On the arithmetic size of linear differential equations

by

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Table of contents

§1. Introduction.

The notion of G-function was first introduced by C. L. Siegel in 1929. Later work of Bombieri, Chudnovsky, André, Dwork clarified the geometric content of that (one variable) notion, as a solution of a special type of linear differential operator (*of arithmetic type* or *G-operator*). A geometric theory of G-functions was established in full generality by André and Baldassarri in [AB].

We recall (a variation of) the classical definition. Let K be a number field and let $\mathcal{V}_K$ be its ring of integers. A *G-function at the origin defined over K* is a formal power series

$$y(x)=\sum_{j\in\mathbb{N}}A_jx^j\in K[\![x]\!]\ ,$$

such that:
1) $Ly=0$ for some non zero $L\in K(x)\left[\frac{d}{dx}\right]$;
2) for each embedding $\sigma:K\hookrightarrow\mathbb{C}$, the formal power series $\sum_{j\in\mathbb{N}}\sigma(A_j)x^j\in\mathbb{C}[\![x]\!]$ has a positive radius of convergence;
3) there exists a sequence of positive integers $\{c_s\}_{s\in\mathbb{N}}$ such that $c_sA_j\in\mathcal{V}_K$ for all $j\leq s$ and

$$\sup_{s\in\mathbb{N}^*}\frac{1}{s}\log c_s\lneqq\infty\ .$$

A first non-trivial example of a G-function is the hypergeometric series

$${}_2F_1(a,b,c;x)=\sum_{j\in\mathbb{N}}\frac{(a)_j(b)_j}{(c)_jj!}x^j\ ,$$

where $a,b,c\in\mathbb{Q}$ and $(a)_j=a(a+1)\cdots(a+j-1)$. The vector

$$((c-b){}_2F_1(a,b,c+1;x),c\,{}_2F_1(a,b,c;x))$$

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is a solution of the differential system

$$(E_{a,b,c}) \qquad \frac{dY}{dx} = YE_{a,b,c}, \quad \text{with } E_{a,b,c} = \begin{pmatrix} \frac{-c}{x} & \frac{c-a}{1-x} \\ \frac{c-b}{x} & \frac{a+b-c}{1-x} \end{pmatrix} .$$

The general definitions on G-connections will be recalled in §2. Of great importance in their study is the finite invariant $\tau = \sigma - \rho$. In the hypergeometric case $\tau(E_{a,b,c}) = \sigma(E_{a,b,c}) - \varrho(E_{a,b,c})$ has been calculated by B. Dwork [D, Cor. 1.2] as

$$\tau(E_{a,b,c}) = 1 - \frac{\eta}{\Phi(N)} .$$

Here N is the least common denominator of a, b and c; $0 \leq A, B \lneqq N$ and $0 \lneqq C \leq N$ are positive integers such that $\left(\frac{A}{N}, \frac{B}{N}, \frac{C}{N}\right) \equiv (a,b,c) \bmod N$; η is the cardinality of the set of all $w \in (\mathbb{Z}/N\mathbb{Z})^{\times}$ such that, if $(A^{(w)}, B^{(w)}, C^{(w)}) \equiv (wA, wB, wC) \bmod N$ and $0 \leq A^{(w)}, B^{(w)} \lneqq N$, $0 \lneqq C^{(w)} \leq N$, we have $A^{(w)} \geq C^{(w)} \gneqq B^{(w)}$ or $B^{(w)} \geq C^{(w)} \gneqq A^{(w)}$; Φ is the Euler function.

The principal theme of this paper is the generalization to connections on arithmetic varieties of the main result [D, Th. 1.1] in the above mentioned paper by Dwork. Naturally, we use the geometric language introduced in [AB]. Our result provides a precise estimate for the invariant $\tau = \sigma - \rho$ of an arithmetic differential equation. This invariant, depending only on the geometric generic fiber of the connection, is highly significant. A consequence of our result is that for a differential equation having $\tau = 0$ is equivalent to having zero p-curvature for a set primes p of Dirichlet density 1. Indeed, this is expected to imply that the p-curvature is zero for all but a finite set of primes. The Grothendieck conjecture predicts that $\tau = 0$ should imply that the geometric generic fiber of the connection is trivial.

We also prove a result on the relation between generic v-adic radius of convergence and order of nilpotence of the reduced equation extending [DGS, III.5.1].

In the appendix, we prove a generalization of the Eisenstein theorem.

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§2. Basic definitions and statement of the main results.

2.1. Let K be a number field and $\mathcal{V}_K$ be its ring of integers. We consider a non-empty open subscheme $S = \mathrm{Spec}(\mathcal{V}_S)$ of $\mathrm{Spec}(\mathcal{V}_K)$. We set

$$\Sigma_S = \{\text{finite places of } K \text{ having center on } \mathcal{V}_S\} = \{\text{closed points of } S\} .$$

For each $v \in \Sigma_S$ we denote:
$|\ |_v=$ the absolute value of K associated to v, normalized as follows:

$$|p|_v = p^{-[K_v:\mathbb{Q}_p]/[K:\mathbb{Q}]}, \text{ if } v|p ;$$

$K_v=$ v-completion of K;
$\mathcal{V}_v=$ ring of integers of K_v;
$k(v)=$ residue field of K_v of characteristic $p = p(v)$;
$\pi_v=$ a uniformizer of $\mathcal{V}_v$.

Moreover, for all $\underline{\alpha}=(\alpha_1,\dots,\alpha_d)\in\mathbb{N}^d$ we set $|\underline{\alpha}|_\infty=\sum_{i=1}^d\alpha_i$, $\underline{\alpha}!=\alpha_1!\cdots\alpha_d!$ and

$$\binom{\underline{\alpha}}{\underline{\beta}}=\prod_{i=0}^{d}\binom{\alpha_i}{\beta_i}\ ,\ \text{for all } \underline{\beta}=(\beta_1,\dots,\beta_d)\in\mathbb{N}^d,\ \text{such that } \beta_i\le\alpha_i \text{ for all } i=1,\dots,d.$$

We denote by $\underline{1}_i$ the element of $\mathbb{N}^d$ having all its entries equal to zero except the i-th one equal to 1.

2.2. Let $\mathcal{F}$ be a function field over K; a *smooth S-model of $\mathcal{F}/K$* is a smooth S-scheme $f:X\longrightarrow S$ of finite type with geometrically connected non-empty fibers such that the field of rational functions of $X_K=X\times\operatorname{Spec}(K)$ is $\mathcal{F}$.

The choice of an S-model of $\mathcal{F}$ privileges, for each $v\in\Sigma_S$, one extension $|\ |_{X,v}$ of $|\ |_v$ to $\mathcal{F}$. In fact, let η_v denote the generic point of the closed fiber $X_{k(v)}=X\times_S\operatorname{Spec}(k(v))$, the local ring $\mathcal{O}_{X,\eta_v}$ is a discrete valuation ring, since it is a local regular domain of dimension one with uniformizer π_v. So we define $|\ |_{X,v}$ as the unique extension of $|\ |_v$ to a non archimedean absolute value of $\mathcal{F}$, such that

$$\mathcal{O}_{X,\eta_v}=\{x\in\mathcal{F}:\ |x|_{X,v}\le1\}\ ,$$

normalized so to extend $|\ |_v$.

Let (M,∇) be a $\mathcal{F}/K$-differential module of finite rank μ (*i.e.* $M\cong\mathcal{F}^\mu$) and let

$$\nabla:M\longrightarrow\Omega^1_{\mathcal{F}/K}\otimes M$$

be its integrable connection. A *model of (M,∇) on X/S* is a locally free $\mathcal{O}_X$-module $\mathcal{M}$ of rank μ with an integrable connection

$$\nabla:\mathcal{M}\longrightarrow\Omega^1_{X/S}\otimes\mathcal{M}\ ,$$

such that $(\mathcal{M},\nabla)_{\eta_X}=(M,\nabla)$, where η_X is the generic point of X.

We define the *generic*

v-adic radius of convergence $R_{X,v}(M)$ of (M,∇) on X/S as follows. We consider an étale coordinate neighborhood $(U,\underline{x})$, with $\underline{x}=(x_1,\dots,x_d)$, of η_v in X and a local basis $\underline{e}=(e_1,\dots,e_\mu)$ of $\mathcal{M}$ in a neighborhood of the generic point of X. Let

$$D_i=\frac{\partial}{\partial x_i}\ ,\ \text{for any } i=1,\dots,d.$$

For any $\underline{\alpha}=(\alpha_1,\dots,\alpha_d)\in\mathbb{N}^d$ we set

$$\underline{D}^{[\underline{\alpha}]}=\frac{1}{\underline{\alpha}!}\underline{D}^{\underline{\alpha}}=\prod_{i=1}^{d}\frac{1}{\alpha_i!}D_i^{\alpha_i}\ ,$$

$$\nabla\left(\underline{D}^{[\underline{\alpha}]}\right)=\prod_{i=1}^{d}\frac{1}{\alpha_i!}\nabla\left(\frac{\partial}{\partial x_i}\right)^{\alpha_i}\ ,$$

and

$$\nabla\left(\underline{D}^{[\underline{\alpha}]}\right)\underline{e}=\underline{e}\mathcal{G}_{[\underline{\alpha}]},\ \text{with } \mathcal{G}_{[\underline{\alpha}]}\in M_{\mu\times\mu}(\mathcal{F}) \text{ and } \mathcal{G}_{[\underline{0}]}=I_\mu\ . \tag{2.2.1}$$

Then

$$R_{X,v}(M) = \left(\max\left(1, \limsup_{|\underline{\alpha}|_\infty \to \infty} |\mathcal{G}_{[\underline{\alpha}]}|_{X,v}{}^{1/|\underline{\alpha}|_\infty}\right)\right)^{-1} . \tag{2.2.2}$$

We define the *global inverse radius of* (M, ∇) *on* X/S as

$$\varrho_{X/S}(M) = \sum_{v \in \Sigma_S} \log \frac{1}{R_{X,v}(M)} \in [0, +\infty] . \tag{2.2.3}$$

We say that (M, ∇) is of *type G* or is a *G-module* if $\varrho_{X/S}(M) \lneqq \infty$, for one choice (hence for all) of S, of the S-model X and of $(\mathcal{M}, \nabla)$.

