi \mapsto R(\xi, U)$ would be constant of value R, and $\xi \mapsto R(\xi, U)$ would then be continuous at $\xi = t_{a,r}$. On the other hand, $\delta_{\mathcal{X}}(t_{a,r}, U) \geq r$, so in particular we assume $\widetilde{R}(t_{a,r}, \Sigma) \leq \delta_{\mathcal{X}}(t_{a,r}, U)$.

Let

$$J = \{i \in I : |a - a_i| = r\} \,,$$

and let $\varepsilon_0 := \min_{i \notin J} |a - a_i|$. We further subdivide J into a disjoint union $J = J_1 \cup J_2$, where

$$J_1 = \{i \in J : r_i < r\} \quad, \quad J_2 = \{i \in J : r_i = r\} \,.$$

We want to construct a (not fundamental) system of affinoid neighborhoods $\{V_\varepsilon\}_{\varepsilon > \varepsilon_0}$ of $t_{a,r} \in U$, with the property that the Shilov boundary $\Gamma(V_\varepsilon)$ of V_ε consists of the points $t_{a_i, r-\varepsilon}$, for $i \in J_1$, $t_{a_i, r} = t_{a,r}$, for $i \in J_2$ and of the point $t_{a, r+\varepsilon}$. Notice that $t_{a, r+\varepsilon} \to t_{a,r}$, and $t_{a_i, r-\varepsilon} \to t_{a,r}$, as $\varepsilon \to 0$, $\forall i \in J_1$. We simply take for $\varepsilon > \varepsilon_0$
(5.3.3)

$$V_\varepsilon = \{\eta \in X : r - \varepsilon \leq |x(\eta) - a| \leq r + \varepsilon\} \setminus \left(\bigcup_{i \in J_1} D(a_i, (r-\varepsilon)^-) \cup \bigcup_{i \in J_2} D(a_i, r^-)\right) .$$

Notice that

$$\text{(5.3.4)} \qquad \delta_{\mathcal{X}}(t_{a, r+\varepsilon}, V_\varepsilon) = r + \varepsilon \;\; ; \;\; \delta_{\mathcal{X}}(t_{a,r}, V_\varepsilon) = r \;\; ; \;\; \delta_{\mathcal{X}}(t_{a_i, r-\varepsilon}, V_\varepsilon) = r - \varepsilon \;\; \forall i \in J \,.$$

Coming back to our differential system (5.0.1) and its iterates

$$\frac{1}{s!}\left(\frac{d}{dx}\right)^s Y = G_{[s]} Y \,, \; G_{[s]} \in M_{\mu \times \mu}(\mathcal{O}(U)),$$

we have, by (4.12.2), $\forall \eta \in V_\varepsilon$, $t_{a_i, r-\varepsilon}$, for $i \in J_1$, $t_{a_i, r} = t_{a,r}$, for $i \in J_2$ and of the point $t_{a, r+\varepsilon}$.

$$\widetilde{R}(\eta, \Sigma) \geq \min\left(\min_{i \in J} R(t_{a_i, r-\varepsilon}, V_\varepsilon)\,, R(t_{a,r}, V_\varepsilon)\right) .$$

The affinoid V_ε contains the annuli $\{\eta \in X \,|\, r < |x(\eta) - a| < r + \varepsilon\}$, $\{\eta \in X \,|\, r - \varepsilon < |x(\eta) - a_i| < r\}$ and analytic functions on V_ε restrict to analytic elements on them. So, we may apply the theorem of Christol-Dwork (5.1) to deduce

$$\lim_{\varepsilon \to 0} R(t_{a_i, r-\varepsilon}, V_\varepsilon) = R(t_{a,r}, V_\varepsilon) \,,$$

$\forall i \in J$, and similarly $\lim_{\varepsilon \to 0} R(t_{a, r+\varepsilon}, V_\varepsilon) = R(t_{a,r}, V_\varepsilon)$. Notice that

$$\lim_{\varepsilon \to 0} \delta_{\mathcal{X}}(t_{a_i, r-\varepsilon}, V_\varepsilon) = \lim_{\varepsilon \to 0} \delta_{\mathcal{X}}(t_{a, r+\varepsilon}, V_\varepsilon) = \delta_{\mathcal{X}}(t_{a,r}, V_\varepsilon) = r \,,$$

$\forall i \in J$. We conclude that $\forall \sigma > 0$, $\exists \varepsilon > \varepsilon_0$ such that

$$\widetilde{R}(\eta, \Sigma) \geq R(t_{a,r}, V_\varepsilon) - \sigma = \min(\widetilde{R}(t_{a,r}, \Sigma), r) - \sigma = \widetilde{R}(t_{a,r}, \Sigma) - \sigma \, ,$$

$\forall \eta \in V_\varepsilon$.

