Action of an endomorphism on (the solutions of) a linear differential equation

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Contents

1 Introduction

The purpose of this survey is to provide the reader with a user friendly introduction to the two articles [8] and [9], which give a Galoisian description of the action of an endomorphism of a differential field (K, ∂) on the solutions of a linear differential equation defined over (K, ∂). Although this paper is totally independent from [6], the combination of the two surveys can give an overview of the topic, which has the default of not being complete and the advantage of being rather short, facilitating the orientation in a literature that has developed relatively quickly.

Parameterized Galois theories start with the seminal works [15] and [2]. In the latter the authors consider the dependence of a full set of solutions of a linear differential equation with respect to a differential parameter, which is incarnated in a derivation linearly independent from the one appearing in the equation. This work has been followed by [12], which considers the problem of differential dependence of a full set of solutions of a linear difference equation. See [11] for a detailed introduction to this topic or [6] for a shorter survey.

The dependence of a full set of solutions of a linear functional equation with respect to a discrete parameter is considered in [8], [9] and [18]. Due to

the intrinsic difficulty of difference algebra, the proofs, but also sometimes the statements, are more complicated than in the analogous continuous theory. We will hide the technicalities, but the reader should not be naive and should pay attention to easy generalizations of notions of usual algebra.

The parameterized Galois theories have given a great impulse to Galois theory of functional equations. Their developments and applications, that we have collected in a separated list of references at the end of the paper, go beyond the scope of this survey. We won't mention any of them.

We introduce the definitions of difference and differential algebra that are essential for the content of this survey. A more detailed, but still quite short, presentation can be found in [8]. For general introductions to these topics, see [4],[16], [13].

2 A quick overview of differential Galois theory

For the reader convenience we give a very short introduction to *classical* differential Galois theory, as a sort of guideline for the pages below. There are numerous introductions to this topic, going from short notes to thick volumes. We cite here a selection of references [1], [17], [22], [21], [5], [20].

2.1 Differential algebra

A ∂-ring (R, ∂) is a ring R equipped with a derivation ∂, i.e., with a linear map $\partial : R \to R$ satisfying the Leibniz rule $\partial(ab) = a\partial(b) + \partial(a)b$ for all $a, b \in R$. For simplicity we will frequently say that R is a ∂-ring. All rings in this paper are supposed to be commutative with 1 and to have characteristic zero. The ring $R^\partial = \{r \in R : \partial(r) = 0\}$ is the subring of ∂-constants of R. A ∂-ideal of R is an ideal which is stable by the action of ∂. A maximal ∂-ideal of R is a ∂-ideal of R that is maximal for the inclusion among the ∂-ideals. A maximal ∂-ideal does not need to be maximal but it is always prime. A ∂-ring is said to be ∂-simple if it has no nontrivial ∂-ideals.

A ∂-ring is called a ∂-field if the underling ring is a field. Its subring of ∂-constants is always a field. Let (K, ∂) be a ∂-field. A ∂-field extension $(L, \partial)/(K, \partial)$ is a field extension L/K such that both L and K are ∂-fields and that the derivation of L extends the derivation of K. If A is a subset of L then $K\{A\}_\partial$ (resp. $K\langle A\rangle_\partial$) is the smallest ∂-ring (resp. ∂-field) containing K and A.

2.2 A crush course in differential Galois theory

Let (K, ∂) be a ∂-field of characteristic 0. One can naturally consider a linear differential system $\partial(y) = Ay$ with coefficient in K, i.e., a linear differential system associated with a matrix A that belongs to the ring $M_n(K)$ of square matrices of order n with coefficients in K.

Let us suppose that the field of ∂-constants of K is algebraically closed. Under this assumption, we know that there exists a Picard-Vessiot extension L/K for $\partial(y) = Ay$, i.e., a ∂-field extension $(L, \partial)/(K, \partial)$, such that

1. there exists $U \in \mathrm{GL}_n(L)$, whose entries generate L over K and verifying $\partial(U) = AU$.

2. $L^\partial = K^\partial =: k$;

The differential Galois group of $\partial(y) = Ay$ is defined as

$$\mathrm{Gal}(L/K) := \{\varphi \text{ is a field automorphim of } L/K, \text{ commuting to } \partial\}.$$

Any automorphism $\varphi \in \mathrm{Gal}(L/K)$ sends U to another invertible matrix of solutions of $\partial y = Ay$, so that $U^{-1}\varphi(U) \in \mathrm{GL}_n(k)$. This gives a (faithful) representation $\mathrm{Gal}(L/K) \to \mathrm{GL}_n(k)$ of $\mathrm{Gal}(L/K)$ as a group of matrices: It turns out that $\mathrm{Gal}(L/K)$ is an algebraic group. One can define a Galois correspondence among the intermediate ∂-fields of L/K and the linear algebraic subgroups of $\mathrm{Gal}(L/K)$.

One of the most important results of the Galois theory of differential equations is that the dimension $\dim_k \mathrm{Gal}(L/K)$ of $\mathrm{Gal}(L/K)$ as an algebraic variety is equal to the transcendence degree of the extention L/K.

3 Examples of situations encompassed by the theory below

We are presenting here a few (baby) examples and problems that the reader should keep in mind reading the sequel.

Example 3.1. Let us consider the field of rational functions $\mathbb{C}(\alpha, x)$ in the variables α and x, equipped with the usual derivation $\partial = \frac{d}{dx}$, acting trivially on $\mathbb{C}(\alpha)$, and the automorphism

$$\begin{array}{rccc} \tau: & \mathbb{C}(\alpha,x) & \to & \mathbb{C}(\alpha,x), \\ & f(\alpha,x) & \mapsto & f(\alpha+1,x). \end{array}$$

A solution of the rank 1 linear differential equation

$$\partial = \frac{\alpha}{1-x} y$$

is given by the hypergeometric series:

$$F_\alpha = \sum_{n\geq 0} \frac{(\alpha)_n}{n!} x^n \in \mathbb{C}(\alpha)[[x]],$$

where $(a)_0 = 1$ and $(\alpha)_{n+1} = (\alpha+n)(\alpha)_n$, for any $n \geq 1$. One can naturally ask whether F_α is also solution of a (linear) τ-equation, which in this setting would be also called a contiguity relation. Of course, it is easy to find out that F_α is solution of

$$\tau(y) = \frac{1}{1-x} y.$$

The question that we address here is the following: would it be possible to read the existence of the τ-equation above on a convenient Galois group?