A few questions on the dependence of $R_{X,v}(M)$ and $\varrho_{X/S}(M)$ on the choice of the model X/S, of $(\mathcal{M}, \nabla)$, of the étale coordinates $\underline{x}$ and of the basis $\underline{e}$ naturally arise at this moment: we will come back to this problem in proposition 2.9.

2.3. An easy fact to show (*cf.* §3 below) is that, if (M, ∇) admits a model on X/S, we have for each $v \in \Sigma_S$

$$R_{X,v}(M) \geq |p|_v^{1/(p-1)} .$$

We obtain a better estimate for $R_{X,v}(M)$ by looking at the properties of the differential module induced by $(\mathcal{M}, \nabla)$ on the closed fiber of X over v:

2.4. Let k be a field of characteristic $p \gneqq 0$ and X_k a smooth k-scheme of finite type. Let $(\mathcal{M}_k, \nabla_k)$ be an integrable X_k/k-connection. We recall that $(\mathcal{M}_k, \nabla_k)$ is said to be *nilpotent of exponent* $\leq n$ if, given étale coordinates $(x_1, \dots, x_d)$ on X_k, one has:

$$\nabla\left(\frac{\partial}{\partial x_1}\right)^{pw_1} \cdots \nabla\left(\frac{\partial}{\partial x_d}\right)^{pw_d} = 0 ,$$

for all $(w_1, \dots, w_d) \in \mathbb{N}^d$, such that $|w|_\infty = n$. If $n = 1$, we say that $(\mathcal{M}_k, \nabla_k)$ has *p-curvature* 0.

The following proposition (*cf.* §3 below for the proof) is the generalisation to the several variable case of a classical estimate (*cf.* [DGS, page 96]):

Proposition 2.5. *Let X/S be a smooth S-model of $\mathcal{F}$, $(\mathcal{M}, \nabla)$ an X/S-connection as before and $v \in \Sigma_S$; then the integrable connection $(\mathcal{M}_{k(v)}, \nabla_{k(v)})$ induced on the closed fiber of X over v is nilpotent if and only if $R_{X,v}(M) \gneqq |p(v)|_v^{1/(p(v)-1)}$.*

Apart from $\varrho_{X/S}(M)$, one can define another invariant $\sigma_{X/S}(M)$, attached to an $\mathcal{F}/K$-differential module. We define $\sigma_{X/S}(M)$, called the size, as follows:

2.6. Let $f : X \longrightarrow S$ and $(\mathcal{M}, \nabla)$ be defined as in (2.2). Let $\mathcal{I}$ be the kernel of the map $\mathcal{O}_X \otimes_{f^{-1}(\mathcal{O}_S)} \mathcal{O}_X \longrightarrow \mathcal{O}_X$, induced by mulptiplication. Then for any $n \geq 0$ one defines $\mathcal{P}^n_{X/S} = \mathcal{O}_X \otimes_{f^{-1}(\mathcal{O}_S)} \mathcal{O}_X/\mathcal{I}^{n+1}$ (*cf.* [BO, §2]).

Since X/S is a smooth S-model, it is possible to give an explicit description of $\mathcal{P}^n_{X/S}$ (*cf.* [BO, 2.2]). Let $(x_1, \dots, x_d)$ be local étale coordinates on X, $\xi_i = x_i \otimes 1 - 1 \otimes x_i \in \mathcal{O}_X \otimes_{\mathcal{O}_S} \mathcal{O}_X$, for $i = 1, \dots, d$, and $\underline{\xi} = (\xi_1, \dots, \xi_d)$. Then, for any $n \geq 1$, $\mathcal{P}^n_{X/S}$ is the $\mathcal{O}_X$-module generated by $\{\underline{\xi}^{\underline{\alpha}} = \xi_1^{\alpha_1} \cdots \xi_d^{\alpha_d} : \underline{\alpha} \in \mathbb{N}^d, |\underline{\alpha}|_\infty \leq n\}$.

We notice that $\mathcal{O}_X \otimes_{\mathcal{O}_S} \mathcal{O}_X$ has a left (resp. right) $\mathcal{O}_X$-module structure defined by the map $\mathcal{O}_X \longrightarrow \mathcal{O}_X \otimes_{\mathcal{O}_S} \mathcal{O}_X$, $a \longmapsto a \otimes 1$ (resp. $a \longmapsto 1 \otimes a$). Then $\mathcal{P}^n_{X/S}$ has a left and a right $\mathcal{O}_X$-module structure induced by the $\mathcal{O}_X$-module structures of $\mathcal{O}_X \otimes_{\mathcal{O}_S} \mathcal{O}_X$.

We consider the *induced stratification data* [BO, 2.11] associated to $(\mathcal{M}, \nabla)$

(2.6.1) $$\Theta^{(n)} : \mathcal{M} \longrightarrow \left(\mathcal{P}^n_{X/S} \otimes_{\mathcal{O}_X} \mathcal{M}\right) \otimes_{\mathcal{V}_S} K \ ,$$

which are linear morphisms with respect to the left $\mathcal{O}_X$-module structure, while the tensor product $\otimes_{\mathcal{O}_X}$ is taken with respect to the right one. These are truncated Taylor expansions of solutions of $(\mathcal{M}, \nabla)$ at the generic point

(2.6.2) $$\underline{e} \longmapsto \sum_{|\underline{\alpha}|_\infty \leq n} \underline{\xi}^{\underline{\alpha}} \otimes \underline{e}\mathcal{G}_{[\underline{\alpha}]} \ ,$$

where $\mathcal{G}_{[\underline{\alpha}]}$ are defined as in (2.2.1).

We consider the ideal $I^{(n)}$ of $\mathcal{V}_S$

(2.6.3) $$I^{(n)} = \left\{a \in \mathcal{V}_S : \ a\Theta_n(\mathcal{M}) \subset \mathcal{P}^n_{X/S} \otimes_{\mathcal{O}_X} \mathcal{M}\right\} \ .$$

We notice that $I^{(n)} \neq 0$ and $I^{(n+1)} \subset I^{(n)}$. We set

(2.6.4) $$h_{X/S}(\mathcal{M}, n) = \frac{\log N(I^{(n)})}{[K : \mathbb{Q}]} \ , \ \text{where } N(I^{(n)}) = \#\left(\mathcal{V}_S/I^{(n)}\right).$$

The *size* of $(\mathcal{M}, \nabla)$ on X/S is defined as

(2.6.5) $$\sigma_{X/S}(M) = \limsup_{n \to \infty} \frac{1}{n} h_{X/S}(\mathcal{M}, n) \in [0, +\infty] \ .$$

If we set $I^{(n)}\mathcal{V}_v = c_{v,n}\mathcal{V}_v$, with $c_{v,n} \in \mathcal{V}_v$, and

(2.6.6) $$h_X(\mathcal{M}, n, v) = \log |c_{v,n}^{-1}|_v = \sup_{|\underline{\alpha}|_\infty \leq n} \log |\mathcal{G}_{[\underline{\alpha}]}|_{X,v} \ ,$$

then we have:

Proposition 2.7.

(2.7.1) $$h_{X/S}(\mathcal{M}, n) = \sum_{v \in \Sigma_S} h_X(\mathcal{M}, n, v) \ .$$

Proof. We have

$$\#\left(\mathcal{V}_S/I^{(n)}\right) = \prod_{v \in \Sigma_S} \#\left(\mathcal{V}_v/c_{n,v}\mathcal{V}_v\right) \ ,$$

with $\mathcal{V}_v/c_{n,v}\mathcal{V}_v = 0$, for almost all $v \in \Sigma_S$. Since

$$\#\left(\mathcal{V}_v/I^{(n)}\mathcal{V}_v\right) = \#\left(\mathcal{V}_v/c_{v,n}\mathcal{V}_v\right) = \#\left(k(v)\right)^{v(c_{n,v})} = |c_{n,v}|_{X,v}^{-[K:\mathbb{Q}]} \ ,$$

we conclude. ∎

For further reference we state the following proposition. It is a generalization of a useful result of André [A, IV §5]. The original proof of André rests on the Dwork-Robba Theorem and on some calculation deriving from the Leibniz formula. There are two proof of the several variables case: the first one, in [B], is based purely on the the theory of spectral norms and on the Leibniz formula, while the second one, in [BD], is based on the generalization of the Dwork-Robba Theorem, and gives more generally continuity of the radius of convergence at points of a Berkovich analytic space.