The conclusion is that $\eta \mapsto \widetilde{R}(\eta, \Sigma)$ is LSC at $t_{a,r}$. Since in the present case, $\eta \mapsto \delta_{\mathcal{X}}(\eta, U)$ is continuous, we conclude that $\eta \mapsto R(\eta, U)$ is LSC at $t_{a,r}$. Since we already know that it is USC, we conclude that it is actually continuous at $t_{a,r}$. $\qquad\square$

5.3 Continuity of $\xi \mapsto R_{\mathcal{X}}(\xi, \Sigma)$ in $\dim X = 1$, when U is a neighborhood of η_X

We assume here that $\mathcal{X}$ is a smooth formal scheme of relative dimension 1 over $\mathrm{Spf}\, k^\circ$, and U is an affinoid neighborhood of the maximal point η_X of $X = \mathcal{X}_\eta$ (always satisfying the requirements in (1.1)). Notice that the special fiber $\mathcal{X}_s$ of $\mathcal{X}$ is a smooth scheme over $\tilde{k}$, which we may assume to be connected. The reduction map $\pi : X \to \mathcal{X}_s$, is such that the fiber at each closed point of $\mathcal{X}_s$ is an open disk of radius one, called *a residue class*, while the inverse image of the generic point $\eta_{\mathcal{X}_s}$ consists only of the maximal point η_X of X. An affinoid $U \subset X$ is a neighborhood of η_X if and only if it contains almost all residue classes in X, and contains a non trivial annulus of outer radius one in each of the remaining residue classes. So, U is the disjoint union of the generic fiber $Y = \mathcal{Y}_\eta$ of a smooth formal scheme $\mathcal{Y}$ (the union of full residue classes) and of a finite number $\{C_1, \ldots, C_r\}$ of analytic subdomains of the open disk of radius one, which are bound to contain some annulus of outer radius one. We are given the system (5.0.1) on U, and must prove continuity of $\xi \mapsto R_{\mathcal{X}}(\xi, \Sigma)$ on U. Notice that, if we call $\Sigma_{|Y}$, $\Sigma_{|C_1}$, \ldots, $\Sigma_{|C_r}$, the restrictions of Σ to the various analytic subdomains of U, we have

$$R_{\mathcal{Y}}(\xi, \Sigma_{|Y}) = R_{\mathcal{X}}(\xi, \Sigma) \, , \ \forall \xi \in Y \, ,$$

because $\delta_{\mathcal{X}}(\xi, Y) = \delta_{\mathcal{Y}}(\xi, Y) = 1$, $\forall \xi \in Y$. Similarly,

$$R_{\mathcal{Y}}(\xi, \Sigma_{|C_i}) = R_{\mathcal{X}}(\xi, \Sigma) \, , \ \forall \xi \in C_i \, , \ \forall i = 1, \ldots, r \, .$$

We already proved continuity of $\xi \mapsto R_{\mathcal{Y}}(\xi, \Sigma_{|Y})$ as a function $Y \to \mathbb{R}$, at the maximal point $\eta_Y = \eta_X$. Notice that $R_{\mathcal{Y}}(\xi, \Sigma_{|Y}) = \min(1, \widetilde{R}(\xi, \Sigma))$, since $\delta_{\mathcal{Y}}(\xi, Y) = 1$. Continuity on Y then follows from theorem (5.3), since the residue classes in Y may be considered independently, except for being glued at η_Y, and the definition of $R_{\mathcal{Y}}(\eta_Y, \Sigma_{|Y}) = \min(1, \widetilde{R}(\eta_Y, \Sigma))$ only depends upon Σ viewed as a differential system on $\mathscr{H}(\eta_Y)^\mu = \mathscr{H}(\eta_X)^\mu$. As for the residue classes containing C_1, \ldots, C_r, respectively, let us fix one of them, which we regard as $D(0, 1^-) \supset C_1$. Notice that C_1 contains some open annulus of outer radius 1. Continuity of $\xi \mapsto R_{\mathcal{X}}(\xi, \Sigma_{|C_1})$ on C_1 also follows from (5.3). We have to prove that

$$(5.3.5) \qquad \lim_{\xi \to \eta_X} R_{\mathcal{X}}(\xi, \Sigma_{|C_1}) = R_{\mathcal{X}}(\eta_X, \Sigma) \, .$$

