

Singular walks in the quarter plane and Bernoulli numbers

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Abstract

We consider singular (aka genus 0) walks in the quarter plane and their associated generating functions $Q(x, y, t)$, which enumerate the walks starting from the origin, of fixed endpoint (encoded by the spatial variables x and y) and of fixed length (encoded by the time variable t). We first prove that the previous series can be extended up to a universal value of t (in the sense that this holds for all singular models), namely $t = \frac{1}{2}$, and we provide a probabilistic interpretation of $Q(x, y, \frac{1}{2})$. As a second step, we refine earlier results in the literature and show that $Q(x, y, t)$ is indeed differentially transcendental for any $t \in (0, \frac{1}{2}]$. Moreover, we prove that $Q(x, y, \frac{1}{2})$ is strongly differentially transcendental. As a last step, we show that for certain models the series expansion of $Q(x, y, \frac{1}{2})$ is directly related to Bernoulli numbers. This provides a second proof of its strong differential transcendence.

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1 Introduction

In combinatorics, scholars organize enumerative data as coefficients of formal power series, usually called *ordinary generating functions*, OGF, for short. In the impossibility of an exact formula for these coefficients, one usually studies their asymptotics and the singularities of their OGFs, or tries to establish whether this latter series is solution of a certain type of functional equations. It is particularly useful to understand whether an OGF is *D-finite*, i.e., solution of a linear differential equation with rational functions coefficients: in this case the coefficients of the OGF are P-recursive, i.e., they satisfy particular linear recurrence relations. The lesson of the applications of Galois theory of functional equations to combinatorics is that it is frequently easier to prove that an OGF *is not* a solution to a non-linear differential equation with rational function coefficients, rather than

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to prove that it is not D-finite, although the latter condition is weaker. It has indeed been proven that many OGFs are not only non-D-finite, but even *differentially transcendental*, i.e., they are not solutions of any non-linear differential equation (we will be more precise soon). In 2003, Klazar [Kla03] proved that the OGF of Bell numbers (which count the number of partitions of a set of given cardinality) is in fact differentially transcendental over any field of meromorphic functions in a neighborhood of 0. In a previous work [BDVR24], we proved that Klazar’s result is actually an instance of a rather common phenomenon and we called such OGFs *strongly differentially transcendental*. While mathematical intuition says that being strongly differentially transcendental is stronger than being merely differentially transcendental, it is not immediately clear what kind of information is encoded in such a property.

In this paper we consider the classical models of *singular walks* in the quarter plane (to be introduced below) and show, extending the results in [DHRS20], that for all specializations of the length-parameter t in $(0, 1/2)$ we obtain the usual differential transcendence of the associated generating function $Q(x, y, t)$, while for $t = 1/2$ we obtain a strongly differentially transcendental series. Using a probabilistic interpretation based on Green functions of random walks, we prove that the point $t = 1/2$ is indeed very special; heuristically, this connects a gap in the various kinds of differential transcendence to a critical probabilistic phenomenon in the underlying killed random walk.

State of the art. We consider the classical models of quarter plane walks in combinatorics: Given a set $\mathcal{S}$ of prescribed steps (see examples in Table 1), describing the possible moves of the walk, one of the main objectives is to enumerate, either exactly or asymptotically, the number of walks starting at the origin $(0, 0)$, staying in the quarter plane $\mathbb{N}^2 = \{0, 1, 2, \dots\}^2$, with jumps in the step set $\mathcal{S}$, of length n , and ending at the point (i, j) ; see Figure 1 for an example of path. Denote by $\#_{\mathcal{S}}\{(0, 0) \xrightarrow{n} (i, j)\}$ the number of such walks. The associated generating function is

$$Q_{\mathcal{S}}(x, y, t) = \sum_{i, j, n \geq 0} \#_{\mathcal{S}}\{(0, 0) \xrightarrow{n} (i, j)\} x^i y^j t^n \in \mathbb{N}[x, y][[t]] \subset \mathbb{Z}[[x, y, t]]. \quad (1.1)$$

Over the past two decades, these models have attracted considerable attention from the mathematical community. The main questions are: First, is it possible to solve this enumeration problem in closed form? If not, asymptotically (e.g., when the path length n tends to infinity)? What is the “complexity” of the generating function (1.1), as a function of the step set $\mathcal{S}$? In other words, can the function $Q_{\mathcal{S}}(x, y, t)$ be rational, algebraic, D-finite or differentially algebraic?

Although these questions can be addressed *a priori* for any step set $\mathcal{S}$, it has been shown by Bousquet-Mélou and Mishna in [BMM10] and in subsequent works that the class of nearest neighbor walks already gives rise to an impressive variety of possible behaviors.



Table 1: The five main models considered in this work, called singular models. Among them, the first three $\mathcal{A} = \{\text{NW}, \text{NE}, \text{SE}\}$, $\mathcal{B} = \{\text{NW}, \text{N}, \text{E}, \text{SE}\}$, $\mathcal{C} = \{\text{NW}, \text{N}, \text{NE}, \text{E}, \text{SE}\}$ are symmetric models with respect to the first diagonal, while the last two $\mathcal{D} = \{\text{NW}, \text{N}, \text{SE}\}$, $\mathcal{E} = \{\text{NW}, \text{N}, \text{NE}, \text{SE}\}$ are not.

According to [BMM10], the set of nearest neighbor walks can be divided into two subclasses, the *singular* and the *non-singular* models. By definition, singular models are those for which the step

set $\mathcal{S}$ is contained in a linear half-plane. Bousquet-Mélou and Mishna proved that there are exactly five singular models that are truly two-dimensional, not equivalent, and can cross the quarter plane on both axes. They are here called $\mathcal{A}$ - $\mathcal{E}$ (following the convention of [MM14]), and they are shown in Table 1. Note that, unlike models $\mathcal{D}$ and $\mathcal{E}$, the step sets of models $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are symmetric.

These models are sometimes called *directed* (see [MR09]), in the sense that they cannot return to the origin once they have left it; see Figure 1 for an example of a path, see also Figure 2. Singular models are also called *genus zero models* (see [FIM17, Chap. 6]) for the following reason: in all five cases appearing on Table 1, defining the characteristic (generating/inventory) polynomial

$$\chi_{\mathcal{S}}(x, y) := \sum_{(i,j) \in \mathcal{S}} x^i y^j, \quad (1.2)$$

the kernel polynomial

$$K_{\mathcal{S}}(x, y, t) := xy(1 - t \cdot \chi_{\mathcal{S}}(x, y)) \quad (1.3)$$

is biquadratic in x and y , irreducible in $\mathbb{C}(t)[x, y]$, which defines a pencil of algebraic curves of genus zero¹ for generic t . In what follows, most of the time, $\mathcal{S}$ will be taken as one of the five step sets listed in Table 1. When no ambiguity can arise, we will simplify our notation by removing the step set $\mathcal{S}$ and the variable t ; therefore $K(x, y)$ and $Q(x, y)$ will stand for $K_{\mathcal{S}}(x, y, t)$ and $Q_{\mathcal{S}}(x, y, t)$, respectively.

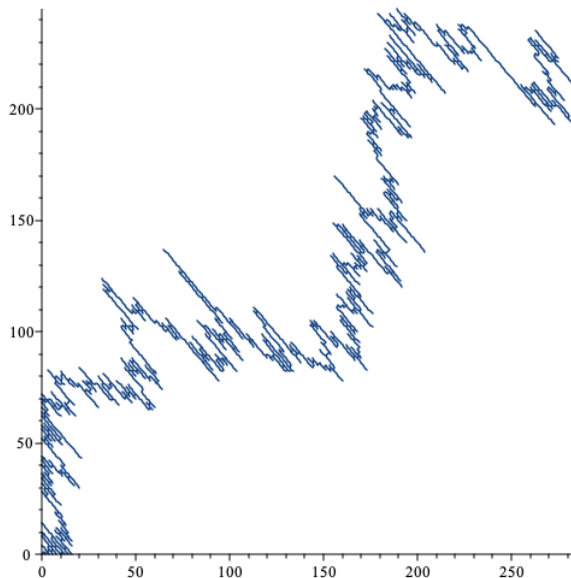


Figure 1: An example of path for the model $\mathcal{A}$ (10,000 steps). According to the classical law of large numbers, almost every path tends to infinity along the first diagonal, with antidiagonal “Brownian” fluctuations.

In the probabilistic setting (i.e., when one specializes $t = \frac{1}{\#\mathcal{S}}$, where $\#\mathcal{S}$ denotes the cardinality of the step set $\mathcal{S}$), singular models were first identified in the book [FIM17, Chap. 6]. More precisely, genus zero models are characterized in [FIM17, Thm 6.1.1 and §6.4], where various

¹This can be seen graphically using an old theorem of Henry Frederick Baker [Bak93] (see also [Bee09]), which says that the genus of a plane curve defined by a bivariate polynomial is upper bounded by the number of grid points inside the Newton polygon of the polynomial. In our case, the corresponding Newton polygons do not contain any grid points.

properties of singular models are derived, such as functional equations and explicit expressions for the generating functions, boundary value problems, uniformizations of the associated kernel curve (1.3), etc. See [DHRS21, FI21] for further insights into possible uniformizations.

In the combinatorial literature, singular models first appeared in Bousquet-Mélou and Mishna's classification [BMM10] of walks with small steps in the quarter plane. They proved that all singular models admit an infinite group (in a sense, this group captures the symmetries of the step set $\mathcal{S}$). See also [Mis09] for a classification of quarter plane walks when the step set has cardinality 3.

In [MR09], Mishna and Reznitzky consider the models $\mathcal{A}$ and $\mathcal{D}$ and express the generating functions $Q(x, 0, t)$, $Q(0, y, t)$ and $Q(x, y, t)$ as infinite sums of algebraic functions. As a consequence, they prove that $Q(1, 1, t)$ is not D-finite, i.e., it does not satisfy any linear differential equation in $\frac{d}{dt}$ with polynomial coefficients in t . The main argument is to show that $Q(1, 1, t)$ has an infinite number of singularities, which is classically incompatible with D-finiteness. Melczer and Mishna extended these results to the models $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{E}$ [MM14].

Singular models are further considered in the work [DHRS20] by Dreyfus, Hardouin, Roques and Singer. The authors show that if $0 < t < \frac{1}{\#\mathcal{S}}$ is a *transcendental* number, then $Q(x, y, t)$ is x - and y -differentially transcendental. As in the probabilistic literature, weights are allowed on the steps in their article [DHRS20]. The main technical novelty in [DHRS20] is the use of the Galois theory of q -difference equations, following their work [DHRS18] in the genus-one setting. Note that in [DHRS20] the weights are normalized so that their sum is 1 (thus they define transition probabilities); comparing their results with ours therefore requires changing the variable $t \mapsto \frac{t}{\#\mathcal{S}}$, in other words their assumption $t \in (0, 1)$ is equivalent to $t \in (0, \frac{1}{\#\mathcal{S}})$ in our notation.

Finally, returning to the probabilistic literature, Janse van Rensburg, Prellberg and Reznitzky [JvRPR08] consider self-avoiding singular walks in the quarter plane, compute their growth constants and prove functional equations for the associated generating functions. In [Poz11] Poznanović obtains an interesting bijection to prove combinatorially some results in [JvRPR08]. In [HRT23], Hoang, Tarrago and the last author of this article compute positive harmonic functions with Dirichlet boundary conditions for singular walks (with possible large jumps). This last paper also contains asymptotic estimates of the Green function and the description of the Martin boundary.

Our contributions and techniques. In our first result we prove that the function $Q(x, y)$ converges for $|x|, |y| < 1$ and $|t| < \frac{1}{2}$ and that it can be defined as a series in $\mathbb{R}[[x, y]]$ up to $t = \pm \frac{1}{2}$. Furthermore, at $t = \frac{1}{2}$, its coefficients are all rational.

Theorem 1 (See Theorem 4 below). *For any model in Table 1, the power series*

$$Q(x, y, \tfrac{1}{2}) = \sum_{i, j, n \geq 0} \frac{\#\mathcal{S}\{(0, 0) \xrightarrow{n} (i, j)\}}{2^n} x^i y^j \in \mathbb{Q}[[x, y]]$$

is well defined, meaning that all its coefficients are finite and define rational numbers.

In order to prove the above theorem, we use, inspired by [MR09], mainly a combinatorial approach to recursively compute the number of paths $\#\mathcal{S}\{(0, 0) \xrightarrow{n} (i, j)\}$. Obviously, any path of a singular model can be decomposed as a sequence of walks on antidiagonal segments

$$S_k = \{(i, j) \in \mathbb{Z}^2 : i \geq 0, j \geq 0, i + j = k\} = \{(k, 0), (k-1, 1), \dots, (0, k)\}, \quad (1.4)$$

followed by diagonal jumps, see Figure 2, and also Figure 1. This decomposition can be used to derive a fairly simple expression for the number of walks, using basic linear algebra. See Section 2.