Proposition 2.8. *With the above notation, we have*

$$\log\frac{1}{R_{X,v}(M)} = \lim_{n\to\infty}\frac{1}{n}h_X(\mathcal{M},n,v)\ . \tag{2.8.1}$$

For lack of references we give a sketch of the proof:

Sketch of the proof of (2.8). By definition of $R_{X,v}(M)$, for all $\varepsilon>0$ there exists $n_0\in\mathbb{N}$ such that for all $\underline{\alpha}\in\mathbb{N}^d$ such that $|\underline{\alpha}|_\infty\geq n_0$ we have

$$\frac{1}{|\underline{\alpha}|_\infty}\log\left|G_{[\underline{\alpha}]}\right|_{X,v}\leq\log\frac{1}{R_{X,v}(M)}+\varepsilon\ .$$

It follows that for all $n\geq n_0$ and all $\underline{\alpha}\in\mathbb{N}^d$ such that $|\underline{\alpha}|_\infty\leq n$ we obtain

$$\frac{1}{|\underline{\alpha}|_\infty}\log\left|G_{[\underline{\alpha}]}\right|_{X,v}\leq\frac{1}{n}h_X(\mathcal{M},n,v)\leq\frac{1}{n}\sup\left(h_X(\mathcal{M},n_0,v),\log\frac{1}{R_{X,v}(M)}+\varepsilon\right)$$

and hence

$$\log\frac{1}{R_{X,v}(M)}=\limsup_{n\to\infty}\frac{1}{n}h_X(\mathcal{M},n,v)\ .$$

Let us prove that

$$\log\frac{1}{R_{X,v}(M)}=\liminf_{n\to\infty}\frac{1}{n}h_X(\mathcal{M},n,v)\ .$$

Let $\{\underline{\beta}^{(j)}:j=1,\dots,N\}$ be the set of all $\underline{\beta}\in\mathbb{N}^d$ such that $|\underline{\beta}|_\infty=n$, for a fix $n\in\mathbb{N}$. Let $\underline{k}\in\mathbb{N}^N$ and $\tau=\sum_{j=1}^N k_j$. Then by induction on $\#\{k_j:k_j\neq0\}$, using the Leibniz formula we can prove (*cf.* [BD]) that for $|\underline{\alpha}|_\infty\lneqq n$ we have

$$\begin{aligned}\sup\left(0,\log\left|\mathcal{G}_{\left[\underline{\alpha}+\sum_{j=1,\dots,N}k_j\underline{\beta}^{(j)}\right]}\right|_{X,v}\right)&\\ \leq(\tau+1)h_X(\mathcal{M},n,v)+Cd(N+1)\log(\tau+1)&+Cd(\tau+N)\log(pn)\ ,\end{aligned}\tag{2.8.2}$$

where C is a constant defined by $|p|_v=p^{-C}$. For all $\underline{\gamma}\in\mathbb{N}^d$ such that $|\underline{\gamma}|_\infty\gneqq n$ we write $\underline{\gamma}$ in the form $\underline{\gamma}=\underline{\alpha}+\sum_{j=1}^N k_j\underline{\beta}^{(j)}$, with $\underline{k}\in\mathbb{N}^N$ and $|\underline{\alpha}|_\infty\lneqq n$. We take $\tau=\sum_{j=1}^N k_j=\left[\frac{|\underline{\gamma}|_\infty}{n}\right]$. By (2.8.2) we have

$$\begin{aligned}\frac{1}{|\underline{\gamma}|_\infty}\sup\left(1,\log|\mathcal{G}_{[\underline{\gamma}]}|_{X,v}\right)&\leq\left(\frac{1}{|\underline{\gamma}|_\infty}+\frac{1}{n}\right)h_X(\mathcal{M},n,v)\\ &+\frac{Cd}{|\underline{\gamma}|_\infty}(N+1)\log\left(\frac{|\underline{\gamma}|_\infty}{n}+1\right)+Cd\left(\frac{N}{|\underline{\gamma}|_\infty}+\frac{1}{n}\right)\log(pn)\ .\end{aligned}$$

Taking the limit for $|\underline{\gamma}|_\infty$ we obtain

$$\log \frac{1}{R_{X,v}(M)} \le \frac{1}{n} h_X(\mathcal{M}, n, v) + \frac{Cd}{n} \log(pn) \ ,$$

and hence

$$\log \frac{1}{R_{X,v}(M)} \le \liminf_{n\to\infty} \frac{1}{n} h_X(\mathcal{M}, n, v).$$

This achieves the proof. ∎

Proposition 2.9. *The generic radius of convergence $R_{X,v}(M)$ only depends on the generic fiber (M,∇) of $(\mathcal{M},\nabla)$, in particular it is independent of the choice of the local basis $\underline{e}$ of $\mathcal{M}$ on X and of the étale coordinates $\underline{x}$ of X. The same therefore holds for $\varrho_{X/S}(M)$ (which, of course, depends of the choice of S and of the S-model X).*

The size $\sigma_{X/S}(M)$ is independent of the particular X/S-model $(\mathcal{M},\nabla)$ of (M,∇).

Proof. The independence of the generic radius of convergence of the choice of the étale coordinates follows from (2.6.3), (2.6.6) and (2.8.1), since the definition of $\Theta^{(n)}$ is independent of $\underline{x}$ and $\underline{e}$ (*cf.* [BO]).

We notice that two X/S-models $(\mathcal{M},\nabla)$ and $(\mathcal{M}',\nabla')$ are isomorphic on an open subscheme of X containing η_v. The fact that the generic radius of convergence $R_{X,v}(M)$ only depends on the generic fiber (M,∇) of $(\mathcal{M},\nabla)$ follows from this remark, (2.6.6) and (2.8.1). Obviously, the same is true for $\varrho_{X/S}(M)$.

We now show that $\sigma_{X/S}(M)$ is independent of the particular X/S-model $(\mathcal{M},\nabla)$ of (M,∇). Let $(\mathcal{M}',\nabla')$ be another X/S-model. Then $(\mathcal{M},\nabla)$ and $(\mathcal{M}',\nabla')$ are isomorphic on an open subscheme of X. By (2.6.6), this means that there exists a finite subset $\{v_1,\dots,v_r\}$ of Σ_S such that $h_X(\mathcal{M},n,v) = h_X(\mathcal{M}',n,v)$ for any $v \in \Sigma_S \smallsetminus \{v_1,\dots,v_r\}$. Then (2.8.1) implies that

$$\begin{aligned}
\sigma_{X/S}(M) &= \limsup_{n\to\infty} \frac{1}{n} \sum_{v\in\Sigma_S} h_X(\mathcal{M},n,v) \\
&= \limsup_{n\to\infty} \frac{1}{n} \sum_{v\in\Sigma_S\smallsetminus\{v_1,\dots,v_r\}} h_X(\mathcal{M}',n,v) + \sum_{v\in\{v_1,\dots,v_r\}} \log \frac{1}{R_{X,v_i}(M)} \\
&= \limsup_{n\to\infty} \frac{1}{n} \sum_{v\in\Sigma_S} h_X(\mathcal{M}',n,v) \ .
\end{aligned}$$

This proves our assertion. ∎

The following theorem, which is our main result, was recently proved in the one variable case by Dwork [D]. It refines a formula of Bombieri-André [A, IV.5]. The proof will be given in §4.

Main theorem 2.10. Let (M,∇) be a differential $\mathcal{F}/K$-module of type G and of rank μ; and let $(\mathcal{M},\nabla)$ be a model of (M,∇) over a smooth S-model X of $\mathcal{F}/K$. If Σ'_S is the subset of Σ_S of all primes v such that the induced connection $(\mathcal{M}_{k(v)},\nabla_{k(v)})$ on the closed fiber $X_{k(v)}$ does *not* have p-curvature 0 and

(2.10.1) $$\Delta(M) = \limsup_{n\to+\infty} \frac{1}{n} \sum_{\substack{v\in\Sigma'_S \\ p(v)\le n}} \log |p(v)|_v^{-1} \ ,$$

then:

(2.10.2) $$\Delta(M) \le \sigma_{X/S}(M) - \varrho_{X/S}(M) \le \left(1 + \frac{1}{2} + \cdots + \frac{1}{\mu-1}\right) \Delta(M) \ .$$

A consequence of the previous theorem is that one may equivalently define G-connections by the boundness of size rather than of global inverse radius.

Remark 2.11. We would like to stress the intrinsic nature of the last statement, in particular:
1) While the global inverse radius of convergence $\varrho_{X/S}(M)$ and the size $\sigma_{X/S}(M)$ depend on the choice of the scheme S and of the smooth S-model X (but *not* on the particular choice of the X/S-model $(\mathcal{M},\nabla)$ of (M,∇)), *their difference* $\sigma_{X/S}(M)-\varrho_{X/S}(M)$ *only depends on the geometric generic fiber of the differential module* $(\mathcal{M},\nabla)$. In fact, if X/S is a smooth model of the function field $\mathcal{F}/K$, then for any open dense subscheme T of S, $X_T = X\times_S T$ is an open subscheme of X and a smooth T-model. Then we recall the definition of the invariant $\tau(M)$:

$$\tau(M) = \inf_{T\hookrightarrow S} \sigma_{X_T/T}(M)\ .$$

Since $\Sigma_S \smallsetminus \Sigma_T$ is finite, by (2.8) we have

$$\begin{aligned}\sigma_{X_T/T}(M) &= \limsup_{n\to+\infty}\frac{1}{n}\sum_{v\in\Sigma_T} h_X(\mathcal{M},n,v)\\ &= \sigma_{X/S}(M) - \lim_{n\to+\infty}\frac{1}{n}\sum_{v\in\Sigma_S\smallsetminus\Sigma_T} h_X(\mathcal{M},n,v)\\ &= \sigma_{X/S}(M) - \sum_{v\in\Sigma_S\smallsetminus\Sigma_T}\log\frac{1}{R_{X,v}(M)}\ .\end{aligned}$$

When we take the infimum both on the left and on the right side, we obtain

$$\sigma_{X/S}(M)-\varrho_{X/S}(M) = \inf_{T\hookrightarrow S}\sigma_{X_T/T}(M) = \tau(M)\ .$$

On the other hand, if we pick two smooth models X/S and X'/S' of the function field $\mathcal{F}/K$, then, replacing X (resp. X') by an open dense subscheme, we may assume that $S=S'$. Two smooth S-models X and X' are generically isomorphic (since the local rings at their generic points are isomorphic), therefore

$$\inf_{T\hookrightarrow S}\sigma_{X_T/T}(M) = \inf_{T\hookrightarrow S}\sigma_{X'_T/T}(M)\ .$$

So $\tau(M)=\sigma_{X/S}(M)-\varrho_{X/S}(M)$ only depends on the $\mathcal{F}/K$-differential module (M,∇).

Now let $\mathcal{F}'$ be a finite extension of $\mathcal{F}$ and K' be the algebraic closure of K in $\mathcal{F}'$. Let S' be the normalization of S in K'. By replacing X by an open submodel, we can suppose that $X' = X\times_S S'$ is smooth over S' and hence that it is an S'-model of the compositum $K'\mathcal{F}=\mathcal{G}\subset\mathcal{F}'$. By our normalization,

$$\sigma_{X/S}(M)=\sigma_{X'/S'}(M_{\mathcal{G}}) \text{ and } \varrho_{X/S}(M)=\varrho_{X'/S'}(M_{\mathcal{G}}).$$

Assume now that $K'=K$, $S'=S$. Then $\mathcal{G}=\mathcal{F}$, and we can find an S-model X' of $\mathcal{F}'/K$ and an étale covering $\varphi: X'\longrightarrow X$. We note that $\sigma_{X/S}(M)=\sigma_{X'/S}(M)$ and $\varrho_{X/S}(M)=\varrho_{X'/S}(M)$ also in this case (*cf.* Appendix).