Since $\delta_{\mathcal{X}}(\xi, U)$ is continuous on U and takes the value 1 at η_X, restricting to points ξ of the form $t_{0,r}$, as $r \to 1$, it follows from (5.1) that

$$(5.3.6) \qquad \lim_{\xi \to \eta_X} R_{\mathcal{X}}(\xi, \Sigma_{|C_1}) = \min(1, \widetilde{R}(\eta_X, \Sigma)) \, ,$$

which only depends upon η_X, and is therefore independent of the residue class containing C_1, chosen to approach η_X. We conclude

Corollary 5.4. *If U is an affinoid neighborhood of η_X, $\xi \mapsto R(\xi, \Sigma)$ is continuous on U.*

Appendix. Valuation polygon of an analytic function in several variables

For the reader's convenience we recall some facts from [15, §2] on the several variable Newton polygon theory, which has been applied in this paper.

We set $v(x) = -\log|x|$ for any x in an extension of k. Let us assume that k is algebraically closed. For any convergent power series $f = \sum_{\underline{\alpha} \in \mathbb{N}^d} a_{\underline{\alpha}} \underline{x}^{\underline{\alpha}} \in k\,[[\underline{x}]] = k\,[[x_1, \ldots, x_d]]$, $i.e.$ for a formal power series such that

$$\liminf_{n \to \infty} \frac{v(a_{\underline{\alpha}})}{|\underline{\alpha}|_\infty} > -\infty \,,$$

we set:

$$(5.4.1) \qquad \mathrm{Conv}(f) = \left\{ \underline{\mu} \in \mathbb{R}^d : v(a_{\underline{\alpha}}) + \sum \alpha_i \mu_i \to +\infty \text{ when } \sum \alpha_i \to +\infty \right\} ;$$

$$(5.4.2) \qquad v(f, \underline{\mu}) = \inf_{\underline{\alpha} \in \mathbb{N}^d, f_{\underline{\alpha}} \neq 0} \left(v(f_{\underline{\alpha}}) + \sum_i \alpha_i \mu_i \right) , \text{ for any } \underline{\mu} \in \mathrm{Conv}(f);$$

$$(5.4.3) \qquad \mathrm{Reg}(f) = \left\{ \underline{\mu} \in \mathrm{Conv}(f) : \exists! \underline{\beta} \in \mathbb{N}^d \text{ s.t. } v(f, \underline{\mu}) = v(a_{\underline{\beta}}) + \sum_i \beta_i \mu_i \right\} ;$$

$$(5.4.4) \qquad \mathrm{Z}(f) = \mathrm{Conv}(f) \smallsetminus \mathrm{Reg}(f) \,.$$

Then the following properties hold:

1. $\mathrm{Conv}(f)$ is a convex subset of $\mathbb{R}^d$;

2. $v(f, -)$ is a concave continuous function on $\mathrm{Conv}(f)$;

3. the graph of $v(f, -)$ on the interior of $\mathrm{Conv}(f)$ is a polyhedron (with possibly infinitely many faces).

Proposition 5.5 (*cf.* [15, 2.12,2.20]).

1. *Let $\underline{\xi} \in k^d$ and $\underline{\mu} = (v(\xi_1), \ldots, v(\xi_d))$ be in $\mathrm{Conv}(f)$. If $\underline{\mu} \in \mathrm{Reg}(f)$ then $f(\underline{\xi}) \neq 0$ and $v(f(\underline{\xi})) = v(f, \underline{\mu})$.*

2. *Let $\underline{\mu} \in \mathrm{Z}(f) \cap v(k)^d$. Then there exists $\underline{\xi} \in k^d$ such that $f(\underline{\xi}) = 0$ and $v(\xi_i) = \mu_i$ for any $i = 1, \ldots, d$.*