Example 3.2. Another instance of the phenomenon above comes from p-adic differential equations. Indeed the action of a Frobenius lift on their solutions is of great help in their study.

Let p be a prime number and let us consider the field $\mathbb{C}_p$ with its norm $|\ |$, such that $|p| = p^{-1}$, and an element $\pi \in \mathbb{C}_p$ verifying $\pi^{p-1} = -p$. Following [10, Chapter II, §6] the series $\theta(x) \in \mathbb{C}_p[[x]]$, defined by $\theta(x) = \exp(\pi(x^p - x))$, has a radius of convergence bigger than 1. Therefore it belongs to the field $\mathcal{E}^\dagger_{\mathbb{C}_p}$, consisting of all series $\sum_{n\in\mathbb{Z}} a_n x^n$ with $a_n \in \mathbb{C}_p$ such that

- $\exists\ \varepsilon > 0$ such that $\forall\ \rho \in]1, 1+\varepsilon[$ we have $\lim_{n\to\pm\infty} |a_n|\rho^n = 0$ and
- $\sup_n |a_n|$ is bounded.

One can endow $\mathcal{E}^\dagger_{\mathbb{C}_p}$ with an endomorphism $F\colon \sum_{n\in\mathbb{Z}} a_n x^n \mapsto \sum_{n\in\mathbb{Z}} a_n x^{pn}$. (For the sake of simplicity we assume here that F is $\mathbb{C}_p$-linear, which has no consequences on this specific example.) The solution $\exp(\pi x)$ of the equation $\partial(y) = \pi y$, where $\partial = \frac{d}{dx}$, does not belong to $\mathcal{E}^\dagger_{\mathbb{C}_p}$, since it has radius of convergence 1. On the other hand, $\exp(\pi x)$ is a solution of an order one linear difference equation with coefficients in $\mathcal{E}^\dagger_{\mathbb{C}_p}$:

$$F(y) = \theta(x)y,\ \theta(x) \in \mathcal{E}^\dagger_{\mathbb{C}_p}.$$

So, here is another very classical situation in which one considers solutions of a linear differential equation and finds difference relations among them.

Example 3.3. Let us consider the field $\mathbb{C}(x)$ of rational functions with complex coefficients, equipped with the derivative $\partial = x\frac{d}{dx}$ and the endomorphism $\sigma : f(x) \mapsto f(x^d)$, where $d \geq 2$ is a fixed integer. Then $x^{1/d}$ is solution of the differential equation

$$\partial(y) = \frac{y}{d},$$

and satisfies a σ-equation, namely $\sigma(y) = x$. This kind of σ-equations is better known as a Mahler equation.

Remark 3.4. Here are some comments:

1. In the examples above, the difference operator is sometimes an automorphism and sometimes an endomorphism. In general we will suppose that we are dealing with an endomorphism acting on the solutions of the differential equation, to include many cases of interest, such as the action of the Frobenius of p-adic differential equations or the case of Mahler equations.

   Notice that a field with an endomorphism can always be embedded in a bigger field with an automorphism, called its inversive closure. So one can always replace (K, σ) with its inversive closure. However, in [8] the authors make great efforts to avoid such an extension, as far as possible. In the theory of p-adic differential equations, for instance, replacing the base field with its inversive closure would erase the distinction between strong and weak Frobenius structures.

2. In the examples above, all the difference relations are linear. This is a coincidence, and, in general we will deal also with the existence of non-linear difference relations among solutions of differential equations.

4 Difference algebra and geometry

4.1 Difference algebra

A σ-ring (R, σ) is a ring R equipped with an endomorphism σ. For simplicity we will frequently say that R is a σ-ring. The ring $R^\sigma = \{r \in R : \sigma(r) = r\}$ is the subring of σ-constants of R. A σ-ideal of R is an ideal which is stable by the action of σ. A maximal σ-ideal of R is a σ-ideal of R that is maximal for the inclusion among the σ-ideals. Notice that a maximal σ-ideal does not need to be either maximal or prime. A σ-ring is said to be σ-simple if it has no nontrivial σ-ideals.

σ-fields. A σ-ring is called a σ-field (resp. a σ-domain) if the underlying ring is a field (resp. a domain and σ is injective). The subring of σ-constants of a σ-field is always a field. Let (k, σ) be a σ-field. A σ-field extension $(L, \sigma)/(k, \sigma)$ is a field extension L/k such that both L and k are σ-fields and that the endomorphism of L extends the endomorphism of k. If A is a subset of L then $k\{A\}_\sigma \subset L$ (resp. $k\langle A\rangle_\sigma \subset L$) is the smallest σ-ring (resp. σ-field) containing k and A.

Definition 4.1 (Definition 4.1.7 in [16])**.** Let $L|k$ be a σ-field extension. Elements $a_1, \ldots, a_n \in L$ are called *transformally (or σ-algebraically) independent over k* if the elements $a_1, \ldots, a_n, \sigma(a_1), \ldots, \sigma(a_n), \ldots$ are algebraically independent over k. Otherwise, they are called transformally dependent over k.

We define the *σ-transcendence degree of $L|k$*, or σ-$\operatorname{trdeg}(L|k)$ for short, as the maximal cardinality of a subset of L whose elements are σ-transformally independent over k.

The ring of σ-polynomials in the indeterminates $x_1, \ldots, x_r$ with coefficients in k, or over k, is the σ-ring $k\{x_1, \ldots, x_r\}_\sigma$, where $x_1, \ldots, x_r$ are σ-algebraically independent over k.

Definition 4.2. [Cf. Def. 3.1, p. 1330 in [14].] A σ-field k is called *linearly σ-closed* if every linear system of difference equations over k has a fundamental solution matrix in k. That is, for every $B \in \mathrm{GL}_n(k)$ there exists $Y \in \mathrm{GL}_n(k)$ with $\sigma(Y) = BY$.

We say that a σ-field k is σ-closed[1] if every system of difference polynomial equations with coefficients in k, which posses a solution in some σ-field extension of k, has a solution in k (see also §1.1 in [3]).