This shows that $\tau(M)$ depends only on the generic geometric fiber of $(\mathcal{M},\nabla)$.

2) First of all we notice that the constant $\Delta(M)$ appearing in the statement of last the theorem is finite by the Prime Numbers Theorem. Moreover $\Delta(M)$ is independent of the choice of the

smooth model X/S and of the choice of S. In fact, let X'/S' be another model of the function field $\mathcal{F}/K$, then there exists $N \in \mathbb{N}$ such that

$$\{v \in \Sigma_S : \ p(v) \geq N\} = \{v \in \Sigma_{S'} : \ p(v) \geq N\} ,$$

and hence that

$$\limsup_{n\to+\infty} \frac{1}{n} \sum_{\substack{v\in\Sigma_S \\ p(v)\leq n}} \log p(v) = \limsup_{n\to+\infty} \frac{1}{n} \sum_{\substack{v\in\Sigma_{S'} \\ p(v)\leq n}} \log p(v) .$$

§3. Generic radius of convergence and nilpotence.

In this section we prove (2.5), which is essentially a result of local type, therefore we are going to introduce some notation, slightly different from that in §2.

Notation 3.1. We consider a number field K equipped with an ultrametric absolute value $|\ |$ such that $|p| \lneqq 1$, for a rational prime p. Let v be the valuation of K associated to $|\ |$, $\mathcal{V}$ the discrete valuation ring of K associated to v, π the uniformizer of $\mathcal{V}$, k the residue field $\mathcal{V}$ of characteristic p, X a smooth $\mathcal{V}$-scheme of finite type, with non empty geometrically connected fibers, X_k the closed fiber of X, $\mathcal{F} = \kappa(X)$ the field of rational functions on X, $|\ |_X$ the extension of $|\ |$ to $\mathcal{F}$ associated to X, normalized so to extend $|\ |$, *i.e.* such that

$$|p| = p^{-[K_v:\mathbb{Q}_p]/[K:\mathbb{Q}]} .$$

Following §2, if (M,∇) is an $\mathcal{F}/K$-differential module admitting a model $(\mathcal{M},\nabla)$ on X, we define as usual

$$\nabla\left(\underline{D}^{[\underline{\alpha}]}\right)\underline{e} = \underline{e}\mathcal{G}_{[\underline{\alpha}]}, \text{ with } \mathcal{G}_{[\underline{\alpha}]} \in M_{\mu\times\mu}(\mathcal{F}) , \tag{3.1.1}$$

and

$$\nabla\left(\underline{D}^{\underline{\alpha}}\right)\underline{e} = \underline{e}\mathcal{G}_{\underline{\alpha}},\ (\ \mathcal{G}_{\underline{\alpha}} = \underline{\alpha}!\mathcal{G}_{[\underline{\alpha}]}\ ) ,$$

where $(U, \underline{x} = (x_1,\ldots,x_d))$ is an étale coordinate neighborhood of the generic point of X_k, $\underline{e}$ is a local basis of $(\mathcal{M},\nabla)$ and $\underline{\alpha} \in \mathbb{N}^d$. We have

$$\mathcal{G}_{\underline{\alpha}+\underline{1}_i} = D_i\mathcal{G}_{\underline{\alpha}} + \mathcal{G}_{\underline{1}_i}\mathcal{G}_{\underline{\alpha}} \ , \text{ for all } i = 1,\ldots,d \text{ and } \underline{\alpha} \in \mathbb{N}^d. \tag{3.1.2}$$

We set, as in the previous section,

$$R_X(M) = \left(\max\left(1, \limsup_{|\underline{\alpha}|_\infty\to\infty} |\mathcal{G}_{[\underline{\alpha}]}|_X{}^{1/|\underline{\alpha}|_\infty}\right)\right)^{-1} . \tag{3.1.3}$$

We will denote by $(\mathcal{M}_k,\nabla_k)$ the integrable connection induced on X_k.

Lemma 3.2. *If (M,∇) has a model $(\mathcal{M},\nabla)$ on X, then*

$$R_X(M) \geq |p|^{1/(p-1)} .$$

Proof. Since $|\underline{\alpha}!| \geq |p|^{\frac{|\underline{\alpha}|_\infty}{p-1}}$, by (3.1.2) we obtain

$$R_X(M) \geq \frac{|p|^{1/(p-1)}}{\max(1, |\mathcal{G}_{\underline{1}_1}|_X,\ldots,|\mathcal{G}_{\underline{1}_d}|_X)} .$$

If $(\mathcal{M},\nabla)$ is a model of (M,∇) on X, there exists an étale coordinate neighborhood $(U,\underline{x})$ such that $\mathcal{M}$ is free over U and $\mathcal{G}_{\underline{1}_i}\in M_{\mu\times\mu}(\mathcal{O}(U))$, therefore

$$|\mathcal{G}_{\underline{1}_i}|_X\leq 1 \ .$$

So we conclude that

$$R_X(M)\geq |p|^{1/(p-1)} \ .$$

■

If $(\mathcal{M}_k,\nabla_k)$ is nilpotent of exponent $\leq n$, we give a lower bound for $R_X(M)$:

Proposition 3.3. *Let (M,∇) be a $\mathcal{F}/K$-connection as in (3.1); then $(\mathcal{M}_k,\nabla_k)$ is nilpotent if and only if the following condition is satisfied:*

$$R_X(M)\gneqq |p|^{1/(p-1)} \ .$$

In particular, if $(\mathcal{M}_k,\nabla_k)$ is nilpotent of exponent $\leq n$ we have:

$$R_X(M)\geq |\pi|^{-1/pn}|p|^{1/(p-1)} \ .$$

First we need a technical lemma.

Lemma 3.4. *If $(\mathcal{M}_k,\nabla_k)$ is nilpotent of exponent $\leq n$ and if $\{\underline{w}^{(i)}:i=1,\dots,N\}$ is the set of all $\underline{w}\in\mathbb{N}^d$ such that $|\underline{w}|_\infty=n$, we have*

$$\left|\mathcal{G}_{\sum_{i=1}^N s_ip\underline{w}^{(i)}}\right|_X\leq|\pi|^{\sum_{i=1}^N s_i} \ .$$

Proof. For all $(w_1,\dots,w_d)\in\mathbb{N}^d$, such that $|\underline{w}|_\infty=n$, we have $|\mathcal{G}_{p\underline{w}}|_X\leq|\pi|\lneqq 1$. We want to prove by induction on $s\in\mathbb{N}^*$ that for all $\underline{\alpha}\in\mathbb{N}^d$ we have

$$\left|\mathcal{G}_{ps\underline{w}+\underline{\alpha}}\right|_X\leq|\pi|^s\left|\mathcal{G}_{\underline{\alpha}}\right|_X\gneqq 1 \ . \tag{3.4.1}$$

By Leibniz formula, we obtain

$$\begin{aligned}\nabla\left(\underline{D}^{p(s+1)\underline{w}+\underline{\alpha}}\right)\underline{e}&=\nabla\left(\underline{D}^{p\underline{w}}\right)\left(\underline{e}\mathcal{G}_{ps\underline{w}+\underline{\alpha}}\right)\\&=\underline{e}\sum_{\underline{0}\leq\underline{\beta}\leq p\underline{w}}\binom{p\underline{w}}{\underline{\beta}}\mathcal{G}_{p\underline{w}-\underline{\beta}}\left(\left(\frac{\partial}{\partial\underline{x}}\right)^{\underline{\beta}}\mathcal{G}_{ps\underline{w}+\underline{\alpha}}\right)\end{aligned}$$

and hence

$$\begin{aligned}\mathcal{G}_{p(s+1)\underline{w}+\underline{\alpha}}&=\sum_{\underline{0}\leq\underline{\beta}\leq p\underline{w}}\binom{p\underline{w}}{\underline{\beta}}\mathcal{G}_{p\underline{w}-\underline{\beta}}\left(\left(\frac{\partial}{\partial\underline{x}}\right)^{\underline{\beta}}\mathcal{G}_{ps\underline{w}+\underline{\alpha}}\right)\\&=\sum_{\underline{0}\leq\underline{\beta}\leq\underline{w}}\binom{p\underline{w}}{p\underline{\beta}}\mathcal{G}_{p\underline{w}-p\underline{\beta}}\left(\left(\frac{\partial}{\partial\underline{x}}\right)^{p\underline{\beta}}\mathcal{G}_{ps\underline{w}+\underline{\alpha}}\right)+\sum_{\substack{\underline{0}\lneqq\underline{\beta}\leq p\underline{w}\\(p,\underline{\beta})=1}}\binom{p\underline{w}}{\underline{\beta}}\mathcal{G}_{p\underline{w}-\underline{\beta}}\left(\left(\frac{\partial}{\partial\underline{x}}\right)^{\underline{\beta}}\mathcal{G}_{ps\underline{w}+\underline{\alpha}}\right) \ ,\end{aligned}$$

where $(p,\underline{\beta})$ is the ideal generated in $\mathbb{Z}$ by $\{p,\beta_1,\dots,\beta_d\}$. Then:

1) if $\underline{0} \lneqq \underline{\beta} \leq \underline{w}$ we have

$$\begin{cases} \left|\binom{p\underline{w}}{p\underline{\beta}}\right| \leq 1 \\ \left|\mathcal{G}_{p\underline{w}-p\underline{\beta}}\right|_X \leq 1 \\ \left|\left(\frac{\partial}{\partial \underline{x}}\right)^{p\underline{\beta}}(\mathcal{G}_{ps\underline{w}+\underline{\alpha}})\right|_X \leq |\pi| \left|\mathcal{G}_{ps\underline{w}+\underline{\alpha}}\right|_X \leq |\pi|^{s+1}\left|\mathcal{G}_{\underline{\alpha}}\right|_X \end{cases},$$

and for $\underline{\beta} = 0$

$$\left|\mathcal{G}_{p\underline{w}}\mathcal{G}_{ps\underline{w}+\underline{\alpha}}\right|_X \leq |\pi|^{s+1}\left|\mathcal{G}_{\underline{\alpha}}\right|_X .$$