Corollary 5.6. *Let $\underline{\xi} \in k^d$ with $\underline{\mu} = (v(\xi_1), \ldots, v(\xi_d)) \in Conv(f)$. We suppose that $|f(\underline{\xi})| < |f(\underline{0})|$ (resp. $|f(\underline{\xi})| > |f(\underline{0})|$). Then there exists $\underline{\zeta} \in k^d$ such that $f(\underline{\zeta}) = 0$ and $|\zeta_i| = |\xi_i|$ (resp. $|\zeta_i| \leq |\xi_i|$) for any $i = 1, \ldots, d$.*

Proof. Let us suppose that $|f(\underline{\xi})| < |f(0)|$. This means that $v(f_{(0,\ldots,0)}) < v(f(\xi))$ and hence that

$$\inf_{\underline{\alpha} \in \mathbb{N}^d, f_{\underline{\alpha}} \neq 0} \left(v(f_{\underline{\alpha}}) + \sum_i \alpha_i \mu_i \right) = v(f_{\underline{\alpha}'}) + \sum_i \alpha_i' \mu_i = v(f_{\underline{\alpha}''}) + \sum_i \alpha_i'' \mu_i \,,$$

for some $\underline{\alpha}', \underline{\alpha}'' \in \mathbb{N}^d$ such that $\underline{\alpha}' \neq \underline{\alpha}''$. Therefore $\underline{\mu} \in \mathrm{Z}(f)$ and the corollary follows from the previous proposition.

If on the contrary $|f(\underline{\xi})| > |f(\underline{0})|$ it is enough to consider the expansion of f at $\underline{\xi}$. $\qquad \square$

References

[1] Yves André. *p*-adic orbifolds and *p*-adic triangle groups. *Sūrikaisekikenkyūsho Kōkyūroku*, (1073):136–159, 1998. Rigid geometry and group actions (Japanese) (Kyoto, 1998).

[2] Vladimir G. Berkovich. Letter to Francesco Baldassarri.

[3] Vladimir G. Berkovich. *Spectral theory and analytic geometry over non-Archimedean fields*, volume 33 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1990.

[4] Vladimir G. Berkovich. Étale cohomology for non-Archimedean analytic spaces. *Institut des Hautes Études Scientifiques. Publications Mathématiques*, 78:5–161 (1994), 1993.

[5] G. Christol and B. Dwork. Modules différentiels sur des couronnes. *Université de Grenoble. Annales de l'Institut Fourier*, 44(3):663–701, 1994.

[6] G. Christol and Z. Mebkhout. Sur le théorème de l'indice des équations différentielles *p*-adiques. II. *Ann. of Math. (2)*, 146(2):345–410, 1997.

[7] G. Christol and Z. Mebkhout. Sur le théorème de l'indice des équations différentielles *p*-adiques. III. *Ann. of Math. (2)*, 151(2):385–457, 2000.

[8] G. Christol and Z. Mebkhout. Sur le théorème de l'indice des équations différentielles *p*-adiques. IV. *Invent. Math.*, 143(3):629–672, 2001.

[9] Gilles Christol and Zoghman Mebkhout. Équations différentielles *p*-adiques et coefficients *p*-adiques sur les courbes. *Astérisque*, (279):125–183, 2002. Cohomologies *p*-adiques et applications arithmétiques, II.

[10] B. Dwork. On exponents of *p*-adic differential modules. *J. reine angew. Math.*, 484:85–126, 1997.

[11] Bernard Dwork, Giovanni Gerotto, and Francis J. Sullivan. *An introduction to G-functions*, volume 133 of *Annals of Mathematics Studies*. Princeton University Press, 1994.

[12] Ryszard Engelking. *General topology*, volume 6 of *Sigma Series in Pure Mathematics*. Heldermann Verlag, Berlin, second edition, 1989. Translated from the Polish by the author.

[13] F. Gachet. Structure fuchsienne pour des modules différentiels sur une polycouronne ultramétrique. *Rendiconti del Seminario Matematico della Università di Padova. The Mathematical Journal of the University of Padova*, 102:157–218, 1999.

[14] E. G. C. Poole. *Introduction to the theory of linear differential equations*. Dover Publications Inc., New York, 1960.

[15] Philippe Robba. Prolongement analytique pour les fonctions de plusieurs variables sur un corps valué complet. *Bulletin de la Société Mathématique de France*, 101:193–217, 1973.