Working with a σ-closed σ-field spares some technicalities, but not all of them, and not the most significant. Moreover being σ-closed is in general quite a strong requirement for a σ-field. Being linearly σ-closed is a weaker assumption, although quite strong. For instance, the field of meromorphic functions over $\mathbb{C}$ (resp $\mathbb{C} \setminus \{0\}$) is σ-closed for $\sigma : f(x) \mapsto f(x+1)$ (resp. $\sigma : f(x) \mapsto f(qx)$, for $q \in \mathbb{C}$, $q \neq 0$ and $|q| \neq 1$). This is not at all a trivial remark. See [19] for a proof.

k-σ-algebras. A k-σ-algebra S is a k-algebra equipped with an endomorphism σ, such that the natural morphism $k \to S$ commutes to σ. If there exists a finite

[1] A σ-closed σ-field is also called a model of ACFA, in model theory language.

set $A \subset S$ such that $S = k\{A\}_\sigma$ then we say that S is a finitely σ-generated k-σ-algebra.

Let k be a σ-field and S a k-σ-algebra. We say that S is σ-separable over k if σ is injective on the k-σ-algebra $S \otimes_k k'$, for every σ-field extension k' of k.

4.2 Breviary on σ-algebraic groups

Definition 4.3. Let k be a σ-field. A *σ-algebraic group over k* is a (covariant) functor G from the category of k-σ-algebras to the category of groups which is representable by a finitely σ-generated k-σ-algebra. I.e., there exists a finitely σ-generated k-σ-algebra $k\{G\}$ such that

$$G \simeq \mathrm{Alg}_k^\sigma(k\{G\}, -).$$

Here Alg_k^σ stands for morphisms of k-σ-algebras. By the Yoneda lemma $k\{G\}$ is unique up to isomorphisms.

The most natural example in this setting is the σ-algebraic group $\mathrm{GL}_{n,k}$ which is represented by the finitely σ-generated k-σ-algebra $k\{X, \det X^{-1}\}_\sigma$, where $X = (X_{i,j})$ is a square matrix of order n.

We say that H is a σ-closed subgroup of G, if it is a subfunctor of G. We will be interested in σ-closed subgroups of $\mathrm{GL}_{n,k}$, i.e. σ-algebraic groups that are represented by quotient of $k\{X, \det X^{-1}\}_\sigma$ by a convenient σ-ideal.

Remark 4.4. A σ-closed subgroup H of a σ-algebraic group G can, of course, be normal, in the usual sense. See Definition A.41 and Theorem A.43 in [8] for the existence of the quotient G/H.

We need to define the σ-dimension $\sigma\text{-}\dim_k(G)$ of a σ-algebraic group G over a σ-field k. We refer to [8, Appendix A.7] for a discussion of the different issues of such a definition.

Definition 4.5. If G is a σ-algebraic group associated with the finitely σ-generated k-σ-algebra $k\{G\}$ we define:

$$\sigma\text{-}\dim_k(G) = \left\lfloor \limsup_{i\to\infty} \left(\frac{\dim(k[a, \ldots, \sigma^i(a)])}{i+1} \right) \right\rfloor,$$

where $\lfloor x \rfloor$ denotes the largest integer not greater than x, $a = (a_1, \ldots, a_m)$ is a σ-generating set of $k\{G\}$ over k and the $\dim(k[a, \ldots, \sigma^i(a)])$ is the usual Krull dimension.

Remark 4.6. If $k\{G\}$ is a σ-domain σ-finitely generated over k, then $\sigma\text{-}\dim_k(G)$ coincides with $\sigma\text{-}\mathrm{trdeg}(k\{G\}|k)$. See [8, Lemma A.26].

We will need also the following definition:

Definition 4.7. Let G be a σ-closed subgroup of $\mathrm{GL}_{n,k}$. We call G a *σ^d-constant* subgroup of $\mathrm{GL}_{n,k}$ if G is contained in the σ-closed subgroup $\mathrm{GL}_{n,k}^{\sigma^d}$ of $\mathrm{GL}_{n,k}$ defined by the σ-ideal generated by $\sigma^d(X) - X$.

If $\widetilde{k}$ is a σ-field extension of k, we say that G *is conjugate over $\widetilde{k}$ to a σ^d-constant group* if there exists $h \in \mathrm{GL}_n(\widetilde{k})$ such that $hG_{\widetilde{k}}h^{-1} \leq \mathrm{GL}_{n,\widetilde{k}}$ is σ^d-constant, where $G_{\widetilde{k}}$ is the restriction of the functor G to the category of $\widetilde{k}$-σ-algebras.

If G is a σ-closed subgroup of $\mathrm{GL}_{n,k}$, then it is defined by a σ-ideal $I(G)$ of $k\{X, \det X^{-1}\}_\sigma$. Notice that $k\{X, \det X^{-1}\}_\sigma$ contains a copy of $k[X, \det X^{-1}]$, which is the ring of rational function of the linear algebraic group $\mathrm{GL}_{n,k}$.[2] The Zariski closure $G[0]$ of G is the linear algebraic subgroup of $\mathrm{GL}_{n,k}$ defined by the ideal $I(G) \cap k[X, \det X^{-1}]$.

4.3 Issues with difference algebras and fields

Let (k, σ) be a characteristic 0 σ-field. The theory below produces a Galois group which is a σ-closed subgroup of $\mathrm{GL}_{n,k}$. The problem with difference geometry is the following: no matter how huge are the σ-field extensions of the σ-field k that we consider, we may never get enough zeros of our ideal to characterize its geometry. A more serious way of restating this problem is the following: one needs to look at the Galois group as a difference group scheme to actually establish a Galois correspondence. The following example should clarify the situation.

Example 4.8. Let us consider the σ-field k and the σ-closed subgourp G of $\mathbb{G}_{m,k}$ defined by $\{t^2 = 1\}$: This means that the Hopf algebra of $\mathbb{G}_{m,k}$ is the σ-ring of σ-polynomials $k\left\{t, \frac{1}{t}\right\}_\sigma$ and that G is the σ-closed subgroup defined by the σ-ideal generated by the σ-polynomial $t^2 - 1$. Let S be any σ-field extension of k. Clearly, the group of S-rational points of G is $G(S) = \{1, -1\}$.

Let H be a σ-closed subgroup of G defined by the σ-ideal generated by $\{t^2-1, \sigma(t)-1\}$. Once more for any σ-field extension S of k we have $H(S) = \{1\}$, although such an ideal is not trivial, in any sense.