In Section 3, we shall use that the generating function $Q(x, y) \equiv Q_{\mathcal{S}}(x, y, t)$ in (1.1) satisfies the functional equation

$$K(x, y)Q(x, y) = xy - tx^2Q(x, 0) - ty^2Q(0, y), \quad (1.5)$$

with the kernel $K(x, y)$ as in (1.3). By manipulating this equation as in [FIM17] and [DHRS20], but with slightly different parametrizations, we get a new functional equation satisfied by the *section* $Q(x, 0)$. This new equation is a q -difference equation for $t \in (0, \frac{1}{2})$ and a finite difference equation for $t = \frac{1}{2}$. Using the Ishizaki-Ogawara theorem (which we recall in Theorem 19) and Theorem 20 we conclude (for the precise definitions see Section 3):

Theorem 2. *For any model $\mathcal{S}$ in the list $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}$ and for any $t \in (0, \frac{1}{2}]$ the formal power series $Q(x, 0) \equiv Q_{\mathcal{S}}(x, 0, t)$ (resp. $Q(0, y), Q(x, y)$) is differentially transcendental over $\mathbb{C}(x)$ (resp. $\mathbb{C}(y), \mathbb{C}(x, y)$). For $t = \frac{1}{2}$, $Q(x, 0)$ (resp. $Q(0, y), Q(x, y)$) is strongly differentially transcendental.*

This result extends the main theorem of [DHRS20], which is only proven for transcendental values of the variable t , and only for $t \in (0, \frac{1}{\#S})$. Note that the paper [DHRS20] considers models with weights. We recover their result in full generality only for the models with step sets of cardinality 3 (namely $\mathcal{A}$ and $\mathcal{D}$), see Section 4.4, as we have three degrees of freedom in our parameters.

Let us give some heuristics about the change in behavior at $t = \frac{1}{2}$ implied by Theorem 2. First, one could argue that if a q -difference equation associated with a homography with two fixed points degenerates into a finite difference equation, whose operator has only one fixed point, the confluence of two special points is expected to increase the complexity of the solution. We can propose another explanation. Namely we show in Section 4 that $Q(x, y, t)$ admits a natural probabilistic as the Green function of a certain random walk, using the idea of the Cramér transform. As it turns out, the latter transform becomes singular at $t = \frac{1}{2}$, so one can also interpret the point $t = \frac{1}{2}$ as critical from this alternative, probabilistic point of view.

We conclude the paper with a small excursion in special function theory, showing that we can compute explicitly the rational coefficients appearing in $Q(x, y, \frac{1}{2})$ in the case of the model $\mathcal{A}$. In fact, *Bernoulli numbers* intriguingly show up:

Theorem 3. *For model $\mathcal{A}$ we have*

$$Q_{\mathcal{A}}(x, 0, \pm \frac{1}{2}) = \sum_{n \geq 0} \frac{T_n}{4^n} x^{2n},$$

where $(T_n)_{n \geq 0} = (1, 2, 16, 272, 7936, \dots)$ is the sequence of the tangent numbers (A000182). Equivalently,

$$Q_{\mathcal{A}}(x, 0, \frac{1}{2}) = 2 \sum_{n \geq 0} (2^{2n+2} - 1) \frac{(-1)^n}{n+1} B_{2n+2} x^{2n},$$

where $(B_n)_{n \geq 0} = (1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, \dots)$ is the sequence of Bernoulli numbers.

We derive a similar result for the models $\mathcal{B}$ and $\mathcal{D}$, in Section 5. As a direct consequence of Theorem 3, and using known results about Bernoulli numbers (e.g., recalled in [BDVR24]), the power series $Q_{\mathcal{A}}(x, 0, \frac{1}{2})$ and $Q_{\mathcal{B}}(x, 0, \frac{1}{2})$ are proved to be strongly differentially transcendental in a direct manner. This provides an alternative proof of Theorem 2 for models $\mathcal{A}, \mathcal{B}$ and $\mathcal{D}$. Moreover, we prove that exponential versions of the power series $Q_{\mathcal{A}}(x, 0, \frac{1}{2})$ and $Q_{\mathcal{D}}(x, 0, \frac{1}{2})$ are D-algebraic; this can be viewed as yet another instance of the Pak-Yeliussizov Open Problem 2.4 in [Pak18].

It would be interesting to find formulas of this kind for all the five singular models and actually establish whether they come from the confluence of some kind of q -Bernoulli numbers, appearing for $t = q/(1+q^2)$ with $q \neq 1$. Notice that the nature of $Q(1, 1, t)$ still remains out of reach with these techniques.

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2 On the convergence of $Q(x, y, t)$ with respect to t

As mentioned in the introduction, one of the original results of our paper concerns the function $Q(x, y, \frac{1}{2})$, that is the evaluation at $t = \frac{1}{2}$ of the generating function $Q(x, y, t)$ introduced in (1.1). While the trivial estimate $\#\mathcal{S}\{(0, 0) \xrightarrow{n} (i, j)\} \leq (\#\mathcal{S})^n$ implies that $Q(x, y, t)$ is well defined and analytic for $|x| \leq 1$, $|y| \leq 1$ and $|t| < \frac{1}{\#\mathcal{S}}$, it is *a priori* unclear that $Q(x, y, t)$ can be continued analytically for larger values of $|t|$. This is the object of this section, in which we will prove the following result:

Theorem 4. *For any model in Table 1, the series $Q(x, y, t)$ converges for $|x|, |y| < 1$ and $|t| < 1/2$. Moreover,*

$$Q(x, y, \tfrac{1}{2}) = \sum_{i, j, n \geq 0} \frac{\#\mathcal{S}\{(0, 0) \xrightarrow{n} (i, j)\}}{2^n} x^i y^j \in \mathbb{Q}[[x, y]]$$

is well defined, meaning that all its coefficients are finite and define rational numbers.

As explained by Mishna and Reznitzer [MR09], singular walks naturally have a recursive construction, as directed steps $(1, 0)$, $(1, 1)$, $(0, 1)$ followed by fluctuations with antidiagonal jumps $(-1, 1)$ and $(1, -1)$; see Figure 2. Reformulating some results of [MR09], we obtain in this section a matrix-product expression for the generating function $Q(x, y, t)$, which immediately leads to the following partial result (proved at the very end of this section, as a consequence of Propositions 8, 13 and 14):

Proposition 5. *For any model in Table 1, the power series $Q(x, y, \frac{1}{2})$ is well defined in $\mathbb{R}[[x, y]]$, meaning that all its coefficients are finite.*

We will complete the proof of Theorem 4 by showing that the coefficients of $Q(x, y, \frac{1}{2})$ are rational (see Proposition 15 below).

2.1 Enumeration of walks on segments

For any $k \geq 1$, define S_k as the antidiagonal segment

$$S_k = \{(i, j) \in \mathbb{Z}^2 : i \geq 0, j \geq 0, i + j = k\} = \{(k, 0), (k-1, 1), \dots, (0, k)\}, \quad (2.1)$$

see Figure 2. For $P, Q \in S_k$ and $n \in \mathbb{N}$, define $w_k(P, Q, n)$ as the number of paths in S_k starting at P , ending at Q and having length n . We further introduce the generating function

$$W_k(P, Q, t) := \sum_{n \geq 0} w_k(P, Q, n) t^n. \quad (2.2)$$

For $n \geq 2$, we consider the matrix

$$F_n = \begin{pmatrix} 1 & -t & 0 & \cdots & \cdots & 0 \\ -t & 1 & -t & 0 & \cdots & 0 \\ 0 & -t & 1 & -t & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \cdots & 0 & -t & 1 & -t \\ 0 & \cdots & \cdots & 0 & -t & 1 \end{pmatrix}^{-1} \in \mathcal{M}_{n,n}(\mathbb{R}(t)), \quad (2.3)$$

which in Lemma 7 below allows us to completely characterize the generating functions $W_k(P, Q, t)$ in (2.2). Recall that the n -th Chebychev polynomial of the second kind $U_n(z) \in \mathbb{Q}[z]$ is defined via the generating function $1/(1 - 2zt + t^2) = \sum_{n=0}^{\infty} U_n(z) t^n$, or alternatively via the equality $U_n(\cos \theta) \sin \theta = \sin((n+1)\theta)$. Then, we have the following well-known statement:

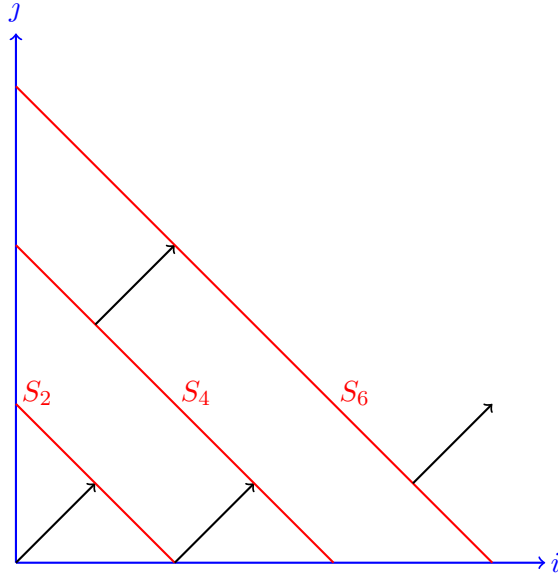


Figure 2: Recursive construction of singular walks. In this example on model $\mathcal{A}$, a walk can be decomposed as a sequence of jumps $(1, 1)$ (black arrows) and antidiagonal jumps $(-1, 1)$ and $(1, -1)$ (red). The segments S_k are defined in (2.1). See Figure 1 for an example of a longer path.

Proposition 6 ([Kam89, Thm 3.2]). *For $1 \leq i, j \leq n$ and $|t| < \frac{1}{2 \cos(\frac{\pi}{n+1})}$, let*

$$f_{i,j} = \frac{(-1)^{i+j+1}}{t} \frac{U_{n-j}(-\frac{1}{2t})U_{i-1}(-\frac{1}{2t})}{U_n(-\frac{1}{2t})} = \frac{1}{t} \frac{U_{n-j}(\frac{1}{2t})U_{i-1}(\frac{1}{2t})}{U_n(\frac{1}{2t})}.$$

Then the coefficients of F_n are the $f_{i,j}$ for $i \leq j$ and $f_{j,i}$ for $i > j$.

For an upcoming use, let us make explicit the matrix F_n at $t = \frac{1}{2}$. Since $\sum_{n=0}^{\infty} U_n(-1)t^n = 1/(1+t)^2$, we have that $U_n(-1) = (-1)^n(n+1)$, so $f_{i,j} = 2^{\frac{i(n-j+1)}{n+1}}$ and thus

$$F_n|_{t=\frac{1}{2}} = \frac{2}{n+1} \begin{pmatrix} n & n-1 & n-2 & \cdots & 2 & 1 \\ n-1 & 2(n-1) & 2(n-2) & \cdots & 4 & 2 \\ n-2 & 2(n-2) & 3(n-2) & \cdots & 6 & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2 & 4 & 6 & \cdots & 2(n-1) & n-1 \\ 1 & 2 & 3 & \cdots & n-1 & n \end{pmatrix}. \quad (2.4)$$

We now state Lemma 7. Although it is not new, we prove it briefly, as this gives us the opportunity to keep our paper self-contained and to introduce some useful notations. Lemma 7 immediately follows from the transfer-matrix method, see e.g. [Sta12, Sec. 4.7], in particular Theorems 4.7.1 and 4.7.2.

Lemma 7. *Let $k \geq 1$. For $0 \leq i \leq k$, denote by $P_i = (k-i, i)$ the points of S_k , see (2.1). The following equality between matrices of generating functions holds:*

$$(W_k(P_i, P_j, t))_{0 \leq i, j \leq k} = F_{k+1}. \quad (2.5)$$

Accordingly, for all $0 \leq i, j \leq k$, the series $W_k(P_i, P_j, t)$ is rational, with first singularity (a pole of order one) at $t = \pm(2 \cos(\frac{\pi}{2+k}))^{-1}$, with $(2 \cos(\frac{\pi}{2+k}))^{-1} > \frac{1}{2}$. In the neighborhood of $(2 \cos(\frac{\pi}{2+k}))^{-1}$ we have

$$W_k(P_i, P_j, t) = \frac{c_{i,j,k}}{(2 \cos(\frac{\pi}{k+2}))^{-1} - t} + O(1), \quad (2.6)$$

with $c_{i,j,k} > 0$. The series $W_k(P_i, P_j, t)$ being even, an expansion similar to (2.6) holds in the neighborhood of $-(2 \cos(\frac{\pi}{k+2}))^{-1}$.