So

$$\left|\binom{p\underline{w}}{p\underline{\beta}}\mathcal{G}_{p\underline{w}-p\underline{\beta}}\left(\left(\frac{\partial}{\partial \underline{x}}\right)^{p\underline{\beta}}\mathcal{G}_{ps\underline{w}+\underline{\alpha}}\right)\right|_X \leq |\pi|^{s+1}\left|\mathcal{G}_{\underline{\alpha}}\right|_X, \ \forall\ \underline{0} \leq \underline{\beta} \leq \underline{w} .$$

2) if $(p,\underline{\beta}) = 1$ and $\underline{0} \lneqq \underline{\beta} \leq p\underline{w}$ we have

$$\begin{cases} \left|\binom{p\underline{w}}{\underline{\beta}}\right| \leq |p| \leq |\pi| \\ \left|\mathcal{G}_{p\underline{w}-\underline{\beta}}\left(\left(\frac{\partial}{\partial \underline{x}}\right)^{\underline{\beta}}\mathcal{G}_{ps\underline{w}+\underline{\alpha}}\right)\right|_X \leq |\pi|^{s}\left|\mathcal{G}_{\underline{\alpha}}\right|_X \end{cases},$$

and hence

$$\left|\binom{p\underline{w}}{\underline{\beta}}\mathcal{G}_{p\underline{w}-\underline{\beta}}\left(\left(\frac{\partial}{\partial \underline{x}}\right)^{\underline{\beta}}\mathcal{G}_{ps\underline{w}p+\underline{\alpha}}\right)\right|_X \leq |\pi|^{s+1}\left|\mathcal{G}_{\underline{\alpha}}\right|_X, \ \forall\ \underline{0} \lneqq \underline{\beta} \leq p\underline{w},\ (p,\underline{\beta}) = 1 .$$

Therefore we obtain $\left|\mathcal{G}_{ps\underline{w}+\underline{\alpha}}\right|_X \leq |\pi|^s \left|\mathcal{G}_{\underline{\alpha}}\right|_X$, for all $s \in \mathbb{N}$, $\underline{\alpha} \in \mathbb{N}^d$ and $\underline{w} \in \mathbb{N}^d$ such that $|\underline{w}|_\infty = n$.

From (3.4.1), by induction on $\#\{i = 1, \ldots, N : s_i \neq 0\}$, it follows that

$$\left|\mathcal{G}_{\sum_{i=1}^N s_i p \underline{w}^{(i)}}\right|_X \leq |\pi|^{\sum_{i=1}^N s_i} .$$

■

Proof of proposition (3.3). Let us suppose that $R_X(M) \gneqq |p|^{1/(p-1)}$ and choose $R \gneqq 0$ such that $R_X(M) \gneqq R \gneqq |p|^{1/(p-1)}$. Then

$$\lim_{|\underline{\alpha}|_\infty \to \infty} \left|\frac{\mathcal{G}_{\underline{\alpha}}}{\underline{\alpha}!}\right|_X R^{|\underline{\alpha}|_\infty} = 0 .$$

For any $n \in \mathbb{N}$, we write n in the form $n = n_k p^k + n_{k-1}p^{k-1} + \ldots + n_0$, with $0 \leq n_i \leq p-1$, for all $i = 0, \ldots, k$, and we have (*cf.* for instance [DGS, page 51])

$$|n!| = |p|^{\frac{n-S_n}{p-1}}, \text{ with } S_n = n_k + n_{k-1} + \ldots + n_0. \tag{3.4.2}$$

If we choose $\underline{\alpha} \in \mathbb{N}^d$ such that $\alpha_i = p^s$, with $s \in \mathbb{N}$, we obtain

$$\frac{R^{|\underline{\alpha}|_\infty}}{|\underline{\alpha}!|} = \left(\frac{R}{|p|^{1/(p-1)}}\right)^{dp^s} |p|^{d/(p-1)} ,$$

which implies that

$$\limsup_{|\underline{\alpha}|_\infty \to \infty} \frac{R^{|\underline{\alpha}|_\infty}}{|\underline{\alpha}!|} = +\infty \ .$$

Therefore we conclude that

$$\lim_{|\underline{\alpha}|_\infty \to \infty} |\mathcal{G}_{\underline{\alpha}}|_X = 0 \ .$$

It follows that there exists an $n \in \mathbb{N}$ such that, for $|\underline{\alpha}|_\infty \geq n$, we have $|\mathcal{G}_{\underline{\alpha}}|_X \lneqq 1$, hence $(\mathcal{M}_k, \nabla_k)$ is nilpotent.

On the other hand, suppose that $(\mathcal{M}_k, \nabla_k)$ is nilpotent of exponent $\leq n$. Let $\underline{\alpha} \in \mathbb{N}^d$, with $|\underline{\alpha}|_\infty \geq dpn$; since there exists $i_\circ = 1, \dots, d$, such that $\alpha_{i_\circ} \geq np$, we can find $\underline{s} \in \mathbb{N}^N$ such that:

$$\underline{\alpha} = \sum_{i=1}^{N} s_i p \underline{w}^{(i)} + \underline{\beta} \ ,$$

where $\{\underline{w}^{(i)} : i = 1, \dots, N\}$ is the set of all $\underline{w} \in \mathbb{N}^d$ such that $|\underline{w}|_\infty = n$, $|\underline{\beta}|_\infty \lneqq dpn$ and

$$\sum_{i=1}^{N} s_i = \frac{|\underline{\alpha}|_\infty - |\underline{\beta}|_\infty}{pn} \gneqq \frac{|\underline{\alpha}|_\infty}{pn} - d \ .$$

By (3.1.2), $|\mathcal{G}_{\underline{\alpha}}|_X \leq |\mathcal{G}_{\underline{\alpha}'}|$, when $\alpha_i \geq \alpha'_i$ for all $i = 1, \dots, d$; therefore the previous lemma implies:

$$|\mathcal{G}_{\underline{\alpha}}|_X \leq \left| \mathcal{G}_{\sum_{i=1}^{N} s_i p \underline{w}^i} \right|_X \leq |\pi|^{\frac{|\underline{\alpha}|_\infty}{pn} - d} \ . \tag{3.4.3}$$

Finally we conclude:

$$\begin{aligned} \limsup_{|\underline{\alpha}|_\infty \to \infty} \left| \frac{\mathcal{G}_{\underline{\alpha}}}{\underline{\alpha}!} \right|_X^{\frac{1}{|\underline{\alpha}|_\infty}} &\leq \limsup_{|\underline{\alpha}|_\infty \to \infty} |\pi|^{\left(\frac{|\underline{\alpha}|_\infty}{pn} - d\right)\frac{1}{|\underline{\alpha}|_\infty}} |p|^{-1/(p-1)} \\ &= |\pi|^{1/pn} |p|^{-1/(p-1)} \lneqq |p|^{-1/(p-1)} \ . \end{aligned}$$

■

§4. Size of G-connections.

In this section we will use the notation introduced in §2. We now prove our main result

Theorem 4.1. *Let (M, ∇) be a differential $\mathcal{F}/K$-module of type G and of finite rank μ; we assume that (M, ∇) admits a model $(\mathcal{M}, \nabla)$ over a smooth S-model X of $\mathcal{F}/K$. If Σ'_S is the subset of Σ_S of primes v such that the induced connection $(\mathcal{M}_{k(v)}, \nabla_{k(v)})$ on the closed fiber $X_{k(v)}$ does not have p-curvature 0 and*

$$\Delta(M) = \limsup_{n \to +\infty} \frac{1}{n} \sum_{\substack{v \in \Sigma'_S \\ p(v) \leq n}} \log |p(v)|_v^{-1} \ ,$$

then:

$$\Delta(M) \leq \sigma_{X/S}(M) - \varrho_{X/S}(M) \leq \left(1 + \frac{1}{2} + \cdots + \frac{1}{\mu - 1}\right) \Delta(M) \ .$$

Before giving the proof of the theorem, we need a lemma:

Lemma 4.2. *Under the hypothesis of the theorem, let Σ''_S be set of all $v \in \Sigma_S$ such that the induced connection $(\mathcal{M}_{k(v)}, \nabla_{k(v)})$ on the closed fiber $X_{k(v)}$ has p-curvature 0. We have:*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{v\in\Sigma''_S} h_X(\mathcal{M}, n, v) = \sum_{v\in\Sigma''_S} \log \frac{1}{R_{X,v}(M)} \ .$$

Proof. The proof is divided in steps. In the first step we prove

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{v\in\Sigma''_S} h_X(\mathcal{M}, n, v) \geq \sum_{v\in\Sigma''_S} \log \frac{1}{R_{X,v}(M)} \ , \tag{4.2.1}$$

while steps from 2 to 5 are devoted to the proof of

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{v\in\Sigma''_S} h_X(\mathcal{M}, n, v) \leq \sum_{v\in\Sigma''_S} \log \frac{1}{R_{X,v}(M)} \ . \tag{4.2.2}$$

Step 1. *Proof of* (4.2.1).
We observe that for any $N \in \mathbb{N}$ we have

$$\frac{1}{n} \sum_{v\in\Sigma''_S} h_X(\mathcal{M}, n, v) \geq \frac{1}{n} \sum_{\substack{v\in\Sigma''_S \\ p(v)\leq N}} h_X(\mathcal{M}, n, v)$$

and therefore by Fatou's Lemma we obtain

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{v\in\Sigma''_S} h_X(\mathcal{M}, n, v) \geq \sum_{\substack{v\in\Sigma''_S \\ p(v)\leq N}} \log \frac{1}{R_{X,v}(M)} \ .$$