Now let us consider the rational points of those groups in some k-σ-algebras. Notice that an endomorphism of a ring (or of a k-algebra) does not need to be injective, as in the case of σ-fields: This allows the set of zeros to be much larger in some k-σ-algebras than in any σ-field. Typically, we can consider the rational points of G in a k-σ-algebra S which is an extension of the k-σ-algebra $k\{t, t^{-1}\}_\sigma/(t^2 - 1)$. Similarly for the group H and the algebra $k\{t, t^{-1}\}_\sigma/(t^2 - 1, \sigma(t) - 1)$.

This justifies the fact that we cannot simply consider a set of zeros in a large σ-field, but we are obliged to look at such groups as difference group schemes, i.e. group functors from the category of k-σ-algebras to the category of groups, as in previous subsection. The problem will become even clearer in the Example 4.9 below.

4.4 Difference-differential algebra

A $\sigma\partial$-ring R is a ring which is both a σ-ring and a ∂-ring and that satisfies the following compatibility condition: We suppose that there exists a ∂-constant $\hbar$ such that $\partial\sigma = \hbar\sigma\partial$.

The notions already introduced above for ∂-rings and σ-rings intuitively generalize to this case, so that we have $\sigma\partial$-ideals, $\sigma\partial$-simple $\sigma\partial$-ring, $\sigma\partial$-fields, $\sigma\partial$-field extensions, K-$\sigma\partial$-algebras, and so on.

[2] To be precise one should introduce a different notation for the σ-algebraic group $\mathrm{GL}_{n,k}$ and the linear algebraic group $\mathrm{GL}_{n,k}$. According to the definition that follows, we could call $\mathrm{GL}_{n,k}[0]$ the linear algebraic group, but it appears as a useless complication of the notation, since the meaning will be always clear from the context.

Example 4.9. Let us consider the $\sigma\partial$-field $K = \mathbb{C}(x)$, with the automorphism $\sigma(x) = x + 1$, and the usual derivation $\partial = \frac{d}{dx}$. Moreover we consider the $\sigma\partial$-field extension of K defined by:

$$L := K\langle\sqrt{x}\rangle_{\sigma,\partial} = K(\sqrt{x+i}; \forall i \in \mathbb{Z},\, i \geq 0)$$

and the group $\mathrm{Aut}^{\sigma\partial}(L/K)$ of automorphisms φ of L over K, that commute to σ and ∂. Since φ commutes with the derivation and $\sqrt{x}$ is solution of the equation $\partial(y) = \frac{y}{2x}$, there exists a ∂-constant c_φ such that $\varphi(\sqrt{x}) = c_\varphi\sqrt{x}$. Moreover, $\varphi(x) = x$ implies that $c_\varphi^2 = 1$. Finally the commutativity with σ imposes that $\varphi(\sqrt{x+i}) = \sigma^i(c_\varphi)\sqrt{x+i}$. Of course the only choice in $\mathbb{C}$ for c_φ is 1 or -1. So:

$$\mathrm{Aut}^{\sigma\partial}(L/K) \cong \{c_\varphi^2 = 1\} = \{1, -1\} \subset \mathbb{G}_m(\mathbb{C}).$$

The invariant σ-field of such a group is $K(\sqrt{x+i}\sqrt{x+j}; \forall i, j \in \mathbb{Z},\, i, j \geq 0)$, which compromises any hope of having a decent Galois correspondence.

Now let us consider the subgroup $\mathbb{G}_m(\mathbb{C})$ defined by $\{c_\varphi^2 = 1, \sigma(c_\varphi) = 1\}$. If we look at its $\mathbb{C}$ points, it coincides with the trivial group $\{1\}$ and therefore its invariant field is the whole field L. Clearly this is not what we want: we really would like to be able to say that the invariant field of the subgroup defined by $\{c_\varphi^2 = 1, \sigma(c_\varphi) = 1\}$ is $K(\sqrt{x+i}; \forall i \in \mathbb{Z},\, i \geq 1)$.

To make sense of the situation, as we have already pointed out, one is obliged to develop a schematic approach and look for rational points not only in σ-field extensions of $\mathbb{C}$ but in the whole category of $\mathbb{C}$-σ-algebras. See next section.

This example is already in [8]. Many more can be found in *loc.cit.*

5 Difference Galois theory of differential equations

The structure of the difference Galois theory of differential equations is not different from the structure of any Galois theory: One needs to construct a splitting ring, the σ-Picard-Vessiot ring, and to construct a group of automorphisms of such a ring, or of its quotient field, if it is a domain. Then one can classify the groups appearing in the theory and recover information on the solutions of the differential system considered in the first place.

Notation 5.1. We consider a $\sigma\partial$-field (K, ∂, σ), with its field of ∂-constants $k = K^\partial$, and we suppose that there exists $\hbar \in k$ such that

$$\partial\sigma = \hbar\sigma\partial, \tag{5.1}$$

so that k is a σ-field. All fields are in characteristic 0. Our object of study will be a linear differential system

$$\partial(y) = Ay, \text{ with } A \in M_n(K). \tag{5.2}$$

Remark 5.2. If we can find a solution column y of (5.2) in a $\sigma\partial$-field extension of K, then $\sigma(y)$ verifies the differential system: $\partial(\sigma(y)) = \hbar\sigma(\partial y) = \hbar\sigma(A)\sigma(y)$. More generally, for any positive integer d we can iterate $\partial(y) = Ay$ and obtain:

$$\partial(\sigma^d(y)) = \hbar_d\sigma^d(\partial(y)) = \hbar_d\sigma^d(A)\sigma^d(y), \tag{5.3}$$

where $\hbar_d = \hbar\sigma(\hbar)\cdots\sigma^{d-1}(\hbar)$.

If U is a fundamental solution of $\partial(y) = Ay$ is some $\sigma\partial$-field extension of K, we can be interested in finding all algebraic relations among the entries of $U, \sigma U, \ldots, \sigma^d U$. This problem can be tackled studying the differential Galois groups of the following linear differential system of order $n(d+1)$:

$$\partial(y^{(d)}) = \begin{pmatrix} A & 0 & \cdots & \cdots & 0 \\ 0 & \sigma(A) & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \hbar_d\sigma^d(A) \end{pmatrix} y^{(d)} \tag{5.4}$$

If we do not want to bound the order d we need to give a meaning to a "limit" of the differential Galois groups constructed for any d. The σ-Galois group of $\partial(y) = Ay$ introduced below incarnates, heuristically, this limit. We are not going to introduce any notion of limit, but this idea is behind all parameterized Galois theories.