Proof. We first look at the row vector of generating functions

$$(W_k(P_0, P_0, t), \dots, W_k(P_k, P_0, t))$$

with fixed ending point at P_0 and varying starting points in S_k . Using a first step decomposition of the walk, we easily obtain the identities

$$\begin{aligned} W_k(P_0, P_0, t) &= 1 + tW_k(P_1, P_0, t), \\ W_k(P_i, P_0, t) &= tW_k(P_{i-1}, P_0, t) + tW_k(P_{i+1}, P_0, t), \quad 1 \leq i \leq k-1, \\ W_k(P_k, P_0, t) &= tW_k(P_{k-1}, P_0, t). \end{aligned}$$

In terms of matrix product, this means that

$$(W_k(P_0, P_0, t), \dots, W_k(P_k, P_0, t))F_{k+1}^{-1} = (1, 0, \dots, 0),$$

the first vector of the canonical basis of $\mathbb{R}^{k+1}$. Similar computations show that

$$(W_k(P_0, P_i, t), \dots, W_k(P_k, P_i, t))F_{k+1}^{-1}$$

is the $(i+1)$ -th vector of the canonical basis of $\mathbb{R}^{k+1}$, which proves Equation (2.5), since F_{k+1} is symmetric.

We now prove the statement on the radius of convergence. For this we use the classical representation of the Chebychev polynomials as determinants of tridiagonal matrices, proving that

$$\det(F_n^{-1}) = (-t)^n U_n \left(-\frac{1}{2t} \right). \quad (2.7)$$

On the other hand, as $U_n(\cos \theta) = \frac{\sin((n+1)\theta)}{\sin \theta}$, the largest zero of $U_n(z)$ happens at $z = \cos(\frac{\pi}{n+1})$. As a consequence, the matrix F_n is defined for $|t| < \frac{1}{2 \cos(\frac{\pi}{n+1})}$, which corresponds to the statement after the change of variable $n = k+1$.

In order to prove (2.6), we perform a series expansion of U_n around its first zero $\cos(\frac{\pi}{n+1})$, namely

$$U_n(z) = \frac{2(n+1) \cos(\frac{\pi}{n+1})}{\sin(\frac{\pi}{n+1})^2} (z - \cos(\frac{\pi}{n+1})) + \dots$$

and use Proposition 6. In this way, we obtain (2.6), with the following expression for the constant

$$c_{i,j,k} = \frac{\sin(\frac{\pi}{k+2})^2 U_{k+1-j}(\cos(\frac{\pi}{k+2})) U_{i-1}(\cos(\frac{\pi}{k+2}))}{(k+2) \cos(\frac{\pi}{k+2})},$$

on which we can read the positivity (recall that $\cos(\frac{\pi}{k+2})$ is the largest root of U_{k+1} and that the roots of the Chebychev polynomials interlace, in such a way that Chebychev polynomials with index lower than $k+1$ are positive on the segment $[\cos(\frac{\pi}{k+2}), 1]$). $\square$

We apply these preliminary results to the generating functions associated to our five models, starting with model $\mathcal{A}$.

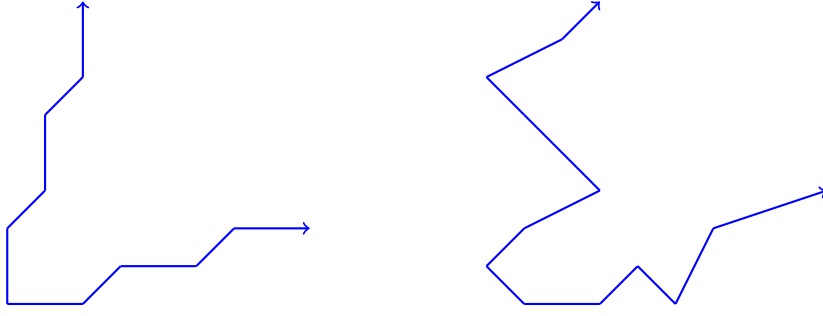


Figure 3: Proposition 8 can be adapted to apply to general domains, not only quarter planes. On the left, a domain with periodic boundary; on the right, an arbitrary domain.

2.2 Preliminary results for singular walks with step set $\mathcal{A}$

In this subsection, we focus on the model $\mathcal{A}$, the first model in Table 1. Notice that for model $\mathcal{A}$ a walk can jump from one segment S_k to another only thanks to the step $(1, 1)$: a walk starting in $(0, 0)$ can only jump to S_2 and then either stay on S_2 or jump to S_4 , and so on. This immediately implies that there are no walks ending on points of the segment S_k , when k is odd. More formally, let us fix $Q_k \in S_k = \{(k - i, i) : i = 0, \dots, k\}$ and consider the series

$$\sum_{n \geq 0} \#_{\mathcal{A}}\{(0, 0) \xrightarrow{n} Q_k\} t^n. \quad (2.8)$$

Then the series (2.8) above is identically zero for any odd value of the integer $k \geq 1$. This proves the first (trivial) part of the following proposition, which is the key-point of this subsection. To state its second part, we need to define the following sequence of matrices $(A_n)_{n \geq 3}$:

$$A_n = \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 1 & 0 \end{pmatrix} \in \mathcal{M}_{n-2, n}(\mathbb{R}). \quad (2.9)$$

Proposition 8. *If k is odd, the series (2.8) is identically zero. Moreover, for any $k \geq 1$, one has*

$$\left(\sum_{n \geq 0} \#_{\mathcal{A}}\{(0, 0) \xrightarrow{n} (2k - i, i)\} t^n \right)_{0 \leq i \leq 2k} = t^k A_3 F_3 \cdots A_{2k+1} F_{2k+1},$$

with A_{2k+1} and F_{2k+1} as in (2.9) and (2.3), respectively.

Remark 9. The expression given in Proposition 8 is robust in the following sense: if the boundary of the quarter plane were replaced by any other boundary, as in Figure 3, then a very similar matrix-product expression for the generating function would hold. This observation applies more generally for all singular step sets $\mathcal{S}$.

Proposition 8 is not new and is equivalent to a result proved in [MR09]: we will compare our formulation with the one contained in *loc. cit.* in Corollary 11 below. To prove Proposition 8 we need the following lemma:

Lemma 10. *For even values of $k \geq 1$, the generating function (2.8) is rational and its first poles are at $t = \pm(2 \cos(\frac{\pi}{2+k}))^{-1}$.*

Proof. Using the obvious decomposition of a walk from $(0, 0)$ to $Q_{2k} \in S_{2k}$ as a sequence of jumps $(1, 1)$ followed by fluctuations on the segments $S_2, \dots, S_{2k}$, see again Figure 2, one has

$$\begin{aligned} \#_{\mathcal{A}}\{(0, 0) \xrightarrow{n+k} Q_{2k}\} &= \sum_{Q_2 \in S_2, Q_4 \in S_4, \dots, Q_{2k-2} \in S_{2k-2}} \\ &\sum_{n_1 + \dots + n_k = n} w_2((1, 1), Q_2, n_1) w_4(Q_2 + (1, 1), Q_4, n_2) \cdots w_{2k}(Q_{2k-2} + (1, 1), Q_{2k}, n_k). \end{aligned}$$

As a consequence,

$$\begin{aligned} \sum_{n \geq 0} \#_{\mathcal{A}}\{(0, 0) \xrightarrow{n} Q_{2k}\} t^n &= \\ t^k \sum_{Q_2 \in S_2, Q_4 \in S_4, \dots, Q_{2k-2} \in S_{2k-2}} W_2((1, 1), Q_2, t) \cdots W_{2k}(Q_{2k-2} + (1, 1), Q_{2k}, t). \end{aligned} \quad (2.10)$$

Using now Lemma 7, it appears that we wrote the series (2.8) as a finite sum of rational functions, with explicitly given radii of convergence.

To show that the singularity of $\sum_{n \geq 0} \#_{\mathcal{A}}\{(0, 0) \xrightarrow{n} Q_{2k}\} t^n$ in (2.10) at $t = (2 \cos(\frac{\pi}{2+k}))^{-1}$ is not removable, we use Equation (2.6) from Lemma 7. More specifically, we perform an expansion of (2.10) at $t = (2 \cos(\frac{\pi}{2+k}))^{-1}$. We have

$$\sum_{n \geq 0} \#_{\mathcal{A}}\{(0, 0) \xrightarrow{n} Q_{2k}\} t^n = \frac{C}{(2 \cos(\frac{\pi}{2+k}))^{-1} - t} + O(1),$$

where C can be computed as follows, setting $t_0 = (2 \cos(\frac{\pi}{2+k}))^{-1}$:

$$C = t_0^{-k} \sum_{Q_2 \in S_2, Q_4 \in S_4, \dots, Q_{2k-2} \in S_{2k-2}} W_2((1, 1), Q_2, t_0) \cdots W_{2k-2}(Q_{2k-4} + (1, 1), Q_{2k-2}, t_0) \tilde{c}_{Q_{2k-2}, Q_{2k}, k}, \quad (2.11)$$

where $\tilde{c}_{Q_{2k-2}, Q_{2k}, k} = c_{i,j,k}$ in (2.6), with the indices i and j corresponding respectively to $P_i = Q_{2k-2} + (1, 1)$ and $P_j = Q_{2k}$. In (2.11), C appears as a sum of terms which are all positive, therefore $C > 0$. The proof is complete. $\square$

Proof of Proposition 8. Given $k \geq 1$, the matrix A_{2k+1} describes how the walk jumps from the segment S_{2k-2} to the next one, S_{2k} , and may thus be called a *jump matrix*. More precisely, in the model $\mathcal{A}$, if the walk exits the segment S_{2k-2} at the point $(2k-2-i, i)$, then it will enter S_{2k} at position $(2k-2-i, i) + (1, 1) = (2k-1-i, i+1)$; see Figure 2. From a matrix point of view, this corresponds to the following computation: given the vector $e_i^{2k-1} := (0, \dots, 0, 1, 0, \dots, 0) \in \mathcal{M}_{1, 2k-1}(\mathbb{R})$ with the 1 at position i , one has

$$e_i^{2k-1} A_{2k+1} = e_{i+1}^{2k+1}.$$

The proof of Proposition 8 is then immediate from (2.10) and using Lemma 7. $\square$

In this part we compare our results to that of Mishna and Rechnitzer [MR09]. They consider the generating function

$$D_k(y) = \sum_{i=0}^k \sum_{n \geq 0} \#_{\mathcal{A}}\{(0, 0) \xrightarrow{n} (k-i, i)\} t^n y^i \in \mathbb{Q}(t)[y] \cap \mathbb{Q}[y][[t]]. \quad (2.12)$$

It is clear that $D_0(y) = 1$ and $D_{2\ell+1}(y) = 0$ for all $\ell \geq 0$. Mishna and Rechnitzer further prove, with the notation $t = \frac{q}{1+q^2}$, that the following result holds.

Corollary 11 ([MR09, Eq. (45)]). *The generating function $D_k(y)$ satisfies the functional equation*

$$(q^{k+2} + 1)(yq - 1)(y - q)D_k(y) = q^3(y^{k+2} + 1)D_{k-2}(q) - qy^2(q^{k+2} + 1)D_{k-2}(y). \quad (2.13)$$

We now show how we can deduce Corollary 11 from our Proposition 8.

Proof of Corollary 11. We shall identify polynomials with row vectors. Therefore, the generating function D_{2k} in (2.12) is exactly the row vector appearing in the left-hand side of the main equation in Proposition 8.

We shall first rewrite (2.13) in an equivalent way, simply by dividing the left- and the right-hand sides by $q(q^{k+2} + 1)/t$, and then replacing k by $2k$. We get

$$(ty^2 - y + t)D_{2k}(y) = tq^2 \frac{y^{2k+2} + 1}{q^{2k+2} + 1} D_{2k-2}(q) - ty^2 D_{2k-2}(y). \quad (2.14)$$

Using the identification between polynomials and row vectors, (2.14) is an equality between row vectors of length $2k + 3$.

Using now Proposition 8, the left-hand side of (2.14) may be written as

$$(ty^2 - y + t)D_{2k}(y) = t^k A_3 F_3 \cdots A_{2k+1} F_{2k+1} \begin{pmatrix} t & -1 & t & \cdots & \cdots & 0 \\ 0 & t & -1 & t & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & t & -1 & t \end{pmatrix},$$

where the last matrix belongs to $\mathcal{M}_{2k+1, 2k+3}(\mathbb{R}[t])$. The product of the last three matrices simplifies remarkably:

$$(t(F_{2k+1})_{[2, 2k], 1} | 0\rangle - I_{2k-1} | 0\rangle t(F_{2k+1})_{[2, 2k], 2k+1}) \in \mathcal{M}_{2k-1, 2k+3}(\mathbb{R}(t)),$$

where we used the following notation: $(F_{2k+1})_{[2, 2k], 1}$ is a column vector of length $2k - 1$ formed by the elements $(2, 1)$ up to $(2k, 1)$ of the matrix F_{2k+1} .