We deduce (4.2.1) by taking the limit for $N \to \infty$.
Step 2. *Let $R'_{X,v} = |\pi_v|_{X,v}^{-1/p} |p|_v^{1/(p-1)}$, with $p = p(v)$. Then*

$$h_X(\mathcal{M}, n, v) \leq n \log \frac{1}{R'_{X,v}} \ .$$

It is enought to prove that for any $v \in \Sigma''_S$ and any $\underline{\alpha} \in \mathbb{N}^d$ we have

$$|\mathcal{G}_{[\underline{\alpha}]}|_{X,v} \leq \left(\frac{1}{R'_{X,v}}\right)^{|\underline{\alpha}|_\infty} \ . \tag{4.2.3}$$

Let $\underline{\alpha} = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$ be such that $\alpha_i < p$ for any $i = 1, \dots, d$. Then (4.2.3) is obviously verified. So let $\underline{\alpha} = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$ be such that there exists $i = 1, \dots, d$ such that $\alpha_i \geq p$. Then there exists $s_1, \dots, s_d, \beta_1, \dots, \beta_d \in \mathbb{N}$, with $0 \leq \beta_i \leq p-1$, such that $\alpha_i = s_i p + \beta_i$, for all $i = 1, \dots, d$, *i.e.*

$$\underline{\alpha} = \sum_{i=1}^{d} s_i p \underline{1}_i + (\beta_1, \dots, \beta_d) \ .$$

By (3.1.2) and (3.4) we have

$$|G_{[\underline{\alpha}]}|_{X,v} \leq \left| \frac{G_{\sum_{i=1}^{d} s_i p \underline{1}_i}}{\underline{\alpha}!} \right|_{X,v} \leq \prod_{i=1}^{d} \frac{|\pi_v|_v^{s_i}}{|\alpha_i!|_v} .$$

Using (3.4.2), we deduce that

$$|G_{[\underline{\alpha}]}|_{X,v} \leq \prod_{i=1}^{d} |\pi_v|_v^{s_i} |p|_v^{\frac{S_{\alpha_i}-\alpha_i}{p-1}} \leq \prod_{i=1}^{d} \frac{|\pi_v|_v^{\left[\frac{\alpha_i}{p}\right]} |\pi_v|_v^{S_{\alpha_i}/(p-1)}}{|p|_v^{\frac{\alpha_i}{p-1}}} .$$

We notice that

$$\frac{\alpha_i}{p} - \left[\frac{\alpha_i}{p}\right] \leq \frac{S_{\alpha_i}}{p} \leq \frac{S_{\alpha_i}}{(p-1)}$$

and hence that

$$|G_{[\underline{\alpha}]}|_{X,v} \leq \prod_{i=1}^{d} \frac{|\pi_v|_v^{\alpha_i/p}}{|p|_v^{\frac{\alpha_i}{p-1}}} = \left(|\pi_v|_v^{1/p} |p|_v^{\frac{-1}{p-1}} \right)^{|\underline{\alpha}|_\infty} ,$$

which proves (4.2.3).
Step 3. *Proof of the inequality*

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{v\in\Sigma''_S} h_X(\mathcal{M},n,v) \leq \sum_{\substack{v\in\Sigma''_S \\ p(v)\leq N}} \log \frac{1}{R_{X,v}(M)} + \limsup_{n\to\infty} \sum_{\substack{v\in\Sigma''_S \\ N\lneqq p(v)\leq n}} \log \frac{1}{R'_{X,v}} . \tag{4.2.4}$$

We notice that for $p(v) \gneqq |\underline{\alpha}|_\infty$ we have

$$|\mathcal{G}_{[\underline{\alpha}]}|_{X,v} \leq 1 ,$$

and hence

$$\sum_{v\in\Sigma''_S} h_X(\mathcal{M},n,v) = \sum_{\substack{v\in\Sigma''_S \\ p(v)\leq n}} h_X(\mathcal{M},n,v) .$$

Therefore, for any $N \in \mathbb{N}$, $N \lneqq n$, by *Step 2* we obtain

$$\begin{aligned} \frac{1}{n} \sum_{v\in\Sigma''_S} h_X(\mathcal{M},n,v) &= \sum_{\substack{v\in\Sigma''_S \\ p(v)\leq N}} \frac{1}{n} h_X(\mathcal{M},n,v) + \sum_{\substack{v\in\Sigma''_S \\ N\lneqq p(v)\leq n}} \frac{1}{n} h_X(\mathcal{M},n,v) \\ &\leq \sum_{\substack{v\in\Sigma''_S \\ p(v)\leq N}} \frac{1}{n} h_X(\mathcal{M},n,v) + \sum_{\substack{v\in\Sigma''_S \\ N\lneqq p(v)\leq n}} \log \frac{1}{R'_{X,v}} . \end{aligned}$$

Proposition 2.8 allows us to deduce (4.2.4) by the previous inequality.
Step 4. $\limsup_{n\to\infty} \sum_{\substack{v\in\Sigma''_S \\ N\lneqq p(v)\leq n}} \log \frac{1}{R'_{X,v}}$ *is finite.*

We have

$$\sum_{\substack{v\in\Sigma''_S \\ p(v)\gneqq N}} \log \frac{1}{R'_{X,v}} = \sum_{\substack{v\in\Sigma''_S \\ p(v)\gneqq N}} \log \left(|p(v)|_v^{1/p(v)e_v} |p(v)|_v^{-1/(p(v)-1)} \right) ;$$

where e_v is the ramification index of v with respect to $p(v)$. Since $e_v = 1$ for almost all v, it is enough to study the convergence of the following series for $N >> 0$

$$\sum_{\substack{v\in\Sigma''_S\\ p(v)\gneq N}} \log\left(|p(v)|_v^{\frac{1}{p(v)}}\ |p(v)|_v^{\frac{-1}{(p(v)-1)}}\right) = \sum_{\substack{v\in\Sigma''_S\\ p(v)\gneq N}} \log\left(|p(v)|_v^{\frac{-1}{p(v)(p(v)-1)}}\right)$$

$$\leq \sum_{p\gneq N}\log\left(\prod_{v|p}|p|_v\right)^{\frac{-1}{p(p-1)}} = \sum_{p\gneq N}\frac{\log p}{p(p-1)} \leq \sum_{p\gneq N}\frac{1}{(p-1)^{3/2}}\ .$$

Step 5. *Conclusion of the proof of* (4.2.2).
By *Step 4*, the inequality (4.2.4) becomes

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{v\in\Sigma''_S} h_X(\mathcal{M},n,v) \leq \sum_{\substack{v\in\Sigma''_S\\ p(v)\leq N}}\log\frac{1}{R_{X,v}(M)} + \sum_{\substack{v\in\Sigma''_S\\ p(v)\gneq N}}\log\frac{1}{R'_{X,v}}\ .$$

We conclude the proof of (4.2.2) taking $N\to+\infty$. ∎

Proof of the theorem 4.1. Because of lemma 4.2, it is enough to prove the following inequalities:

$$\Delta(M) + \sum_{v\in\Sigma'_S}\log\frac{1}{R_{X,v}(M)} \leq \limsup_{n\to\infty}\frac{1}{n}\sum_{v\in\Sigma'_S} h_X(\mathcal{M},n,v) \tag{4.2.5}$$

and

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{v\in\Sigma'_S} h_X(\mathcal{M},n,v) \leq \left(1+\frac{1}{2}+\cdots+\frac{1}{\mu-1}\right)\Delta(M) + \sum_{v\in\Sigma'_S}\log\frac{1}{R_{X,v}(M)}\ . \tag{4.2.6}$$

We first prove (4.2.5). By definition of Σ'_S, if $n\geq p(v)$, we have

$$h_X(\mathcal{M},n,v) \geq \log|p(v)|_v^{-1}\ .$$

For $n\gneq N$, we obtain

$$\frac{1}{n}\sum_{v\in\Sigma'_S} h_X(\mathcal{M},n,v) \geq \frac{1}{n}\sum_{\substack{v\in\Sigma'_S\\ p(v)\leq N}} h_X(\mathcal{M},n,v) + \frac{1}{n}\sum_{\substack{v\in\Sigma'_S\\ N\lneq p(v)\leq n}}\log|p(v)|_v^{-1}$$

$$= \frac{1}{n}\sum_{\substack{v\in\Sigma'_S\\ p(v)\leq N}} h_X(\mathcal{M},n,v) + \frac{1}{n}\left(\sum_{\substack{v\in\Sigma'_S\\ p(v)\leq n}}\log|p(v)|_v^{-1} - \sum_{\substack{v\in\Sigma'_S\\ p(v)\leq N}}\log|p(v)|_v^{-1}\right)\ ;$$

hence we deduce that

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{v\in\Sigma'_S} h_X(\mathcal{M},n,v) \geq \sum_{\substack{v\in\Sigma'_S\\ p(v)\leq N}}\log\frac{1}{R_{X,v}(M)} + \Delta(M)\ .$$

Finally, we find (4.2.5) taking the limit for $N\to+\infty$.