5.1 Construction of σ-Picard-Vessiot rings

Definition 5.3. A K-$\sigma\partial$-algebra R is called a σ-Picard-Vessiot ring for $\partial(y) = Ay$ if:

- there exists $U \in \mathrm{GL}_n(R)$ such that $\partial(U) = AU$ and $R = K\{U, \det U^{-1}\}_\sigma$;
- R is ∂-simple.

A σ-Picard-Vessiot ring (over K) is a K-$\sigma\partial$-algebra that is a σ-Picard-Vessiot ring for a differential system $\partial y = Ay$, with coefficients in K.

The definition above is subtle and needs some comments. Indeed let us try to construct the $\sigma\partial$-Picard-Vessiot ring naively. We take the ring of σ-polynomials $K\{X, \det X^{-1}\}_\sigma$, where $X = (X_{i.j})$ is a square matrix of order n, and we define a derivation extending ∂. Since the entries of X are σ-algebraically independent, we can do that as we see fit. We set $\partial X = AX$, $\partial(\sigma(X)) = \hbar\sigma(\partial X) = \hbar\sigma(A)\sigma(X)$, and more generally, for any positive integer d:

$$\partial(\sigma^d(X)) = \hbar_d\sigma^d(\partial(X)) = \hbar_d\sigma^d(A)\sigma^d(X). \tag{5.5}$$

We have endowed $K\{X, \det X^{-1}\}_\sigma$ with a structure of $\sigma\partial$-ring. Now we can consider a maximal $\sigma\partial$-ideal $\mathfrak{M}$ of $K\{X, \det X^{-1}\}_\sigma$. The ring $K\{X, \det X^{-1}\}_\sigma/\mathfrak{M}$ almost satisfies the conditions of the definition above, apart that it is $\sigma\partial$-simple and most likely not ∂-simple: It has no proper ideals invariant under both σ and ∂, but it may have a proper ideal which is invariant under the action of ∂. The point behind the definition of $\sigma\partial$-Picard-Vessiot ring is that there exist maximal $\sigma\partial$-ideals that are also maximal ∂-ideals. This means that the construction of R must be more sophisticated than the one that we have sketched above.

On the other hand, asking that R is ∂-simple has some advantages: A maximal ∂-ideal is at least prime, while a maximal $\sigma\partial$-ideal does not need to be prime, so that a σ-Picard-Vessiot ring is always a domain and even a σ-domain.

Proposition 5.4 ([23, Lemma 2.16, p. 1392] and [8, Proposition 1.12]). *Let K be a $\sigma\partial$-field, $k := K^{\partial}$ be algebraically closed and $A \in M_n(K)$. Then there exists a σ-Picard-Vessiot extension R for $\partial y = Ay$ such that $R^{\partial} = K^{\partial} = k$.*

Idea of the proof. For each one of the systems $\partial(y) = A_d y$, as in (5.4), we are able to construct a (classical) Picard-Vessiot ring by taking the quotient of a ring of polynomials in the $(d+1)n^2$ indeterminates $X, \sigma(X), \ldots, \sigma^d(X)$:

$$S_d := K\left[X, \tfrac{1}{\det(X)}, \sigma(X), \tfrac{1}{\sigma(\det(X))}, \ldots, \sigma^d(X), \tfrac{1}{\sigma^d(\det(X))}\right]$$

by some maximal ∂-ideal $\mathfrak{m}_d$ of S_d. Here X is an $n \times n$-matrix of σ-indeterminates and the action of ∂ on S_d is determined by $\partial(X) = AX$ and the commutativity relation (5.1). The difficulty is to make this construction compatible with the natural injection $S_{d-1} \to S_d$ and the action of σ: Namely we need to construct the ideals $\mathfrak{m}_d$ so that $\mathfrak{m}_{d-1} \subset \mathfrak{m}_d$ and $\sigma(\mathfrak{m}_{d-1}) \subset \mathfrak{m}_d$. This difficulty can be resolved by a recourse to the prolongation lemma for difference kernels (see [4, Lemma 1, Chapter 6, p. 149]). We set $\mathfrak{m} := \bigcup_{d \geq 0} \mathfrak{m}_d$ and $R := k\{X, \frac{1}{\det(X)}\}_{\sigma}/\mathfrak{m}$. So R is the union of the ∂-simple rings $R_d := S_d/\mathfrak{m}_d$. One concludes using some standard theorems of differential algebra. □

The uniqueness of the σ-Picard-Vessiot rings is a subtle matter, which may require some technical assumptions. We recall only one statement and refer to [8, §1.1.2] for a deeper discussion of the problem.

Corollary 5.5 ([8, Corollary 1.17]). *Let K be a $\sigma\partial$-field such that K^{∂} is a σ-closed σ-field. Let R_1 and R_2 be two σ-Picard-Vessiot rings for $\partial(y) = Ay$ with $A \in M_n(K)$. Then there exists an integer $l \geq 1$ such that R_1 and R_2 are isomorphic as K-$\partial\sigma^l$-algebras.*

We remind some first properties of σ-Picard-Vessiot ring:

Lemma 5.6. *Let R be a σ-Picard-Vessiot ring over K. We have:*

1. *R is $\sigma\partial$-simple.*
2. *R is a σ-domain. In particular σ and ∂ extend to the field of fractions L of R and $L^{\partial} = R^{\partial}$.*
3. *In the notation above, let R be the σ-Picard-Vessiot ring of $\partial y = Ay$, with $A \in M_n(K)$, and L be the field of fractions of R. If $Y \in GL_n(L)$ is a solution matrix of $\partial y = Ay$, then for any integer $d \geq 0$:*

$$K\left[Y, \frac{1}{\det Y}, \sigma(Y), \frac{1}{\det \sigma(Y)}, \ldots, \sigma^d(Y), \frac{1}{\det \sigma^d(Y)}\right] \subset L$$

is a (classical) Picard-Vessiot ring for the differential system (5.4).

Proof. The first assertion is a tautology. For the second assertion see [8, Lemma 1.4]. The third assertion is proved in *loc.cit.*, Lemma 1.3. □

5.2 σ-Picard-Vessiot extensions

Definition 5.7. Let K be a $\sigma\partial$-field and $A \in M_n(K)$. A $\sigma\partial$-field extension L of K is called a *σ-Picard-Vessiot extension for $\partial(y) = Ay$* if

1. there exists $Y \in \mathrm{GL}_n(L)$ such that $\partial(Y) = AY$ and $L = K\langle Y_{ij} |\ 1 \leq i, j \leq n\rangle_\sigma$;
2. $L^\partial = K^\partial$.