The last term in (2.14) is

$$-ty^2 D_{2k-2}(y) = -t^{k-1} A_3 F_3 \cdots A_{2k-1} F_{2k-1} t(0|0|I_{2k-1}|0\rangle|0\rangle).$$

We can now state a matrix identity which is a lifting of (2.14) at the matrix level (somehow a factorization by the common prefix $t^k A_3 F_3 \cdots A_{2k-1} F_{2k-1}$):

$$A_{2k+1} F_{2k+1} \begin{pmatrix} t & -1 & t & \cdots & \cdots & 0 \\ 0 & t & -1 & t & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & t & -1 & t \end{pmatrix} + (0|0|I_{2k-1}|0\rangle|0\rangle) - qt A_{2k+1} F_{2k+1} \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & & \vdots & \vdots \\ \vdots & \vdots & & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} = 0.$$

This concludes the proof. □

2.3 Preliminary results for the models $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$

In this subsection we denote by $\mathcal{S}$ any of the following step sets: $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ or $\mathcal{E}$, thereby extending the results to all models of Table 1. As above, the number of n -length $\mathcal{S}$ -walks from $(0,0)$ to an arbitrary point $Q_k \in S_k$ is denoted by $\#_{\mathcal{S}}\{(0,0) \xrightarrow{n} Q_k\}$. We begin with a statement analogous to Lemma 10.

Lemma 12. *If $\mathcal{S}$ is not $\mathcal{A}$, then for any $k \geq 1$ and any $Q_k \in S_k$, the generating function*

$$\sum_{n \geq 0} \#_{\mathcal{S}}\{(0,0) \xrightarrow{n} Q_k\} t^n \quad (2.15)$$

is rational, with first pole at $t = (2 \cos(\frac{\pi}{2+k}))^{-1}$.

The rationality property remains valid for model $\mathcal{A}$, see Lemma 10; however, due to parity issues the series (2.15) is zero when $\mathcal{S} = \mathcal{A}$ and k is odd, hence a separate statement.

Proposition 8 may be extended immediately to models $\mathcal{B}$ and $\mathcal{D}$, replacing the jump matrices A_n by B_n and D_n in $\mathcal{M}_{n-1,n}(\mathbb{R})$, as follows:

$$D_n = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix} \quad \text{and} \quad B_n = \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ 1 & 0 & 1 & \ddots & & \vdots \\ 0 & 1 & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & 1 & 0 & 1 \\ 0 & \cdots & 0 & 0 & 1 & 0 \end{pmatrix}. \quad (2.16)$$

More precisely:

Proposition 13. *For any $k \geq 1$, one has*

$$\begin{aligned} \left(\sum_{n \geq 0} \#_{\mathcal{B}}\{(0,0) \xrightarrow{n} (k-i,i)\} t^n \right)_{0 \leq i \leq k} &= t^k B_2 F_2 \cdots B_{k+1} F_{k+1}, \\ \left(\sum_{n \geq 0} \#_{\mathcal{D}}\{(0,0) \xrightarrow{n} (k-i,i)\} t^n \right)_{0 \leq i \leq k} &= t^k D_2 F_2 \cdots D_{k+1} F_{k+1}, \end{aligned}$$

with B_n , D_n and F_n as in (2.16) and (2.3), respectively.

Models $\mathcal{C}$ and $\mathcal{E}$ contain some more intrinsic complexity, due to the following: if a walk is on the segment S_k , the next step could bring it either on S_{k+1} or on S_{k+2} .

Given $k \geq 1$, we will call $\mathcal{C}_k$ the set of compositions of k with 1's and 2's, i.e., the set of all (ordered) p -tuples $(n_1, \dots, n_p)$ such that $n_i \in \{1, 2\}$ and $n_1 + \dots + n_p = k$. Though not crucial for our analysis, we observe that $\mathcal{C}_k$ is a set of cardinal given by the k -th Fibonacci number.

Let us introduce the following notation: given $n \in \{1, 2\}$ and $m \geq 1$, $J(n)_m$ denotes the following matrix:

$$J(n)_m = \begin{cases} A_m & \text{if } n = 2, \\ B_m & \text{if } n = 1 \text{ and the model is } \mathcal{C}, \\ D_m & \text{if } n = 1 \text{ and the model is } \mathcal{E}. \end{cases}$$

Proposition 14. *For the models $\mathcal{C}$ and $\mathcal{E}$, one has*

$$\begin{aligned} \left(\sum_{n \geq 0} \# \{(0,0) \xrightarrow{n} (k-i,i)\} t^n \right)_{0 \leq i \leq k} &= \\ \sum_{(n_1, \dots, n_p) \in \mathcal{C}_k} t^p J(n_1)_{n_1+1} F_{n_1+1} J(n_2)_{n_1+n_2+1} F_{n_1+n_2+1} \cdots J(n_p)_{n_1+\dots+n_p+1} F_{n_1+\dots+n_p+1}. \end{aligned}$$

2.4 End of the proof of Theorem 4

The matrix-expressions of the generating functions obtained in Propositions 8, 13 and 14 immediately entail Proposition 5, namely, that the series $Q(x, y, \frac{1}{2})$ in the variables x and y has finite coefficients:

Proof of Proposition 5. Let $\mathcal{S}$ be any step set in Table 1. We can reformulate (1.1) as

$$Q_{\mathcal{S}}(x, y, t) = \sum_{k \geq 0} \sum_{i=0}^k x^{k-i} y^i \sum_{n \geq 0} \#_{\mathcal{S}}\{(0, 0) \xrightarrow{n} (k-i, i)\} t^n.$$

Using now Propositions 8, 13 and 14, we obtain that for any $Q_k = (k-i, i) \in S_k$, the series $\sum_{n \geq 0} \#_{\mathcal{S}}\{(0, 0) \xrightarrow{n} Q_k\} t^n$ has non-negative coefficients and converges uniformly on the closed disk $|t| \leq \frac{1}{2}$, as indeed for any $k \geq 0$,

$$\frac{1}{2} < \frac{1}{2 \cos(\frac{\pi}{2+k})}.$$

Therefore the series $Q_{\mathcal{S}}(x, y, \frac{1}{2})$ is well defined in $\mathbb{R}[[x, y]]$. $\square$

We can actually say more:

Proposition 15. *For any step set $\mathcal{S}$ and any $Q_k \in S_k$, we have*

$$\sum_{n \geq 0} \frac{\#_{\mathcal{S}}\{(0, 0) \xrightarrow{n} Q_k\}}{2^n} \in \mathbb{Q}.$$

Proof. The matrix-expressions obtained in Propositions 8, 13 and 14 entail that $\sum_{n \geq 0} \#_{\mathcal{S}}\{(0, 0) \xrightarrow{n} Q_k\} t^n$ can be computed as the coefficient of a product of matrices involving the F_n in (2.3) and other matrices with integer coefficients. Being the inverse of a matrix whose coefficients belong to $\mathbb{Q}(t)$, the coefficients of F_n should be in $\mathbb{Q}(t)$ as well. As an immediate consequence, all the coefficients of F_n are rational numbers as soon as $t \in \mathbb{Q}$, in particular at $t = \frac{1}{2}$, as claimed in Proposition 15.

Notice that this statement can be made more precise, using the explicit coefficients of F_n computed in Proposition 6 for general values of t , and their simplification at $t = \frac{1}{2}$ derived in (2.4). In particular, the latter identity implies that for all $k \geq 1$ and all $P, Q \in S_k$, the quantity $W_k(P, Q, \frac{1}{2}) \in \frac{1}{k+2} \cdot \mathbb{N}$ (recall from (2.5) that $W_k(P, Q, \frac{1}{2})$ are the coefficients of F_k). $\square$

By Lemmas 10 and 12, the poles $t = (2 \cos(\frac{\pi}{2+k}))^{-1}$ of the series $\sum_{n \geq 0} \#_{\mathcal{S}}\{(0, 0) \xrightarrow{n} Q_k\} t^n$ have an accumulation point at $\frac{1}{2}$ as $k \rightarrow \infty$. Accordingly, one cannot evaluate the series $Q(x, y, t)$ at any point $t > \frac{1}{2}$.

3 On the nature of the series $Q(x, y, t)$ for $t \in (0, \frac{1}{2}]$

In this section, for any fixed step set $\mathcal{S}$ in Table 1, we explore the nature of the ordinary generating function $Q(x, y) \equiv Q_{\mathcal{S}}(x, y, t)$, for all the values of $t \in (0, \frac{1}{2}]$. As we proved in the previous section, it makes sense to evaluate t from 0 to $\frac{1}{2}$, but one cannot go further on the real line. Recall that to simplify the notation we will omit writing the dependence in $\mathcal{S}$ and t , when the context is clear.

We are going to show that the series $Q(x, y)$ is differentially transcendental over $\mathbb{Q}(x, y)$ for $t \in (0, \frac{1}{2})$ and that it is strongly transcendental for $t = \frac{1}{2}$, in the sense that it is transcendental over the germs of bivariate meromorphic functions at 0, as formalized in the following definition:

Definition 16. Let $f \in \mathbb{C}[[x]]$ (resp. $\mathbb{C}[[x, y]]$). We say that f is differentially algebraic over $\mathbb{C}(x)$ (resp. $\mathbb{C}(x, y)$) if there exists an integer n and a non-zero polynomial P with coefficients in $\mathbb{C}(x)$ (resp. $\mathbb{C}(x, y)$) in $n + 2$ (resp. $n^2 + 3$) variables such that:

$$P\left(x, f, \dots, \frac{d^n f}{dx^n}\right) = 0 \quad \text{and} \quad P\left(x, y, f, \frac{df}{dx}, \frac{df}{dy}, \dots, \frac{d^{2n} f}{dy^n dx^n}\right) = 0.$$

We say that f is differentially transcendental over $\mathbb{C}(x)$ (resp. $\mathbb{C}(x, y)$) if it is not differentially algebraic.

Mutatis mutandis, one defines strong differential transcendence, i.e., the differential transcendence of a formal power series over the field of germs of meromorphic functions $\mathbb{C}(\{x\})$ (resp. $\mathbb{C}(\{x, y\})$), that is the field of functions that are meromorphic in an open, non-specified neighborhood of 0.

The main result of this section is:

Theorem 17. For any model in the list $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}$ and for any $t \in (0, \frac{1}{2}]$, the formal power series $Q(x, 0)$ (resp. $Q(0, y)$, $Q(x, y)$) is differentially transcendental over $\mathbb{C}(x)$ (resp. $\mathbb{C}(y)$, $\mathbb{C}(x, y)$). For $t = \frac{1}{2}$, the power series $Q(x, 0)$ (resp. $Q(0, y)$, $Q(x, y)$) is strongly differentially transcendental.

Remark 18. The results of Theorem 17 extend those of in [DHRS20], where the authors consider only *transcendental* values of $t \in (0, \frac{1}{2})$. Note that they consider models with weights, whereas we do not. On the other hand, we consider strong differential transcendence, while [DHRS20] deals with differential transcendence.

We will treat the five singular models one by one, using difference Galois theory. At the end of the section, after the proof, we will comment on the fact that for $t \in (0, \frac{1}{2})$ one cannot expect to find strongly differentially transcendental generating functions.

Strategy of the proof of Theorem 17. The peculiarity of the five models of Table 1 is that the kernel curve $K(x, y, t) = 0$ has genus zero for any $t \in (0, \frac{1}{2}]$, which is equivalent to the existence of a rational parametrization. We will consider the cases $t \in (0, \frac{1}{2})$ and $t = \frac{1}{2}$ separately, since we did not find a parameterization with a non-trivial specialization for any value of t . In the proof below, it will be convenient to set $t = \frac{v}{1+v^2}$, with $v \in (0, 1]$; this substitution has proven useful in other related contexts, see e.g. [MR09, Sec. 2.4].

For $v \in (0, 1)$ we will find $x_0(s), \tilde{x}_0(s), y_0(s) \in \mathbb{C}(s)$ such that $(x_0(s), y_0(s))$ and $(\tilde{x}_0(s), y_0(s))$ are both parametrizations of $K(x, y, v/(v^2 + 1)) = 0$ and $\tilde{x}_0(s) = x_0(v^2 s)$. Plugging the two parametrizations in the functional equation (1.5) and taking the difference of the two expressions, we obtain

$$\frac{v}{v^2 + 1} \cdot (x_0^2 Q(x_0, 0) - \tilde{x}_0^2 Q(\tilde{x}_0, 0)) = (x_0 - \tilde{x}_0) y_0. \quad (3.1)$$

Therefore, as an element of $\mathbb{C}[[s]]$, the formal power series

$$G_0(s) := \frac{v}{v^2 + 1} \cdot (x_0(s))^2 \cdot Q(x_0(s), 0) \quad (3.2)$$

satisfies a v^2 -difference equation of the form $G_0(v^2 s) - G_0(s) = R(s) \in \mathbb{C}(s)$. We prove that such an equation does not have any rational solution and conclude that $G_0(s)$ is differentially transcendental over $\mathbb{C}(s)$, thanks to the following result:

Theorem 19 ([Ish98], [Oga14, Thm 2]). Let $q \neq 0$ be a complex number, not a root of unity. If $f \in \mathbb{C}((s)) \setminus \mathbb{C}(s)$ satisfies the functional equation

$$f(qs) = Q(s)f(s) + R(s), \quad \text{for some } Q(s), R(s) \in \mathbb{C}(s), Q(s) \neq 0,$$

then f is differentially transcendental over $\mathbb{C}(s)$.