We now discuss the upper bound (4.2.6). We notice that $|\mathcal{G}_{\underline{\alpha}}|_{X,v} \leq 1$ for all $\underline{\alpha} \in \mathbb{N}^d$. Then the Dwork-Robba theorem [BD]* affirms that

$$|\mathcal{G}_{[\underline{\alpha}]}|_{X,v} \leq \{|\underline{\alpha}|_\infty, (\mu-1)\}_v \frac{1}{R_{X,v}(M)^{|\underline{\alpha}|_\infty}} ,$$

where

$$\{n,s\}_v = \sup_{\substack{1 \leq \lambda_1 \lneq \lambda_2 \lneq \cdots \lneq \lambda_s \leq n \\ \lambda_1, \lambda_2, \ldots, \lambda_s \in \mathbb{N}}} \left(\frac{1}{|\lambda_1 \cdots \lambda_s|_v} \right) .$$

We set

$$\begin{cases} a_v = |p(v)|_v^{-1} \\ \Theta_{\Sigma'_S}(n) = \displaystyle\sum_{\substack{v \in \Sigma'_S \\ p(v) \leq n}} \log a_v \end{cases}$$

and, for $n \geq p(v)$,

$$\begin{aligned} \theta(n,v) &= (\mu-1)\left[\frac{\log n}{\log p(v)}\right] \log a_v - \log\{n,(\mu-1)\}_v \\ &= \frac{[K_v:\mathbb{Q}_p]}{[K:\mathbb{Q}]} \left((\mu-1)\left[\frac{\log n}{\log p(v)}\right] \log p(v) - \log\{n,(\mu-1)\}_{p(v)} \right) . \end{aligned}$$

Let $p(v) \geq \mu$ and let $i = 1, \ldots, \mu-1$. If $(\mu-i-1)p(v)^{\left[\frac{\log n}{\log p(v)}\right]} \leq n \lneq (\mu-i)p(v)^{\left[\frac{\log n}{\log p(v)}\right]}$ then

$$\log\{n,(\mu-1)\}_{p(v)} = (\mu-i-1)\left[\frac{\log n}{\log p(v)}\right] \log a_v + i\left(\left[\frac{\log n}{\log p(v)}\right] - 1\right) \log a_v = i \log a_v$$

and if $(\mu-1)p(v)^{\left[\frac{\log n}{\log p(v)}\right]} \leq n \lneq p(v)^{\left[\frac{\log n}{\log p(v)}\right]+1}$ then

$$\log\{n,(\mu-1)\}_{p(v)} = (\mu-1)\left[\frac{\log n}{\log p(v)}\right] \log a_v .$$

We conclude that for all $i = 1, \ldots, \mu-1$ we have

$$\theta(n,v) = i \log a_v \quad \text{if } (\mu-i-1)p(v)^{\left[\frac{\log n}{\log p(v)}\right]} \leq n \lneq (\mu-i)p(v)^{\left[\frac{\log n}{\log p(v)}\right]} ,$$

and

$$\theta(n,v) = 0 \quad \text{if } (\mu-1)p(v)^{\left[\frac{\log n}{\log p(v)}\right]} \leq n \lneq p(v)^{\left[\frac{\log n}{\log p(v)}\right]+1} .$$

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* We learned the proof of the generalization of the Dwork-Robba theorem from Professor Dwork's course held at Padova University in the academic year 1994/95. The effective estimate

$$|\mathcal{G}_{[\underline{\alpha}]}|_{X,v} \leq \sup_{|\underline{\beta}|_\infty \leq \mu-1} \left(|G_{\underline{\beta}}|_{X,v} R_{X,v}(M)^{|\underline{\beta}|_\infty} \right) \{|\underline{\alpha}|_\infty, (\mu-1)\}_v \frac{1}{R_{X,v}(M)^{|\underline{\alpha}|_\infty}} ,$$

in the several variable case, can be deduce by [DGS, IV, 3.2], using an argument of generic line. In [G], Gachet uses the same idea to prove an analogous effective estimate on a polyannulus.

Hence for $n \geq \mu^2$ we deduce

$$\begin{aligned}\sum_{\substack{v\in\Sigma'_S\\ p(v)\leq n}} \theta(n,v) &= \sum_{\substack{v\in\Sigma'_S\\ p(v)\leq\sqrt{n}}} \theta(n,v) + \sum_{h=1}^{(\mu-2)} h\left(\Theta_{\Sigma'_S}\left(\frac{n}{\mu-h-1}\right) - \Theta_{\Sigma'_S}\left(\frac{n}{\mu-h}\right)\right)\\ &= \sum_{h=1}^{(\mu-2)} h\left(\Theta_{\Sigma'_S}\left(\frac{n}{\mu-h-1}\right) - \Theta_{\Sigma'_S}\left(\frac{n}{\mu-h}\right)\right) + o(n)\\ &= (\mu-1)\Theta_{\Sigma'_S}(n) - \left(\Theta_{\Sigma'_S}(n) + \Theta_{\Sigma'_S}\left(\frac{n}{2}\right)\cdots + \Theta_{\Sigma'_S}\left(\frac{n}{\mu-1}\right)\right) + o(n)\ .\end{aligned}$$

Since $\{n,(\mu-1)\}_v = 1$ for $p(v) \gneqq n$, we obtain

$$h_X(\mathcal{M},n,v) = \sup_{|\underline{\alpha}|_\infty\leq n} \log|\mathcal{G}_{[\underline{\alpha}]}|_{X,v} \leq n\log\frac{1}{R_{X,v}(M)} + \begin{cases}\log\{n,(\mu-1)\}_v & \text{if } p(v)\leq n\\ 0 & \text{if } \ p(v)\gneqq n\end{cases},$$

and therefore

$$\begin{aligned}\limsup_{n\to\infty}\frac{1}{n}\sum_{v\in\Sigma'_S} h_X(\mathcal{M},n,v) &\leq \sum_{v\in\Sigma'_S}\log\frac{1}{R_{X,v}(M)} + \limsup_{n\to\infty}\frac{1}{n}\sum_{\substack{v\in\Sigma'_S\\ p(v)\leq n}}\log\{n,(\mu-1)\}_v\\ &\leq \sum_{v\in\Sigma'_S}\log\frac{1}{R_{X,v}(M)} + \limsup_{n\to\infty}\frac{1}{n}\sum_{\substack{v\in\Sigma'_S\\ p(v)\leq n}}\left((\mu-1)\left[\frac{\log n}{\log p(v)}\right]\log a_v - \theta(n,v)\right)\\ &\leq \sum_{v\in\Sigma'_S}\log\frac{1}{R_{X,v}(M)} + \limsup_{n\to\infty}\frac{1}{n}\left(\Theta_{\Sigma'_S}(n) + \Theta_{\Sigma'_S}\left(\frac{n}{2}\right) + \cdots + \Theta_{\Sigma'_S}\left(\frac{n}{\mu-1}\right)\right)\\ &\qquad + \limsup_{n\to\infty}\frac{1}{n}\sum_{\substack{v\in\Sigma'_S\\ p(v)\leq n}}(\mu-1)\left(\left[\frac{\log n}{\log p(v)}\right]\log a_v - \log a_v\right)\\ &\leq \sum_{v\in\Sigma'_S}\log\frac{1}{R_{X,v}(M)} + \left(1+\frac{1}{2}+\cdots+\frac{1}{\mu-1}\right)\Delta(M) + A(\mu-1)\ ,\end{aligned}$$

where

$$A = \limsup_{n\to\infty}\frac{1}{n}\sum_{\substack{v\in\Sigma'_S\\ p(v)\leq n}}\left(\left[\frac{\log n}{\log p(v)}\right]-1\right)\log a_v\ .$$

For $\sqrt{n} \lneqq p(v) \leq n$ we observe that

$$1 \leq \frac{\log n}{\log p(v)} \lneqq 2$$

and hence that

$$\left(\left[\frac{\log n}{\log p(v)}\right]-1\right) = 0\ .$$

We deduce that

$$\begin{aligned} 0 \le A &= \limsup_{n\to\infty} \frac{1}{n} \sum_{\substack{v\in\Sigma'_S \\ p(v)\le\sqrt{n}}} \left(\left[\frac{\log n}{\log p(v)} \right] - 1 \right) \log a_v \\ &\le \limsup_{n\to\infty} \frac{1}{n} \sum_{\substack{v\in\Sigma'_S \\ p(v)\le\sqrt{n}}} \left(\frac{\log n}{\log p(v)} - 1 \right) \log p(v) \\ &\le \limsup_{n\to\infty} \frac{\log n}{n} \sum_{\substack{v\in\Sigma'_S \\ p(v)\le\sqrt{n}}} \left(1 - \frac{\log p(v)}{\log n} \right) \\ &\le [K:\mathbb{Q}] \left(\limsup_{n\to\infty} \frac{\log n}{n} \sum_{p\le\sqrt{n}} \left(1 - \frac{\log p}{\log n} \right) \right) \\ &\le [K:\mathbb{Q}] \limsup_{n\to\infty} \frac{\log n}{\sqrt{n}} = 0 \ . \end{aligned}$$

We conclude that $A = 0$ and therefore we have proved (4.2.6). ■

§5. Nilpotence and lower bounds.

Following [D], we give another inequality, more precise then (4.2.5), related to the order of nilpotence.

Proposition 5.1. *Let (M, ∇) be a differential $\mathcal{F}/K$-module of type G and of rank μ; we assume that (M, ∇) admits a model $(\mathcal{M}, \nabla)$ over a smooth S-model X of $\mathcal{F}/K$. If $\Sigma_S^{(m)}$ is the subset of Σ_S of all primes v such that the induced connection $(\mathcal{M}_{k(v)}, \nabla_{k(v)})$ on the closed fiber $X_{k(v)}$ is nilpotent of order m (i.e. nilpotent of order $\le m$, but not of order $\le m-1$) and*

$$\Delta^{(m)}(M) = \liminf_{n\to+\infty} \frac{1}{n} \sum_{\substack{v\in\Sigma_S^{(m)} \\ p(v)\le n}} \log |p(v)|_v^{-1} \ ,$$

then we have

(5.1.1)

$$\liminf_{n\to+\infty} \frac{1}{n} \sum_{v\in\Sigma_S^{(m)}} h_X(\mathcal{M}, n, v) \ge \sum_{p\in\Sigma_S^{(m)}} \log \frac{1}{R_{X,v}(M)} + \left(1 + \frac{1}{2} + \cdots + \frac{1}{(m-1)} \right) \Delta^{(m)}(M) \ .$$

Proof. There exists $\underline{w} \in \mathbb{N}^d$ such that $|\underline{w}|_\infty = m-1$ and

$$\left| \mathcal{G}_{[p\underline{w}]} \right|_{X,v} = \left| \frac{1}{(p\underline{w})!} \right|_v \ .$$