Proposition 5.8 ([8, Proposition 1.5])**.** *Let K be a $\sigma\partial$-field and $A \in M_n(K)$. If $L|K$ is a σ-Picard-Vessiot extension for $\partial(y) = Ay$ with solution matrix $Y \in \mathrm{GL}_n(L)$, then $R := K\{Y, \frac{1}{\det(Y)}\}_\sigma$ is a σ-Picard-Vessiot ring for $\partial(y) = Ay$. Conversely, if R is a σ-Picard-Vessiot ring for $\partial(y) = Ay$ with $R^\partial = K^\partial$, then the field of fractions of R is a σ-Picard-Vessiot extension for $\partial(y) = Ay$.*

As far as the uniqueness is concerned we recall only the statement below. Notice that two σ-field extensions L_1 and L_2 of a σ-field K are compatible if there exists a σ-field M which is a σ-field extension of K and two endomorphisms of σ-fields $L_i \to M$, for $i = 1, 2$.

Proposition 5.9 ([8, Corollary 1.18 and Proposition 1.19])**.** *Let K be a $\sigma\partial$-field such that K^∂ is a σ-closed σ-field. Let L_1 and L_2 be two σ-Picard-Vessiot extensions for $\partial(y) = Ay$ with $A \in M_n(K)$. Then*

1. *there exists an integer $l \geq 1$ such that $L_1|K$ and $L_2|K$ are isomorphic as $\partial\sigma^l$-field extensions of K.*
2. *the $\sigma\partial$-fields L_1 and L_2 are isomorphic (as $\sigma\partial$-field extensions of K) if and only if L_1 and L_2 are compatible σ-field extensions of K.*

To conclude this subsection we consider the following very natural situation:

Proposition 5.10 ([8, Proposition 1.14])**.** *Let k be a σ-field and let $K = k(x)$ denote the field of rational functions in one variable x over k. Extend σ to K by setting $\sigma(x) = x$ and consider the derivation $\partial = \frac{d}{dx}$. Thus K is a $\sigma\partial$-field with $\hbar = 1$ and $K^\partial = k$. Then for every $A \in M_n(K)$, there exists a σ-Picard-Vessiot extension $L|K$ for $\partial(y) = Ay$.*

Proof. Since we are in characteristic zero, there exists an $a \in k^\sigma$ which is a regular point for $\partial(y) = Ay$. That is, no denominator appearing in the entries of A vanishes at a. We consider the field $k((x-a))$ of formal Laurent series in $x-a$ as a $\sigma\partial$-field by setting $\partial(\sum b_i(x-a)^i) = \sum i b_i(x-a)^{i-1}$ and $\sigma(\sum b_i(x-a)^i) = \sum \sigma(b_i)(x-a)^i$. Then $k((x-a))$ is naturally a $\sigma\partial$-field extension of K. By choice of a, there exists a solution matrix $Y \in \mathrm{GL}_n(k((x-a)))$ for $\partial(y) = Ay$. Since $k((x-a))^\partial = k$ it is clear that $L := K\langle Y\rangle_\sigma \subset k((x-a))$ is a σ-Picard-Vessiot extension for $\partial(y) = Ay$. □

5.3 The σ-Galois group and its properties

If $R \subset R'$ is an inclusion of $\sigma\partial$-rings, we denote by $\mathrm{Aut}^{\sigma\partial}(R'|R)$ the automorphisms of R' over R in the category of $\sigma\partial$-rings, i.e., the automorphisms are required to be the identity on R and to commute with ∂ and σ.

Let us suppose that R is a $\sigma\partial$-ring and that S is a k-σ-algebra. If we endow S with a trivial action of ∂, then we can define a natural structure of $\sigma\partial$-ring over the tensor product $R \otimes_k S$: We have $\partial(r \otimes s) = \partial(r) \otimes s$ for $r \in R$ and $s \in S$. Now we are ready to introduce the notion of σ-Galois group:

Definition 5.11. Let $L|K$ be a σ-Picard-Vessiot extension with σ-Picard-Vessiot ring $R \subset L$ and field of ∂-constants $k = K^\partial$. We define $\sigma\text{-}\operatorname{Gal}(L|K)$ to be the functor from the category of k-σ-algebras to the category of groups given by

$$\sigma\text{-}\operatorname{Gal}(L|K)(S) := \operatorname{Aut}^{\sigma\partial}(R \otimes_k S | K \otimes_k S)$$

for every k-σ-algebra S. The functor $\sigma\text{-}\operatorname{Gal}(L|K)$ is defined on morphisms by base extension. We call $\sigma\text{-}\operatorname{Gal}(L|K)$ the *σ-Galois group of $L|K$*.

We are interested in the geometrical properties of $\sigma\text{-}\operatorname{Gal}(L|K)$.

Proposition 5.12 ([8, Proposition 2.5])**.** *Let $L|K$ be a σ-Picard-Vessiot extension with σ-Picard-Vessiot ring $R \subset L$. Then $\sigma\text{-}\operatorname{Gal}(L|K)$ is a σ-algebraic group over $k = K^\partial$. The choice of matrices $A \in M_n(K)$ and $Y \in \operatorname{GL}_n(L)$ such that $L|K$ is a σ-Picard-Vessiot extension for $\partial(y) = Ay$ with fundamental solution matrix Y defines a σ-closed embedding*

$$\sigma\text{-}\operatorname{Gal}(L|K) \hookrightarrow \operatorname{GL}_{n,k}$$

of σ-algebraic groups.

Indeed, if $\varphi \in \sigma\text{-}\operatorname{Gal}(L|K)(S)$, for some k-σ-algebra S, then $Y^{-1}\varphi(Y)$ must be an invertible square matrix with coefficients in S, so an element of $\operatorname{GL}_{n,k}(S)$. We will identify $\sigma\text{-}\operatorname{Gal}(L|K)$ with its image in $\operatorname{GL}_{n,k}$.

Notice that another choice of fundamental solution matrix yields a conjugated representation of $\sigma\text{-}\operatorname{Gal}(L|K)$ in $\operatorname{GL}_{n,k}$. Therefore sometimes, we will consider $\sigma\text{-}\operatorname{Gal}(L|K)$ as a σ-closed subgroup of $\operatorname{GL}_{n,k}$ without mentioning the fundamental solution matrix Y.