Since $x_0(s) \in \mathbb{C}(s)$ for any $v \in (0, 1)$, $G_0(s)$ is differentially transcendental over $\mathbb{C}(s)$ if and only if $Q(x, 0)$ is differentially transcendental over $\mathbb{C}(x)$. Moreover, from (1.5), the differential transcendence of $Q(x, 0)$ over $\mathbb{C}(x)$ and that of $Q(0, y)$ over $\mathbb{C}(y)$ are equivalent, since:

$$x_0(s)y_0(s) - tx_0(s)^2Q(x_0(s), 0) = ty_0(s)^2Q(0, y_0(s)).$$

Finally, (1.5) allows to conclude that $Q(x, y)$ is differentially transcendental over $\mathbb{C}(x, y)$, using that x and y are algebraically independent.

Let us now turn to the case $v = 1$, for which we shall reason in the same way. We find $x_1(s), \tilde{x}_1(s), y_1(s) \in \mathbb{C}(s)$ such that $(x_1(s), y_1(s))$ and $(\tilde{x}_1(s), y_1(s))$ are both parametrizations of $K(x, y, \frac{1}{2}) = 0$ and $\tilde{x}_1(s) = x_1(\tau(s))$, where τ is a homography with a single fixed point. Therefore, the formal power series

$$G_1(s) := \frac{1}{2} \cdot (x_1(s))^2 \cdot Q(x_1(s), 0, \frac{1}{2}) \quad (3.3)$$

satisfies a difference equation of the form $G_1(\tau(s)) - G_1(s) = R(s) \in \mathbb{C}(s)$. We prove that such an equation does not have rational solutions and conclude, as for $v \in (0, 1)$, thanks to the following result.

Theorem 20 ([BDVR24, Thm 34 and Cor. 35]). *Let τ be an automorphism of $\mathbb{C}(s)$, having 0 as unique fixed point. If $G(s) \in \mathbb{C}((s)) \setminus \mathbb{C}(s)$ satisfies a functional equation of the form*

$$\tau(G(s)) = Q(s)G(s) + R(s), \quad \text{for some } Q(s), R(s) \in \mathbb{C}(s), Q(s) \neq 0,$$

then $G(s)$ is differentially transcendental over the field $\mathbb{C}(\{s\})$ of germs of meromorphic functions at 0.

As in the previous case, one can recover the information on $Q(x, 0)$, $Q(0, y)$ and $Q(x, y)$ from the strongly differential transcendence of $G(s)$. $\square$

We will now detail the proof of the above theorem for the five genus-zero models, one by one. Model $\mathcal{A}$ is a bit easier than the others. Then we consider models $\mathcal{B}$ and $\mathcal{D}$, which are essentially similar. Finally we look at models $\mathcal{C}$ and $\mathcal{E}$, which are a bit more complicated.

Proof of Theorem 17 in the case of model $\mathcal{A}$. In this case, for $v \in (0, 1)$, we have:

$$\begin{aligned} K(x, y) &= xy - t(y^2 + x^2y^2 + x^2); \\ x_0(s) &= \frac{(1-v^2)s}{v(s^2+1)} \quad \text{and} \quad y_0(s) = \frac{(1-v^2)s}{v^2s^2+1}; \\ \tilde{x}_0(s) &= \frac{(1-v^2)vs}{v^4s^2+1}, \quad \text{with } \tilde{x}_0(s) = x_0(v^2s). \end{aligned} \quad (3.4)$$

Therefore, the formal power series $G_0(s)$ in (3.2) satisfies the v^2 -difference equation:

$$\begin{aligned} G_0(v^2s) - G_0(s) &= \frac{(v^2-1)^3}{v} \frac{s^2(v^2s^2-1)}{(s^2+1)(v^4s^2+1)(v^2s^2+1)} \\ &= \frac{(v^2-1)}{v} \left(\frac{1}{s^2+1} - \frac{2}{v^2s^2+1} + \frac{1}{v^4s^2+1} \right). \end{aligned}$$

By Theorems 19 and 20, we only need to prove that, for any $v \in (0, 1)$, the power series $G_0(s)$ is not rational. We set $\tilde{G}_0(s) = \frac{v}{2(v^2-1)}G_0(s) - \frac{1}{2(s^2+1)}$, so that

$$\tilde{G}_0(v^2s) - \tilde{G}_0(s) = \frac{1}{s^2+1} - \frac{1}{v^2s^2+1}. \quad (3.5)$$

Of course, $G_0(s)$ is rational if and only if $\tilde{G}_0(s)$ is rational. Let us suppose by contradiction that (3.5) has a rational solution $\tilde{G}_0 \in \mathbb{C}(s)$. If $\tilde{G}_0 \in \mathbb{C}(s)$ then the left-hand side of (3.5) would have at least two poles whose quotient is in $v^{2\mathbb{Z}}$. The poles of the right-hand side of (3.5) are $\pm i, \pm i/v$. If the quotient of two of them is in $v^{2\mathbb{Z}}$, then v is a root of unity. Since $v \in (0, 1)$, we have obtained a contradiction and therefore we can conclude that $G_0(s)$ is differentially transcendental over $\mathbb{C}(s)$.

Now we consider the case $t = \frac{1}{2}$, or equivalently the case $v = 1$. For such value of v the parametrization (3.4) degenerates, however the kernel curve becomes reducible:

$$xy - \frac{1}{2}(x^2 + y^2 + x^2y^2) = \frac{1}{2}(ix - iy + xy)(ix - iy - xy).$$

The first factor provides the parametrization $(x, y(x) := \frac{x}{1+ix})$ of the kernel curve. Notice that the diagonal symmetry of model $\mathcal{A}$ implies that $Q(x, 0) = Q(0, x)$, therefore we deduce from (1.5) that $G_1(x)$ in (3.3) satisfies the functional equation:

$$G_1(\tau(x)) = -G_1(x) + \frac{x^2}{1+ix}, \quad \text{with } \tau(x) = \frac{x}{1+ix}. \quad (3.6)$$

As above, we need to prove that $G_1(x)$ cannot be rational. The inhomogeneous term $\frac{x^2}{1+ix}$ in (3.6) has a single pole, which is not a fixed point for $x \mapsto \frac{x}{1+ix}$. As a consequence, if $G_1(x)$ was rational, $G_1\left(\frac{x}{1+ix}\right) - G_1(x)$ would have at least two poles in its τ -orbit. We have obtained a contradiction, hence $G_1(x)$ is not rational. $\square$

Proof of Theorem 17 in the case of model $\mathcal{B}$. We have:

$$\begin{aligned} K(x, y) &= xy - t(y^2 + xy^2 + x^2y + x^2); \\ x_0(s) &= -\frac{(v+1)(1-v)^2s}{v(s-1)(vs-1)} \quad \text{and} \quad y_0(s) = -\frac{(v+1)(1-v)^2s}{(sv^2-1)(vs-1)}; \\ \tilde{x}_0(s) &= -\frac{vs(v+1)(1-v)^2}{(sv^3-1)(sv^2-1)}, \quad \text{with } \tilde{x}_0(s) = x_0(v^2s). \end{aligned} \quad (3.7)$$

Therefore, we deduce from (3.1) that the formal power series $G_0(s)$ in (3.2) satisfies the v^2 -difference equation:

$$\begin{aligned} G_0(s) - G_0(v^2s) &= \frac{-v^4 - v^3 + 2v^2 + v - 1}{(vs-1)v^2} + \frac{v^2-1}{(s-1)v} \\ &\quad + \frac{v^2-1}{(sv^3-1)v} + \frac{-v^4 + 2v^2 - 1}{(vs-1)^2v^2} + \frac{v^4 - 2v^2 + 1}{(sv^2-1)^2v^2} + \frac{v^4 - v^3 - 2v^2 + v + 1}{(sv^2-1)v^2}. \end{aligned}$$

If we set

$$\tilde{G}_0 := G_0 + \frac{v^4 - 2v^2 + 1}{(s-1)^2v^2} + \frac{v^4 - v^3 - 2v^2 + v + 1}{(s-1)v^2} + \frac{v^2-1}{(vs-1)v},$$

then the functional equation becomes

$$\begin{aligned} \tilde{G}_0(s) - \tilde{G}_0(v^2s) &= \frac{-v^4 - v^3 + 2v^2 + v - 1}{(vs-1)v^2} + \frac{v^2-1}{(s-1)v} \\ &\quad + \frac{-v^4 + 2v^2 - 1}{(vs-1)^2v^2} + \frac{v^2-1}{(vs-1)v} + \frac{v^4 - 2v^2 + 1}{(s-1)^2v^2} + \frac{v^4 - v^3 - 2v^2 + v + 1}{(s-1)v^2}. \end{aligned}$$

For any $v \in (0, 1)$, the right-hand side of the equation has only one pole per v^2 -orbit, hence the functional equation has no rational solution.

For $v = 1$, a parametrization of $K(x, y, \frac{1}{2})$ is given by:

$$x_1(s) = -\frac{s^2}{s+2} \quad \text{and} \quad y_1(s) = -\frac{s^2}{(s+1)(s+2)}. \quad (3.8)$$

Moreover $\tilde{x}_1(s) := -\frac{s^2}{(s+1)(3s+2)}$ and $\tilde{x}_1(s) = x_1\left(\frac{s}{s+1}\right)$. Therefore, the formal power series $G_1(s)$ in (3.3) satisfies the difference equation:

$$G_1(s) - G_1\left(\frac{s}{s+1}\right) = s - \frac{16}{3} - \frac{16}{(s+2)^2} - \frac{4}{3(3s+2)} + \frac{20}{s+2} + \frac{1}{(s+1)^2} - \frac{1}{s+1}.$$

If we set $\tilde{G}_1(s) = G_1(s) + \frac{16}{(s+2)^2} - \frac{20}{s+2}$, then

$$\tilde{G}_1(s) - \tilde{G}_1\left(\frac{s}{s+1}\right) = s - \frac{4}{9} + \frac{1}{(s+1)^2} + \frac{16}{9(3s+2)} - \frac{16}{9(3s+2)^2} - \frac{1}{s+1}.$$

One verifies that the iterates of $1/(s+1)$ and of $1/(3s+2)$ with respect to the homography $s \mapsto \frac{s}{s+1}$ are respectively:

$$\left\{ \frac{ns+1}{(n+1)s+1}, n \geq 0 \right\} \quad \text{and} \quad \left\{ \frac{ns+1}{(2n+3)s+2}, n \geq 0 \right\},$$

which implies that the poles of the right-hand side of the equation above are not on the same orbit, and hence that G cannot be a rational function. $\square$

Proof of Theorem 17 in the case of model $\mathcal{D}$. We have:

$$\begin{aligned} K(x, y) &= xy - t(y^2 + xy^2 + x^2); \\ x_0(s) &= \frac{s(v-1)^2(v+1)^2}{v^2(s+1)^2} \quad \text{and} \quad y_0(s) = \frac{s(v-1)^2(v+1)^2}{v(s+1)(sv^2+1)}; \\ \tilde{x}_0(s) &= \frac{s(v-1)^2(v+1)^2}{(sv^2+1)^2}, \quad \text{with } \tilde{x}_0(s) = x_0(v^2s). \end{aligned} \quad (3.9)$$

Therefore, we deduce from (3.1) that the formal power series $G_0(s)$ in (3.2) satisfies the v^2 -difference equation:

$$\begin{aligned} G_0(s) - G_0(v^2s) &= \frac{-v^6 + 3v^4 - 3v^2 + 1}{(s+1)^3 v^3} + \frac{v^4 - 2v^2 + 1}{(s+1)v^3} + \frac{v^6 - 4v^4 + 5v^2 - 2}{(s+1)^2 v^3} \\ &\quad + \frac{-v^4 + 2v^2 - 1}{(sv^2+1)v} + \frac{-v^6 + 3v^4 - 3v^2 + 1}{(sv^2+1)^3 v^3} + \frac{2v^6 - 5v^4 + 4v^2 - 1}{(sv^2+1)^2 v^3}. \end{aligned}$$

If we set

$$\tilde{G}_0 := G_0 - \frac{-v^6 + 3v^4 - 3v^2 + 1}{(s+1)^3 v^3} - \frac{v^4 - 2v^2 + 1}{(s+1)v^3} - \frac{v^6 - 4v^4 + 5v^2 - 2}{(s+1)^2 v^3},$$

then the functional equation becomes

$$\tilde{G}_0(s) - \tilde{G}_0(v^2s) = \frac{-v^6 + 3v^4 - 3v^2 + 1}{(sv^2+1)v^3} + \frac{-2v^6 + 6v^4 - 6v^2 + 2}{(sv^2+1)^3 v^3} + \frac{3v^6 - 9v^4 + 9v^2 - 3}{(sv^2+1)^2 v^3}.$$

For any $v \in (0, 1)$, the right-hand side of the equation has only one pole per v^2 -orbit, hence the functional equation has no rational solution.