Therefore for any $n \in \mathbb{N}$ and for $j = \min\left(\left[\frac{n}{p} + 1 \right], m \right)$ (*i.e.* either j is an integer smaller then m such that $\frac{n}{j} \lneqq n \le \frac{n}{j-1}$ either $j = m$) we have

$$h_X(\mathcal{M}, n, v) \ge (j-1) \log |p(v)|_v^{-1} \ .$$

For a fixed $N \in \mathbb{N}$, $N \gneqq 0$, and for $n \ge Nm$ we deduce that

$$\frac{1}{n} \sum_{\substack{v\in\Sigma_S^{(m)} \\ p(v)\gneqq N}} h_X(\mathcal{M}, n, v) = \frac{1}{n} \sum_{\substack{v\in\Sigma_S^{(m)} \\ N\lneqq p(v)\le \frac{n}{m-1}}} h_X(\mathcal{M}, n, v) + \frac{1}{n} \sum_{j=2}^{m-1} \sum_{\substack{v\in\Sigma_S^{(m)} \\ \frac{n}{j}\lneqq p(v)\le \frac{n}{j-1}}} h_X(\mathcal{M}, n, v) \ .$$

If we set

$$\Theta_{\Sigma_S^{(m)}}(n) = \sum_{\substack{v\in\Sigma_S^{(m)}\\ p(v)\le n}} \log|p(v)|_v^{-1} \ ,$$

then we obtain

$$\frac{1}{n}\sum_{\substack{v\in\Sigma_S^{(m)}\\ p(v)\gneq N}} h_X(\mathcal{M},n,v) \ge (m-1)\frac{1}{n}\left(\Theta_{\Sigma_S^{(m)}}\left(\frac{n}{m-1}\right)-\Theta_{\Sigma_S^{(m)}}(N)\right)$$

$$+\frac{1}{n}\sum_{j=2}^{m-1}(j-1)\left(\Theta_{\Sigma_S^{(m)}}\left(\frac{n}{j-1}\right)-\Theta_{\Sigma_S^{(m)}}\left(\frac{n}{j}\right)\right)$$

$$=\frac{1}{n}\sum_{j=1}^{m-1}\Theta_{\Sigma_S^{(m)}}\left(\frac{n}{j-1}\right)-\frac{m-1}{n}\Theta_{\Sigma_S^{(m)}}(N) \ ,$$

and therefore

$$\liminf_{n\to+\infty}\frac{1}{n}\sum_{\substack{v\in\Sigma_S^{(m)}\\ p(v)\gneq N}} h_X(\mathcal{M},n,v) \ge \left(1+\frac{1}{2}+\cdots+\frac{1}{(m-1)}\right)\Delta^{(m)}(M) \ .$$

We have proved that

$$\liminf_{n\to+\infty}\frac{1}{n}\sum_{v\in\Sigma_S^{(m)}} h_X(\mathcal{M},n,v) \ge \sum_{\substack{p\in\Sigma_S^{(m)}\\ p(v)\le N}} \log\frac{1}{R_{X,v}(M)} + \left(1+\frac{1}{2}+\cdots+\frac{1}{(m-1)}\right)\Delta^{(m)}(M) \ .$$

We get (5.1.1) by taking the limit for $N\to+\infty$. ∎

Corollary 5.2. *Under the hypothesis of the previous proposition we have*

$$\sigma_{X/S}(M)-\varrho_{X/S}(M) \ge \sum_{m=2}^{\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{(m-1)}\right)\Delta^{(m)}(M) \ . \tag{5.2.1}$$

Proof. Let $\widetilde{\Sigma}_S$ be the set of all primes $v\in\Sigma_S$ such that $(\mathcal{M},\nabla)$ has not nilpotent reduction or zero p-curvature. By Fatou's Lemma, we have

$$\sigma_{X/S}(M) \ge \liminf_{n\to+\infty}\frac{1}{n}\sum_{v\in\Sigma_S} h_X(\mathcal{M},n,v)$$

$$\ge \sum_{m=2}^{\infty}\liminf_{n\to+\infty}\frac{1}{n}\sum_{v\in\Sigma_S^{(m)}} h_X(\mathcal{M},n,v) + \liminf_{n\to+\infty}\sum_{v\in\widetilde{\Sigma}_S}\frac{1}{n}h_X(\mathcal{M},n,v)$$

$$\ge \sum_{m=2}^{\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{(m-1)}\right)\Delta^{(m)}(M) + \sum_{v\in\Sigma_S}\log\frac{1}{R_{X,v}(M)} \ .$$

∎

Remark 5.3. We point out that in the last inequality we have

$$\liminf_{n\to+\infty}\sum_{v\in\widetilde{\Sigma}_S}\frac{1}{n}h_X(\mathcal{M},n,v) = \sum_{v\in\widetilde{\Sigma}_S}\log\frac{1}{R_{X,v}(M)} \ .$$

Actually, the left hand side is greater or equal the right hand side by Fatou's lemma. To prove the other inequality we consider two cases:
1) if $(\mathcal{M}_{k(v)},\nabla_{k(v)})$ is not nilpotent, then:

$$\frac{1}{n}h_X(\mathcal{M},n,v)=\frac{1}{n}\log\left|\frac{1}{n!}\right|_{X,v}\leq\frac{\log|p(v)|_{X,v}}{p-1}=\log\frac{1}{R_{X,v}(M)}\ .$$

So

$$\liminf_{n\to+\infty}\sum_{\text{non nilpotent}}\frac{1}{n}h_X(\mathcal{M},n,v)\leq\sum_{\text{non nilpotent}}\log\frac{1}{R_{X,v}(M)}\ ,$$

where the sum is taken over the $v\in\Sigma_S$ such that $(\mathcal{M}_{k(v)},\nabla_{k(v)})$ is not nilpotent.
2) if $(\mathcal{M}_{k(v)},\nabla_{k(v)})$ has zero p-curvature, then the equality is just a consequence of (4.2).

Dwork has conjectured in his article [D] that the last lower bound (5.2.1) is, in fact, an equality.

Appendix. Generalization of Eisenstein's theorem to the several variables case.

We have defined for a smooth $\overline{\mathbb{Q}}^{alg}$-variety V a full subcategory $\mathbf{G}(V)$ of the abelian category $\mathbf{MIC}(V)$ of quasi-coherent $\mathcal{O}_V$-modules with an integrable $V/\overline{\mathbb{Q}}^{alg}$-connection, whose objects are coherent $\mathcal{O}_V$-modules equipped with a G-connection. It is easy to show (*cf.* [AB] or [B]) that $\mathbf{G}(V)$ is a thick tannakian subcategory of $\mathbf{MIC}(V)$. So, it contains $(\mathcal{O}_V,V)$ and is stable by $-\otimes_{\mathcal{O}_V}-$, $\mathrm{Hom}_{\mathcal{O}_V}(-,-)$, duality and extensions. Moreover, it is stable by taking subquotients.

If $f:V\longrightarrow W$ is any morphism of smooth $\overline{\mathbb{Q}}^{alg}$-varieties, f^*, in the sense of $\mathcal{O}$-modules, induces a functor

$$f^*:\mathbf{G}(W)\longrightarrow\mathbf{G}(V)\ .$$

If $f:V\longrightarrow W$ is an étale covering (*i.e.* a finite étale morphism), then f_* induces a functor

$$f_*:\mathbf{G}(V)\longrightarrow\mathbf{G}(W)\ .$$

This is some sort of generalization of the Eisenstein's theorem:

Proposition A.1. *In the notation of §2, let $\varphi:X\longrightarrow Y$ be an étale covering of smooth S-models X and Y. Then:*
1) If $(\mathcal{M},\nabla)$ is a model of (M,∇) over X/S, then $R_{X,v}(M)=R_{Y,v}(\varphi_*M)$, $\sigma_{X/S}(M)=\sigma_{Y/S}(\varphi_*M)$ and $\varrho_{X/S}(M)=\varrho_{Y/S}(\varphi_*M)$.
2) If $(\mathcal{N},\nabla)$ is a model of (N,∇) over Y/S, then $R_{X,v}(\varphi^*N)=R_{Y,v}(N)$, $\sigma_{X/S}(\varphi^*N)=\sigma_{Y/S}(N)$ and $\varrho_{X/S}(\varphi^*N)=\varrho_{Y/S}(N)$.

Proof.
1) We notice that

$$\varphi_*\mathcal{P}^n_{X/S}\cong\mathcal{P}^n_{Y/S}\otimes\varphi_*\mathcal{O}_X\ .$$

Let us consider the stratification data

$$\Theta^{(n)}:\mathcal{M}\longrightarrow\left(\mathcal{P}^n_{X/S}\otimes\mathcal{M}\right)\otimes K\ .$$

We obtain

$$\varphi_*\Theta^{(n)}:\varphi_*\mathcal{M}\longrightarrow\varphi_*\left(\mathcal{P}^n_{X/S}\otimes\mathcal{M}\right)\otimes K\cong\left(\mathcal{P}^n_{Y/S}\otimes\varphi_*\mathcal{M}\right)\otimes K\ ,$$

and hence the ideal $I^{(n)}$ (*cf.* (2.6.3)) does not change. Therefore we have

$$R_{X,v}(M) = R_{Y,v}(\varphi_* M),\ \sigma_{X/S}(M) = \sigma_{Y/S}(\varphi_* M) \text{ and } \varrho_{X/S}(M) = \varrho_{Y/S}(\varphi_* M).$$

2) Since $\varphi^*\mathcal{P}^n_{Y/S} \cong \mathcal{P}^n_{X/S}$, applying the functor φ^* to

$$\Theta'^{(n)} : \mathcal{N} \longrightarrow \left(\mathcal{P}^n_{Y/S} \otimes \mathcal{N}\right) \otimes K\ ,$$

we obtain

$$\varphi^*\Theta'^{(n)} : \varphi^*\mathcal{N} \longrightarrow \varphi^*\left(\mathcal{P}^n_{Y/S} \otimes \mathcal{N}\right) \otimes K \cong \left(\mathcal{P}^n_{X/S} \otimes \varphi^*\mathcal{N}\right) \otimes K\ .$$

This proves that $R_{X,v}(\varphi^* N) = R_{Y,v}(N)$, $\sigma_{X/S}(\varphi^* N) = \sigma_{Y/S}(N)$ and $\varrho_{X/S}(\varphi^* N) = \varrho_{Y/S}(N)$. ∎

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