Now we state an important property of the σ-Galois group, which is extensively used in applications.

Proposition 5.13 ([8, Proposition2.17])**.** *Let $L|K$ be a σ-Picard-Vessiot extension with σ-Galois group G and constant field $k = K^\partial$. Then*

$$\sigma\text{-}\operatorname{trdeg}(L|K) = \sigma\text{-}\dim_k(G).$$

Finally we want to state a result that gives the relation between the σ-Galois group and the usual Galois group of a differential equation.

Proposition 5.14 ([8, Proposition 2.15])**.** *Let $L|K$ be a σ-Picard-Vessiot extension with σ-field of ∂-constants $k = K^\partial$. Let $A \in M_n(K)$ and $Y \in \operatorname{GL}_n(L)$ such that $L|K$ is a σ-Picard-Vessiot extension for $\partial(y) = Ay$ with fundamental solution matrix Y. We consider the σ-Galois group G of $L|K$ as a σ-closed subgroup of $\operatorname{GL}_{n,k}$ via the embedding associated with the choice of A and Y. Set $L_0 = K\langle Y \rangle \subset L$.*

Then $L_0|K$ is a (classical) Picard-Vessiot extension for the linear system $\partial(y) = Ay$. The (classical) Galois group of $L_0|K$ is naturally isomorphic to $G[0]$, the Zariski closure of G inside $\operatorname{GL}_{n,k}$.

Remark 5.15. On can defined a d-th order Zariski closure $G[d]$ of G and compare it to the Galois group of (5.4). By now, the reader has probably an intuition on the kind of statement that one could obtain generalizing the proposition above. The details can be found in [8, §A5 and Proposition 2.15]

5.4 Galois correspondence

In the notation of Proposition 5.14, let S be a k-σ-algebra, $\tau \in G(S)$ and $a \in L$. By definition, τ is an automorphism of $R \otimes_k S$. If we write $a = \frac{r_1}{r_2}$ with $r_1, r_2 \in R$, $r_2 \neq 0$ then, we say that a is invariant under τ if and only if $\tau(r_1 \otimes 1) \cdot r_2 \otimes 1 = r_1 \otimes 1 \cdot \tau(r_2 \otimes 1)$ in $R \otimes_k S$.

If H is a σ-closed subgroup of G, we say that $a \in L$ is invariant under H if a is invariant under every element of $H(S) \subset G(S)$, for every k-σ-algebra S. The set of all elements in L, invariant under H, is denoted with L^H. Obviously L^H is an intermediate $\sigma\partial$-field of $L|K$.

If M is an intermediate $\sigma\partial$-field of $L|K$, then it is clear from Definition 5.7 that $L|M$ is a σ-Picard-Vessiot extension with σ-Picard-Vessiot ring MR, the ring compositum of M and R inside L. There is a natural embedding $\sigma\text{-}\operatorname{Gal}(L|M) \hookrightarrow \sigma\text{-}\operatorname{Gal}(L|K)$ of σ-algebraic groups (in the sense that the first is identified to a subfunctor of the second), whose image consists of precisely those automorphisms that leave invariant every element of M.

Theorem 5.16 (σ-Galois correspondence; [8, Theorem 3.2])**.** *Let $L|K$ be a σ-Picard-Vessiot extension with σ-Galois group $G = \sigma\text{-}\operatorname{Gal}(L|K)$. Then there is an inclusion reversing bijection between the set of intermediate $\sigma\partial$-fields M of $L|K$ and the set of σ-closed subgroups H of G given by*

$$M \mapsto \sigma\text{-}\operatorname{Gal}(L|M) \text{ and } H \mapsto L^H.$$

Theorem 5.17 (Second fundamental theorem of σ-Galois theory; [8, Theorem 3.3])**.** *Let $L|K$ be a σ-Picard-Vessiot extension with σ-Galois group G. Let $K \subset M \subset L$ be an intermediate $\sigma\partial$-field and $H \leq G$ a σ-closed subgroup of G such that M and H correspond to each other in the σ-Galois correspondence.*

Then M is a σ-Picard-Vessiot extension of K if and only if H is normal in G. If this is the case, the σ-Galois group of $M|K$ is the quotient G/H.

6 Integrability

In [9], the authors consider several applications of the discrete parameterized Galois theory of difference equations. We won't mention all of them, in particular we won't mention those concerning rank one differential equations. Indeed they are not so surprising for those who know other parameterized Galois theories. We will focus on the integrability and its applications to differential systems having almost simple Galois groups.

6.1 Definition and first properties

Definition 6.1. Let K be a $\sigma\partial$-field, $A \in M_n(K)$, for some positive integer n, and $d \in \mathbb{Z}_{>0}$. We say that $\partial(y) = Ay$ is *σ^d-integrable (over K)*, if there exists

$B \in \mathrm{GL}_n(K)$, such that

$$\left\{ \begin{array}{l} \partial(y) = Ay \\ \sigma^d(y) = By \end{array} \right. \tag{6.1}$$

is compatible, i.e.,

$$\partial(B) + BA = \hbar_d \sigma^d(A)B, \tag{6.2}$$

where $\hbar_d = \hbar\sigma(\hbar)\cdots\sigma^{d-1}(\hbar)$.

The following proposition interprets the compatibility relation (6.2) in terms of solutions of the system (6.1).

Proposition 6.2 ([9, Proposition 5.2])**.** *Let K be a $\sigma\partial$-field, $\partial(y) = Ay$ be a linear differential equation with $A \in M_n(K)$ and L be a $\sigma\partial$-field extension of K.*

1. *If there exist $B \in \mathrm{GL}_n(K)$ and $Y \in \mathrm{GL}_n(L)$ such that $\partial(Y) = AY$ and $\sigma^d(Y) = BY$ (i.e., Y is a fundamental solution of (6.1)), then B satisfies (6.2).*

2. *Conversely, assume that L is a σ-Picard-Vessiot extension for $\partial(y) = Ay$ such that $k = K^\partial$ is linearly σ^d-closed (see Definition 4.2). If there exists a matrix $B \in \mathrm{GL}_n(K)$ verifying (6.2), then there exists a fundamental solution $Y \in \mathrm{GL}_n(L)$ of (6.1).*

The following result on σ^d-integrability is an analogue of Proposition 2.9 in [12], and §1.2.1 in [7]. The statement below may seem more general than the cited results, because it contains the descent [9, Proposition 5.8].