For $v = 1$, $K(x, y, \frac{1}{2}) = 0$ can be parameterized by:

$$x_1(s) = -\frac{4s^2}{(s-1)^2} \quad \text{and} \quad y_1(s) = \frac{4s^2}{(s+1)(s-1)}.$$

Moreover $\tilde{x}_1(s) = -\frac{4s^2}{(s+1)^2} = x_1\left(\frac{s}{2s+1}\right)$. Therefore, the formal power series $G_1(s)$ in (3.3) satisfies the difference equation:

$$G_1(s) - G_1\left(\frac{s}{2s+1}\right) = -\frac{8}{(1+s)^3} + \frac{8}{(1-s)^3} + \frac{28}{(1+s)^2} - \frac{28}{(1-s)^2} - \frac{32}{1+s} + \frac{32}{1-s}.$$

If we set $\tilde{G}_1(s) = G_1(s) + \frac{8}{(s-1)^3} + \frac{28}{(s-1)^2} + \frac{32}{s-1}$, then

$$\tilde{G}_1(s) - \tilde{G}_1\left(\frac{s}{2s+1}\right) = 16 - \frac{16}{(s+1)^3} + \frac{48}{(s+1)^2} - \frac{48}{s+1}.$$

Since the right-hand side has only one pole, $\tilde{G}_1$ and G_1 cannot be rational functions. $\square$

Proof of Theorem 17 in the case of model C. The data of this model are:

$$\begin{aligned} K(x, y) &= xy - t(y^2 + xy^2 + x^2y^2 + x^2y + x^2); \\ x_0(s) &= \frac{(v+1)(v-1)^2s}{(s^2 + (v+1)s + v^2 - v + 1)v} \quad \text{and} \quad y_0(s) = \frac{(v+1)(v-1)^2s}{v^2s^2 + (v^2 + v)s + v^2 - v + 1}; \\ \tilde{x}_0(s) &= \frac{vs(v+1)(v-1)^2}{v^4s^2 + (v^3 + v^2)s + v^2 - v + 1}, \quad \text{with } \tilde{x}_0(s) = x_0(v^2s). \end{aligned} \quad (3.10)$$

As in the previous cases, we find the functional equation:

$$\begin{aligned} G_0(s) - G_0(v^2s) &= \frac{sv^5 - sv^3 + v^4 - v^3 + v - 1}{(s^2v^4 + sv^3 + v^2s + v^2 - v + 1)v} \\ &\quad + \frac{-sv^4 - sv^3 - 2v^4 + v^2s + 2v^3 + vs - 2v + 2}{(s^2v^2 + v^2s + vs + v^2 - v + 1)v} + \frac{v^4 + v^2s - v^3 - s + v - 1}{(s^2 + vs + v^2 + s - v + 1)v}. \end{aligned}$$

If we set $\tilde{G}_0(s) = G_0(s) - \frac{v^4 + v^2s - v^3 - s + v - 1}{(s^2 + vs + v^2 + s - v + 1)v}$, then we obtain the functional equation:

$$\tilde{G}_0(s) - \tilde{G}_0(v^2s) = r(s) - r(vs), \quad \text{where } r(s) = \frac{-sv^4 - sv^3 - 2v^4 + v^2s + 2v^3 + vs - 2v + 2}{(s^2v^2 + v^2s + vs + v^2 - v + 1)v}.$$

To conclude it is enough to check that the denominator $p(s) := s^2v^2 + v^2s + vs + v^2 - v + 1$ of $r(s)$ has the property that $p(s)$ and $p(v^{2n+1}s)$ have no common roots, for any $v \in (0, 1)$ and $n \in \mathbb{Z}$. Let us introduce a parameter a and calculate the resultant of $p(s)$ and $p(v \cdot a \cdot s)$ with respect to s :

$$v^8 (v^2 - v + 1) (av - 1)^2 (a^2v^4 - a^2v^3 + a^2v^2 + av^3 - 4av^2 + av + v^2 - v + 1).$$

The polynomial above has an obvious double root $a = v^{-1}$ and vanishes at the following values of a : $\frac{i\sqrt{3}v^2 - i\sqrt{3} - v^2 + 4v - 1}{2(v^2 - v + 1)v}$ and $-\frac{i\sqrt{3}v^2 - i\sqrt{3} + v^2 - 4v + 1}{2(v^2 - v + 1)v}$. We need to prove that there is no $v \in (0, 1)$ and no $n \in \mathbb{Z}$ such that one of the latter is equal to v^{2n} . For the first root, suppose by contradiction that

$$\frac{i\sqrt{3}v^2 - i\sqrt{3} - v^2 + 4v - 1}{2(v^2 - v + 1)v} = v^{2n}.$$

This is equivalent to say that for $v \in (0, 1)$, we have $i\sqrt{3}(v^2 - 1) = 2v^{2n+1}(v^2 - v + 1) + v^2 - 4v + 1$, which cannot happen because we have a non-zero purely imaginary number of the left-hand side and a real number on the right-hand side.

For $v = 1$, the kernel curve defined by $xy - \frac{1}{2}(y^2 + xy^2 + x^2y^2 + x^2y + x^2) = 0$ is parameterized by:

$$x_1(s) = \frac{-s^2}{\frac{s^2}{2} - s + 2} \quad \text{and} \quad y_1(s) = \frac{-s^2}{\frac{s^2}{2} + s + 2}. \quad (3.11)$$

We have $\tilde{x}_1(s) = \frac{-s^2}{\frac{3}{2}s^2 + 3s + 2}$, so that $\tilde{x}_1(s) = x_1\left(\frac{s}{1+s}\right)$. Therefore, the formal power series $G_1(s)$ in (3.3) satisfies the difference equation:

$$G_1(s) - G_1\left(\frac{s}{s+1}\right) = \frac{8}{3} + \frac{4(3s+4)}{3(3s^2+6s+4)} - \frac{8}{s^2-2s+4} + \frac{4(s-2)}{s^2+2s+4}.$$

We need to prove that this functional equation has no rational function solutions. If we set $\tilde{s} = 1/s$ then we obtain:

$$G_1(\tilde{s}) - G_1(\tilde{s}+1) = -\frac{4(\tilde{s}+1)}{4\tilde{s}^2+6\tilde{s}+3} - \frac{2(2\tilde{s}-1)}{4\tilde{s}^2-2\tilde{s}+1} + \frac{2(4\tilde{s}+1)}{4\tilde{s}^2+2\tilde{s}+1}.$$

We define $\tilde{G}_1(\tilde{s}) = G_1(\tilde{s}) + (2(2\tilde{s}-1))/(4\tilde{s}^2-2\tilde{s}+1)$, so that:

$$\tilde{G}_1(\tilde{s}) - \tilde{G}_1(\tilde{s}+1) = \frac{2(4\tilde{s}+1)}{4\tilde{s}^2+2\tilde{s}+1} - \frac{2(4\tilde{s}+3)}{4\tilde{s}^2+6\tilde{s}+3}.$$

The poles of the denominator of the right-hand side are

$$-\frac{3}{4} + \frac{i\sqrt{3}}{4}, \quad -\frac{3}{4} - \frac{i\sqrt{3}}{4}, \quad -\frac{1}{4} + \frac{i\sqrt{3}}{4}, \quad -\frac{1}{4} - \frac{i\sqrt{3}}{4},$$

Since any pair of them does not differ by an integer, the functional equation above cannot have a rational solution. We conclude that $G_1(s)$ is differentially transcendental over $\mathbb{C}(\{s\})$. $\square$

Proof of Theorem 17 in the case of model $\mathcal{E}$. We have

$$\begin{aligned} K(x, y) &= xy - t(y^2 + x^2y^2 + xy^2 + x^2); \\ x_0(s) &= \frac{s(v-1)^2(v+1)^2}{v(s^2+2vs+v^4-v^2+1)} \quad \text{and} \quad y_0(s) = \frac{s(v-1)^2(v+1)^2}{v^2s^2+(v^3+v)s+v^4-v^2+1}; \\ \tilde{x}_0(s) &= \frac{vs(v-1)^2(v+1)^2}{v^4s^2+2v^3s+v^4-v^2+1}, \quad \text{with } \tilde{x}_0(s) = x_0(v^2s). \end{aligned} \quad (3.12)$$

As in the previous cases, we find the functional equation:

$$\begin{aligned} G_0(s) - G_0(v^2s) &= \frac{v^6 + sv^3 - 2v^4 - vs + 2v^2 - 1}{(v^4 + s^2 + 2vs - v^2 + 1)v} \\ &\quad + \frac{-sv^5 - 2v^6 + 4v^4 + vs - 4v^2 + 2}{(s^2v^2 + sv^3 + v^4 + vs - v^2 + 1)v} + \frac{sv^5 + v^6 - sv^3 - 2v^4 + 2v^2 - 1}{(s^2v^4 + 2sv^3 + v^4 - v^2 + 1)v}. \end{aligned}$$

If we set $\tilde{G}_0(s) = G_0(s) - \frac{v^6+sv^3-2v^4-vs+2v^2-1}{(v^4+s^2+2vs-v^2+1)v}$, we obtain the functional equation:

$$\tilde{G}_0(s) - \tilde{G}_0(v^2s) = \frac{2sv^5 + 2v^6 - 2sv^3 - 4v^4 + 4v^2 - 2}{(s^2v^4 + 2sv^3 + v^4 - v^2 + 1)v} + \frac{-sv^5 - 2v^6 + 4v^4 + vs - 4v^2 + 2}{(s^2v^2 + sv^3 + v^4 + vs - v^2 + 1)v}.$$

To conclude, we need to prove that the roots of the denominator of the right-hand side have distinct v^2 -orbits, for any $v \in (0, 1)$. The roots are:

$$\alpha_1 = \frac{(v^2-1)i-v}{v^2}, \quad \alpha_2 = \frac{(1-v^2)i-v}{v^2}, \quad \alpha_3 = \frac{\sqrt{3}(v^2-1)i-(v^2+1)}{2v}, \quad \alpha_4 = \frac{\sqrt{3}(1-v^2)i-(v^2+1)}{2v}.$$

Let us suppose by contradiction that there exist $n \in \mathbb{Z}$ and $v \in (0, 1)$, such that $\alpha_1 = v^{2n} \alpha_3$, i.e., $2(v^2-1)i-2v^2 = v^{2n+1}(\sqrt{3}(v^2-1)i-(v^2+1))$. Since v is real we must have $(\sqrt{3}v^{2n+1}-2)(v^2-1) = 0$, which is impossible for any $v \in (0, 1)$. Therefore the v^2 -orbits of α_1 and α_3 are disjoint for any $v \in (0, 1)$. One can reason in the same way for all the other 5 possible quotients α_i/α_j , with $4 \geq j > i \geq 1$, to prove that $\tilde{G}_0$ (and hence G_0) cannot be a rational function, and hence that it is differentially transcendental over $C(s)$.

For $v = 1$, the kernel curve defined by $xy - \frac{1}{2}(y^2 + xy^2 + x^2y^2 + x^2) = 0$ is parameterized by:

$$x_1(s) = \frac{-s^2}{s^2+1} \quad \text{and} \quad y_1(s) = \frac{-s^2}{s^2+s+1}.$$

We have $\tilde{x}_1(s) = -\frac{s^2}{2s^2+2s+1}$, so that $\tilde{x}_1(s) = x_1\left(\frac{s}{1+s}\right)$. Therefore, the power series $G_1(s)$ in (3.3) satisfies the difference equation:

$$G_1(s) - G_1\left(\frac{s}{s+1}\right) = \frac{1}{2} + \frac{s-1}{s^2+s+1} + \frac{1}{2(2s^2+2s+1)} - \frac{s}{s^2+1}.$$

We need to prove that this functional equation has no rational solutions. If we set $\tilde{s} = 1/s$, we get

$$G_1(\tilde{s}) - G_1(\tilde{s}+1) = \frac{2\tilde{s}+1}{\tilde{s}^2+\tilde{s}+1} - \frac{\tilde{s}}{\tilde{s}^2+1} - \frac{\tilde{s}+1}{\tilde{s}^2+2\tilde{s}+2}.$$

We define $\tilde{G}_1(\tilde{s}) = G_1(\tilde{s}) + \frac{\tilde{s}}{\tilde{s}^2+1}$, so that:

$$\tilde{G}_1(\tilde{s}) - \tilde{G}_1(\tilde{s}+1) = \frac{2\tilde{s}+1}{\tilde{s}^2+\tilde{s}+1} - \frac{2(\tilde{s}+1)}{\tilde{s}^2+2\tilde{s}+2}.$$

The poles of the denominator of the right-hand side are

$$-\frac{1}{2} + \frac{i\sqrt{3}}{2}, \quad -\frac{1}{2} - \frac{i\sqrt{3}}{2}, \quad -1+i, \quad -1-i.$$

Since any pair of them does not differ by an integer, the functional equation above cannot admit a rational solution. We conclude that $G_1(s)$ is differentially transcendental over $\mathbb{C}(\{s\})$. $\square$

Remark 21. The idea of considering a parametrization as an important tool to solve probabilistic or combinatorial functional equations goes back to [FIM17]. The parametrization (3.4) of the kernel curve of model $\mathcal{A}$ and the parametrization (3.9) of model $\mathcal{D}$ coincide with the ones used in [DHRS20]. In the case of model $\mathcal{B}$, one can recover the parametrization used in [DHRS20] from parametrization (3.7) applying the variable change $s \mapsto \frac{s}{\sqrt{v}}$. The variable change $s \mapsto \frac{s}{\sqrt{v^2-v+1}}$ allows to recover the parametrization in [DHRS20] from (3.10), while the variable change $s \mapsto \frac{s}{\sqrt{v^4-v^2+1}}$ works for (3.12). So, up to minor variable changes, for $t \in (0, 1/2)$, we essentially use the same parametrization used in [DHRS20].