Theorem 6.3. *[[9, Proposition 5.11]] Let $L|K$ be a σ-Picard-Vessiot extension for $\partial(y) = Ay$, with $A \in M_n(K)$. Then $\partial(y) = Ay$ is σ^d-integrable over K if and only if there exists a σ-separable σ-field extension $\widetilde{k}$ of $k := K^\partial$, such that the σ-Galois group $\sigma\text{-}\mathrm{Gal}(L|K)$ is conjugate over $\widetilde{k}$ to a σ^d-constant subgroup of $\mathrm{GL}_{n,\widetilde{k}}$.*

The theorem above is of no help if one does not have a handy criterion. In [9, Appendix], the authors prove some structure theorems for difference groups having simple and almost simple Zariski closure, generalizing a theorem in [3]. We only state the final criteria that can be deduced from those geometric statements.

6.2 Simple and almost simple groups

A linear algebraic group H over a field k is called simple if it is non-commutative, connected and every normal closed subgroup is trivial. If H is non-commutative, connected and every normal closed connected subgroup is trivial, then H is called almost simple. We say that H is absolutely (almost) simple if the base extension of H to the algebraic closure of k is (almost) simple.

Now we state the two criteria that are useful in applications.

Proposition 6.4 ([9, Proposition 6.1]). *Let K be an* inversive *$\sigma\partial$-field, $A \in M_n(K)$ and $L|K$ a σ-Picard-Vessiot extension for $\partial(y) = Ay$. We assume that the Zariski closure H of $\sigma\text{-}\mathrm{Gal}(L|K)$ inside $\mathrm{GL}_{n,k}$ is an absolutely simple algebraic group of dimension $t \geq 1$ over $k = K^{\partial}$. Then the following statements are equivalent:*

1. *$\sigma\text{-}\mathrm{Gal}(L|K)$ is a proper σ-closed subgroup of H.*
2. *The σ-transcendence degree of $L|K$ is strictly less than t.*
3. *There exists $d \in \mathbb{Z}_{>0}$ such that the system $\partial(y) = Ay$ is σ^d-integrable.*

Theorem 6.5 ([9, Proposition 6.4]). *Let K be an* inversive *$\sigma\partial$-field, $\partial(y) = Ay$ a differential system with $A \in M_n(K)$ and $L|K$ a σ-Picard-Vessiot extension for $\partial(y) = Ay$. We assume that the Zariski closure H of $\sigma\text{-}\mathrm{Gal}(L|K)$ inside $\mathrm{GL}_{n,k}$ is an absolutely almost simple algebraic group of dimension $t \geq 1$ over $k = K^{\partial}$. Let K' be the relative algebraic closure of K inside L. Then the following statements are equivalent:*

1. *$\sigma\text{-}\mathrm{Gal}(L|K')$ is a proper σ-closed subgroup of H.*
2. *The σ-transcendence degree of $L|K$ is strictly less than t.*
3. *There exists $d \in \mathbb{Z}_{>0}$ such that the system $\partial(y) = Ay$ is σ^d-integrable over K'.*

Remark 6.6. Compare to the situation with a differential parameter, in this context we are obliged to make a field extension from K to K' to obtain the integrability in the case of an almost simple group. This comes from the fact that finite cyclic σ-algebraic groups have many σ-closed subgroups, while cyclic finite differential groups have a simpler geometry. See Example 4.8. In other words, the extension K'/K corresponds to the largest finite σ-closed subgroups of $\sigma\text{-}\mathrm{Gal}(L|K)$. By finite, we mean that Zariski closure is a finite algebraic group.

For $n = 2$ the theorem above can be restated in a quite explicit way:

Corollary 6.7 ([9, Proposition 6.6]). *Let $K = k(x)$ be a field of rational functions equipped with the derivation $\partial = \frac{d}{dx}$ and an automorphism σ commuting with ∂, such that $k \subset \mathbb{C}$ be an algebraically closed inversive σ-field. We assume that the differential equation $\partial^2(y) - ry = 0$, with $r(x) \in K$, has (usual) Galois group $\mathrm{Sl}_2(k)$ and we denote by $L|K$ one of its σ-Picard-Vessiot extensions. Let K' be the relative algebraic closure of K in L. We have:*

- *If the σ-transcendence degree of $L|K$ is strictly less than 3, there exists $s \in \mathbb{Z}_{>0}$ such that the differential system*

$$\left\{ \begin{array}{l} \partial^2(b) + (\sigma^s(r) - r)b = 2\partial(d) \\ \\ \partial^2(d) + (\sigma^s(r) - r)d = 2\sigma^s(r)\partial(b) + \partial(\sigma^s(r))b \end{array} \right. \tag{6.3}$$

has a non-zero algebraic solution $(b, d) \in (K')^2$.

- *If we can find a solution* $(b,d) \in (K')^2$ *of* (6.3)*, such that the matrix* $B = \begin{pmatrix} d-\partial(b) & b \\ \sigma^s(r)b-\partial(d) & d \end{pmatrix}$ *is invertible, then the* σ*-transcendence degree of* $L|K$ *is strictly less than* 3*.*

We apply Corollary 6.7 to the case of the Airy equation

$$\partial^2(y) - xy = 0. \tag{6.4}$$

Notice that it has an irregular singularity at ∞, and that all the other points of $\mathbb{A}^1_{\mathbb{C}}$ are ordinary. This immediately implies that (6.4) admits a basis of solutions $A(x)$ and $B(x)$ in the field M of meromorphic functions over $\mathbb{C}$.

Corollary 6.8 ([9, Proposition 6.10])**.** *Let* $\mathbb{C}(x)$ *be the field of rational functions over the complex numbers, equipped with the derivation* $\partial = \frac{d}{dx}$ *and the automorphism* $\sigma : f(x) \mapsto f(x+1)$*, and* M *be the field of meromorphic functions over* $\mathbb{C}$*. In the notation above, let* $L = \mathbb{C}(x)\langle A(x), B(x), \partial(A(x)), \partial(B(x))\rangle_\sigma \subset M$ *be the* σ*-Picard-Vessiot extension for the Airy equation contained in* M*. Then,* $\sigma\text{-}\mathrm{Gal}(L|\mathbb{C}(x))$ *is equal to* $\mathrm{Sl}_{2,\mathbb{C}}$ *and the functions* $A(x)$*,* $B(x)$ *and* $\partial(B(x))$ *are transformally independent over* $\mathbb{C}(x)$*.*

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