The situation is more articulated for $v = 1$. As far as the models $\mathcal{A}$, $\mathcal{C}$ and $\mathcal{E}$ are concerned, we use a specific parametrization valid only for $v = 1$. For models $\mathcal{B}$ and $\mathcal{D}$, a variable change allows to obtain a parametrization that specialize properly at $v = 1$. The parametrization (3.7) of

model $\mathcal{B}$ becomes trivial for $v = 1$. However, we can consider another parametrization that has a “good” specialization at $v = 1$, obtained applying a variable change:

$$x_0\left(\frac{s}{s+1-v^2}\right) = -\frac{s(s-v^2+1)}{v(s+v+1)} \quad \text{and} \quad y_0\left(\frac{s}{s+1-v^2}\right) = -\frac{s(s-v^2+1)}{(s+1)(s+v+1)}. \quad (3.13)$$

Moreover we have:

$$\tilde{x}_0\left(\frac{s}{s+1-v^2}\right) = -\frac{s(s-v^2+1)v}{(s+1)((v^2+v+1)s+v+1)},$$

so that

$$\tilde{x}_0\left(\frac{s}{s+1-v^2}\right) = x_0\left(\frac{s}{s+1-v^2} \circ \frac{v^2s}{s+1}\right).$$

Also in the case of model $\mathcal{D}$ the parametrization (3.9) becomes trivial for $v = 1$. However, we can consider another parametrization that has a “good” specialization at $v = 1$, obtained applying a variable change:

$$x_0\left(\frac{s}{v-1-vs}\right) = -\frac{(v+1)^2 s(1+(s-1)v)}{(s-1)^2 v^2} \quad \text{and} \quad y_0\left(\frac{s}{v-1-vs}\right) = \frac{(v+1)^2 s(1+(s-1)v)}{(sv+1)(s-1)v}. \quad (3.14)$$

Moreover we have:

$$\tilde{x}_0\left(\frac{s}{v-1-vs}\right) = -\frac{(v+1)^2 s(sv-v+1)}{(sv+1)^2},$$

so that

$$\tilde{x}_0\left(\frac{s}{v-1-vs}\right) = x_0\left(\frac{s}{v-1-vs} \circ \frac{sv^2}{sv(v+1)+1}\right).$$

Specializing $v = 1$ at (3.13) and (3.14), one finds (3.8) and (3.11), respectively.

We would like to discuss the dichotomy in the behavior of $Q(x, y)$ in Theorem 17, with respect to the values of t . First of all, we notice that we cannot expect solutions of q -difference equations to be strongly transcendental; in fact, solutions of q -difference equations can be (and one could even say tend to be) meromorphic in a neighborhood of 0, as the next example shows.

Example 22. For $q \in \mathbb{C}$, $|q| > 1$, the q -exponential series

$$e_q(t) := \sum_{n \geq 0} \frac{t^n}{\prod_{i=1}^n (1-q^i)}$$

is an entire function, solution of the functional equation $y(qt) = (1+t)y(t)$. One derives from the equation itself that it has a half spiral of simple zeros. Taking the logarithmic derivative with respect to $t \frac{d}{dt}$, one proves that the meromorphic function $\epsilon_q(t) := t \frac{e_q(t)'}{e_q(t)}$ over $\mathbb{C}$ is solution of:

$$y(qt) = y(t) + \frac{t}{1+t}.$$

As we have shown in the proofs above, for $t \in (0, \frac{1}{2})$ we find a q -difference equation for the generating function (after replacing the variable by the uniformizing variable), i.e., a functional equation with respect to a homography with two fixed points. For $t = \frac{1}{2}$, the two fixed points coincide, and we get a functional equation with respect to a homography with one fixed point. One can heuristically argue that the confluence of two special points to one causes an increase in the complexity of the solutions, i.e., the fact of jumping from a usual differential transcendence to a strong differential transcendence. In the next section we give a probabilistic interpretation of $Q(x, y)$ for any $t \in (0, \frac{1}{2}]$, which makes the point $\frac{1}{2}$ appear to be critical from a probabilistic point of view as well. This corroborates the change in behavior at this point.

4 Probabilistic interpretation using Green functions

As explained in our introduction, one of the main messages of the present work is that for all the models listed in Table 1, the point $t = \frac{1}{2}$ is very special for the generating function $Q(x, y, t)$, in particular because the differential transcendence degenerates to a strong differential transcendence. One can also make the following observations to reinforce the idea that something special happens for $t = 1/2$:

- At $t = \frac{1}{2}$ the kernel $K(x, y, t) = xy(1 - t \cdot \chi_S(x, y))$ introduced in (1.3) degenerates, in some cases even becomes a reducible polynomial;
- As shown in Proposition 15, the coefficients of the series $Q(x, y, t)$ are actually rational numbers at $t = \frac{1}{2}$;
- Strong refinements of the above statement will be obtained later on, see e.g. Theorem 3 (to be proved in Section 5), where we will prove a connection between the coefficients of $Q(x, y, \frac{1}{2})$ and (rational) Bernoulli numbers.

In this section we give a probabilistic interpretation of why $t = \frac{1}{2}$ is a very special point, using the classical concept of the Green function for random walks.

4.1 Green function for random walks on segments

Let $(X(n))_{n \geq 0}$ be a classical simple random walk on $\mathbb{Z}$, with uniform jump probabilities to the left and right nearest neighbors. Let τ_k define the first exit time of the random walk from the segment $\tilde{S}_k = \{-k, \dots, k\}$, in bijection with the set S_k introduced in (2.1), i.e.,

$$\tau_k = \inf\{n \geq 0 : X(n) \notin S_k\}.$$

By definition (see e.g. [Woe00, Chap. 1]), the Green function for the random walk killed when exiting the segment S_k is

$$G_k(P, Q|t) = \sum_{n \geq 0} \mathbb{P}_P(X(n) = Q, \tau_k > n) t^n, \quad (4.1)$$

where $P \in \tilde{S}_k$ and $Q \in \tilde{S}_k$ are arbitrary starting and ending points, respectively, and under $\mathbb{P}_P$ the random walk $(X(n))_{n \geq 0}$ starts at P almost surely, i.e., $X(0) = P$.

It is well known (see again [Woe00, Chap. 1]) that the radius of convergence of $G_k(P, Q|t)$ in (4.1) as a power series in t is independent of P and Q ; it is denoted by $\frac{1}{\rho}$, where $\rho \in (0, 1]$ is often called the *spectral radius* of the random walk.

In the particular case $t = 1$, the Green function (4.1) admits the following classical interpretation in terms of a mean number of visits before exiting S_k :

$$G_k(P, Q) := G_k(P, Q|1) = \mathbb{E}_P \left(\sum_{n \geq 0} \mathbf{1}_{\{X(n)=Q, \tau_k > n\}} \right),$$

where $\mathbf{1}$ denotes the indicator function. Using the above notation, we immediately deduce the following reformulation of the series $W_k(P, Q, t)$ introduced in (2.2):

Lemma 23. *Let $k \geq 1$. For any $P, Q \in S_k$, let $\tilde{P}, \tilde{Q}$ be the corresponding points on $\tilde{S}_k$. For any $|t| < \frac{1}{2 \cos(\frac{\pi}{2+k})}$, we have*

$$W_k(P, Q, t) = G_k(\tilde{P}, \tilde{Q}|2t).$$

Let us do a series of four remarks.

- The coefficient 2 in the Green function G_k simply comes from normalising the transition probabilities of the random walk $(X(n))_{n \geq 0}$ to 1, while in the enumerative series W_k the weights add up to 2.
- In particular, $W_k(P, Q, \frac{1}{2}) = G_k(P, Q|1)$. With this perspective, the case $t = \frac{1}{2}$ is rather special as it corresponds exactly to the Green function of the simple random walk with $t = 1$.
- Together with the identity (2.10), which expresses the series $\sum_{n \geq 0} \#_{\mathcal{A}}\{(0, 0) \xrightarrow{n} Q_{2k}\} t^n$ as a finite sum of terms $W_k(P, Q, t)$, we deduce that $\sum_{n \geq 0} \frac{\#_{\mathcal{A}}\{(0, 0) \xrightarrow{n} Q_{2k}\}}{2^n}$ is a finite sum of products of Green functions as above.
- The point $\frac{1}{2}$ appears as the smallest upper bound of all radii of convergence of the series in Lemma 23.

4.2 An interpretation using Cramér's transform

In this paragraph, we provide a concrete interpretation of the series $Q(x, y, \frac{1}{2})$, using Green functions and the idea of Cramér's transform. Here we will work directly with the two-dimensional model, without using the decomposition with walks on segments S_k as in Figure 2.

Given a step set $\mathcal{S}$, it is convenient to introduce the associated random walk $(Z(n))_{n \geq 0}$ with uniform probability distribution on $\mathcal{S}$, killed when leaving the positive quarter plane. By definition, and similarly to (4.1), the Green function at (i, j) is

$$G(i, j|t) = \sum_{n \geq 0} \mathbb{P}_{(0,0)}(Z(n) = (i, j), \tau > n) t^n, \quad (4.2)$$

where τ is the first time the walk goes out of the quadrant.

The most natural value of t in (4.2) is 1, because in this particular case we have an interpretation of the Green function as the expected number of visits of the random walk at the given site (i, j) :

$$G(i, j|1) = \mathbb{E}_{(0,0)} \left(\sum_{n \geq 0} \mathbb{1}_{\{Z(n) = (i, j)\}} \right).$$

Using the obvious identity (see for instance [BRS14, Eq. (1)])

$$\mathbb{P}_{(0,0)}(Z(n) = (i, j), \tau > n) = \frac{\#_{\mathcal{S}}\{(0, 0) \xrightarrow{n} (i, j)\}}{\#\mathcal{S}^n},$$

we deduce a direct relation between Green functions and the generating function $Q(x, y, t)$, which reads

$$Q(x, y, t) = \sum_{i, j \geq 0} G(i, j|t) x^i y^j. \quad (4.3)$$

We can first observe that evaluating the above identity at $t = \frac{1}{\#\mathcal{S}}$ gives

$$Q(x, y, \frac{1}{\#\mathcal{S}}) = \sum_{i, j \geq 0} G(i, j|1) x^i y^j.$$

Is it possible to evaluate (4.3) at a value of $t > \frac{1}{\#\mathcal{S}}$? Ultimately at $t = \frac{1}{2}$? (And thus the Green function at $\frac{\#\mathcal{S}}{2}$.)

To answer these questions, we will use Cramér transform, a classical tool in random walk theory, recalled in [BRS14, Sec. 2.3] in our combinatorial context. Introduce a new random walk,

say $(W(n))_{n \geq 0}$, whose step set $\mathcal{S}$ is the same as that of $(Z(n))_{n \geq 0}$, but whose distribution is no longer uniform: instead the probability of moving in direction (i, j) is given by

$$\frac{\alpha^i \beta^j}{\chi_{\mathcal{S}}(\alpha, \beta)},$$

for some positive quantities α, β (simply take $\alpha = \beta = 1$ to get the uniform distribution) and $\chi_{\mathcal{S}}$ defined in (1.2).

The local probabilities of the random walks $(Z(n))_{n \geq 0}$ and $(W(n))_{n \geq 0}$ are related as follows (see again [BRS14])

$$\mathbb{P}_{(0,0)}(W(n) = (i, j), \tau > n) = \alpha^i \beta^j \left(\frac{\#\mathcal{S}}{\chi_{\mathcal{S}}(\alpha, \beta)} \right)^n \mathbb{P}_{(0,0)}(Z(n) = (i, j), \tau > n),$$

and thus, denoting $G_{(\alpha, \beta)}$ the Green function for the random walk $(W(n))_{n \geq 0}$, one has

$$G(i, j|t) = \alpha^{-i} \beta^{-j} G_{(\alpha, \beta)}(i, j | \frac{\chi_{\mathcal{S}}(\alpha, \beta)}{\#\mathcal{S}} t). \quad (4.4)$$

The above identity allows to interpret the Green function $G(i, j|t)$ for a general value of t in terms of a classical Green function at $t = 1$, provided in the right-hand side of (4.4) it is possible to adjust the parameters so as to have $\frac{\chi_{\mathcal{S}}(\alpha, \beta)}{\#\mathcal{S}} t = 1$.

For our purposes, we want to evaluate (4.4) at $t = \frac{\#\mathcal{S}}{2}$, because indeed

$$[x^i y^j] Q(x, y, \frac{1}{2}) = G(i, j | \frac{\#\mathcal{S}}{2}) = \alpha^{-i} \beta^{-j} G_{(\alpha, \beta)}(i, j | \frac{\chi_{\mathcal{S}}(\alpha, \beta)}{2}). \quad (4.5)$$

Unfortunately, for any given singular model, we cannot find values of the parameters α, β such that $\frac{\chi_{\mathcal{S}}(\alpha, \beta)}{2} = 1$. Indeed, since $\mathcal{S}$ contains the jumps $(-1, 1)$ and $(1, -1)$, we have

$$\chi_{\mathcal{S}}(\alpha, \beta) > \frac{\alpha}{\beta} + \frac{\beta}{\alpha} \geq 2.$$

As a conclusion, it is possible to interpret $[x^i y^j] Q(x, y, t)$ as a classical Green function for any $t \in [\frac{1}{\#\mathcal{S}}, \frac{1}{2})$, but not at the limiting point $t = \frac{1}{2}$. However, we can take $\alpha = \beta = \varepsilon \rightarrow 0$, and in this case $\chi_{\mathcal{S}}(\alpha, \beta) \rightarrow 2$. So a possible interpretation of $[x^i y^j] Q(x, y, \frac{1}{2})$ is the following:

$$[x^i y^j] Q(x, y, \frac{1}{2}) = \lim_{\varepsilon \rightarrow 0} \frac{G_{(\varepsilon, \varepsilon)}(i, j | \frac{\chi_{\mathcal{S}}(\varepsilon, \varepsilon)}{2})}{\varepsilon^{i+j}}.$$

4.3 Branching random walks

To give an interpretation of $[x^i y^j] Q(x, y, \frac{1}{2})$ as the Green function of a concrete probabilistic model, we will use branching random walks. While this model is very classical in probability theory, see e.g. [Shi15], it has been less studied from a combinatorial perspective. See [dPLW19] for a connection with non-deterministic paths.

Let us briefly describe the model. We introduce an offspring distribution ν on $\{0, 1, 2, \dots\}$. The first particle (called the *ancestor*) starts at the origin and jumps (according to one given model $\mathcal{S}$, for instance $\mathcal{A}$); then it immediately splits into a random number of particles, distributed according to ν . Then each individual of the new generation makes independently one jump from $\mathcal{S}$, and splits into a random number of new individuals. All individuals exiting the quarter plane are killed. See Figure 4 for an illustration.

The mean number of visits made by the branching random walk at a fixed point (i, j) starting from the origin is given by

$$G_{\nu}^{\text{BRW}}(i, j) = \mathbb{E}_{(0,0)} \left(\sum \mathbf{1}_{\{\text{BRW}=(i,j)\}} \right).$$

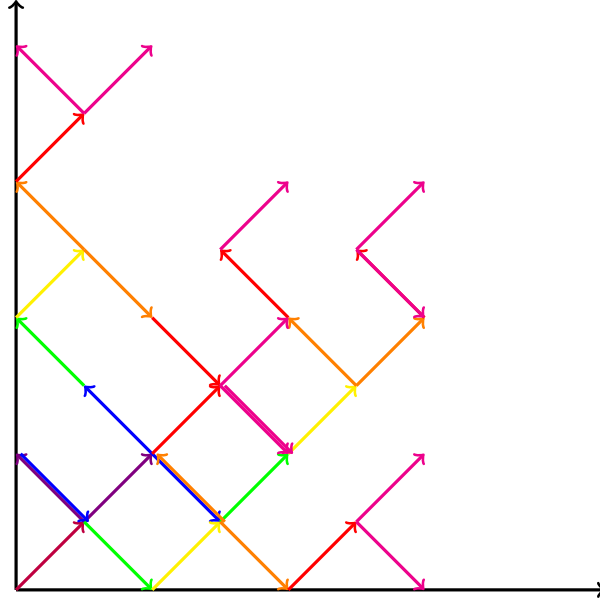


Figure 4: A path of branching random walk. Each particle jumps according to $\mathcal{A}$ and then splits into a random number of particles, uniformly distributed in $\{1, 2\}$. Individuals exiting the quarter plane are killed.

The reason why we introduced this model is that there is a strong connection between the Green function of the branching random walk and the t -Green function of one single random walk, as introduced in (4.2):

$$G_{\nu}^{\text{BRW}}(i, j) = G(i, j | \mathbb{E}\nu),$$

with $\mathbb{E}\nu$ denoting the mean value of the distribution ν . Accordingly,

$$[x^i y^j] Q(x, y, \tfrac{1}{2}) = G(i, j | \tfrac{\#\mathcal{S}}{2}) = G_{\nu}^{\text{BRW}}(i, j),$$

provided $\mathbb{E}\nu = \frac{\#\mathcal{S}}{2}$. For example, in the case $\mathcal{S} = \mathcal{A}$, we can take ν to be the uniform distribution over $\{1, 2\}$, and get $\mathbb{E}\nu = \frac{3}{2} = \frac{\#\mathcal{S}}{2}$. See Figure 4.

This reformulation has the advantage of giving a clear probabilistic interpretation of the coefficients of $Q(x, y, \frac{1}{2})$; on the other hand, it does not immediately explain why the point $t = \frac{1}{2}$ is critical.

4.4 Weighted models

While the results presented in this paper concern unweighted walks (i.e., each step from the step set $\mathcal{S}$ is taken with weight 1), they can be made more general using the following simple idea. Given any step set $\mathcal{S}$ and positive constants α_1 , α_2 and β , the generating function $Q(\alpha_1 x, \alpha_2 y, \beta t)$ turns out to be the generating function $Q_a(x, y, t)$ of the step set $\mathcal{S}$ with weights

$$a_s = \beta \alpha_1^{s_1} \alpha_2^{s_2}, \quad (4.6)$$

for $s = (s_1, s_2) \in \mathcal{S}$. Such families of weights are similar to those used in Section 4.2 to construct the Cramér transform of the random walk; the only difference is that in (4.6) we have one more parameter, because we don't force the sum of the weights to be 1.

The weights (4.6) are constructed using three parameters; consequently, given any weights $(w_s)_{s \in \mathcal{S}}$ on the models $\mathcal{A}$ and $\mathcal{D}$ (which both have cardinality three), we can find values of α_1 , α_2 and β such that $a_s = w_s$. For the other models with four or five steps, the weights (4.6) are not sufficient to cover all possible weights, but allow us to reduce the number of parameters by three.

A consequence of the above remarks and of the equality $Q(\alpha_1 x, \alpha_2 y, \beta t) = Q_a(x, y, t)$ of generating functions is that all transcendence results immediately extend to models weighted according to (4.6).

5 Explicit expressions and Bernoulli numbers

In the case of the models $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{D}$, we can directly get access to the nature of the generating function $Q(x, y, t)$ via explicit formulas for Q evaluated at $t = \frac{1}{2}$. These formulas are expressed in terms of the *Bernoulli numbers* $(B_n)_{n \geq 0}$, classically defined via their exponential generating function:

$$\sum_{n \geq 0} B_n \frac{x^n}{n!} = \frac{x}{e^x - 1}. \quad (5.1)$$

We recall that the EGF (5.1) is D-algebraic but that the ordinary generating function $\sum_{n \geq 0} B_n x^n$ is D-transcendental [BDVR24, Prop. 11], and even strongly D-transcendental [BDVR24, Thm 3]. Note that Bernoulli numbers admit nice closed-form expressions (see for instance [Gou72]), such as

$$B_n = \sum_{k=0}^n (-1)^k \binom{n+1}{k+1} \frac{0^n + 1^n + \dots + k^n}{k+1}, \quad (5.2)$$

therefore Theorems 24, 26 and 28 below provide closed-form expressions for $Q_{\mathcal{A}}(x, 0, \frac{1}{2})$, $Q_{\mathcal{B}}(x, 0, \frac{1}{2})$ and $Q_{\mathcal{D}}(x, 0, \frac{1}{2})$.

We leave as natural questions for further investigation whether $Q_{\mathcal{C}}(x, 0, \frac{1}{2})$ and $Q_{\mathcal{E}}(x, 0, \frac{1}{2})$ can equally be expressed in terms of Bernoulli numbers, and for which models $\mathcal{S}$ can the generating function $Q_{\mathcal{S}}(x, 0, q/(1+q^2))$ be expressed in terms of some q -deformations of the Bernoulli numbers.

5.1 Model $\mathcal{A}$

Theorem 24. *In the case of model $\mathcal{A}$, we have:*

$$Q_{\mathcal{A}}(x, 0, \frac{1}{2}) = 2 \cdot \sum_{n \geq 0} (2^{2n+2} - 1) \frac{(-1)^n}{n+1} B_{2n+2} x^{2n}. \quad (5.3)$$

In particular:

- (a) $Q_{\mathcal{A}}(x, 0, \frac{1}{2})$ is strongly D-transcendental;
- (b) the coefficients sequence $(a_n)_{n \geq 0} = (1, 0, \frac{1}{2}, 0, 1, 0, \frac{17}{4}, 0, 31, 0, \frac{691}{2}, \dots)$ of $Q_{\mathcal{A}}(x, 0, \frac{1}{2})$ satisfies

$$2a_{n+1} = \sum_{\ell=1}^n \binom{n}{\ell} a_{\ell-1} a_{n-\ell} \quad \text{for all } n \geq 0.$$

Proof. Since equation (3.6) admits a unique solution in $\mathbb{C}[[x]]$, namely $\frac{x^2}{2} \cdot Q_{\mathcal{A}}(x, 0, \frac{1}{2})$, it is enough to prove that the power series $f(x) := \sum_{n \geq 0} (2^{2n+2} - 1) \frac{(-1)^n}{n+1} B_{2n+2} x^{2n+2}$ also satisfies (3.6).

We introduce $T(x) := \sum_{n \geq 0} T_n x^{2n+1}$, with $T_n = 2^{2n+1} (2^{2n+2} - 1) \frac{(-1)^n}{n+1} B_{2n+2}$. It follows from [BDVR24, §2.2.1], and in particular from Eq. (2.7) in *loc. cit.*, that $T(x) = x + 2x^3 + 16x^5 + 272x^7 + 7936x^9 + \dots$ satisfies the difference equation

$$T\left(\frac{x}{1+2ix}\right) + (1+2ix)T(x) = 2x.$$

Since $f(2x) = 2xT(x)$, the ordinary generating function f satisfies the equation

$$f\left(\frac{x}{1+ix}\right) + f(x) = \frac{x^2}{1+ix},$$

which coincides with (3.6). Then, assertion (a) follows from the strong D-transcendence of $\sum_{n \geq 0} B_n x^n$ and from the closure properties of D-algebraic power series, while assertion (b) follows from the differential equation $\tan'(x) = 1 + \tan^2(x)$ satisfied by the tangent function $\tan(x) = \sum_{n \geq 0} \frac{T_n}{(2n+1)!} x^{2n+1}$. $\square$

Remark 25. The last part of the proof can be used to show that for any $r \in \mathbb{N}$, the power series $\sum_{n \geq 0} \frac{a_n}{(n+r)!} x^n$ is D-algebraic. For instance, $F_A(x) := \sum_{n \geq 0} \frac{a_n}{(n+1)!} x^n$ satisfies

$$x^2 F_A(x)^2 - 4x F_A'(x) - 4F_A(x) + 4 = 0.$$

In fact, $F_A(x) = \frac{2}{x} \tan\left(\frac{x}{2}\right)$.

5.2 Model $\mathcal{D}$

Theorem 26. *In the case of model $\mathcal{D}$, we have:*

$$Q_{\mathcal{D}}(x, 0, \frac{1}{2}) = 2 \sum_{n \geq 0} (2n+3) B_{2n+2} (-x)^n. \quad (5.4)$$

In particular:

- (a) $Q_{\mathcal{D}}(x, 0, \frac{1}{2})$ is strongly D-transcendental;
- (b) the coefficients sequence $(d_n)_{n \geq 0} = (1, \frac{1}{3}, \frac{1}{3}, \frac{3}{5}, \frac{5}{3}, \frac{691}{105}, 35, \frac{3617}{15}, \dots)$ of $Q_{\mathcal{D}}(x, 0, \frac{1}{2})$ satisfies

$$4(n+2)(2n+3) d_n = \sum_{\ell=0}^{n-1} \binom{2n+4}{2\ell+3} d_{\ell} d_{n-1-\ell} \quad \text{for all } n \geq 0.$$

Proof. Recall that the power series $Q_{\mathcal{D}}(x, y, t)$ satisfies the functional equation

$$Q_{\mathcal{D}}(x, y, t) = 1 + t \left(y + \frac{y}{x} + \frac{x}{y} \right) Q_{\mathcal{D}}(x, y, t) - t \frac{x}{y} Q_{\mathcal{D}}(x, 0, t) - t \frac{y}{x} Q_{\mathcal{D}}(0, y, t),$$

therefore the following “kernel equation” holds

$$\frac{x^2}{2} Q_{\mathcal{D}}(x, 0, \frac{1}{2}) + \frac{y^2}{2} Q_{\mathcal{D}}(0, y, \frac{1}{2}) = xy, \quad (5.5)$$

whenever (x, y) is a point on the “kernel curve” $xy^2 + x^2 + y^2 = 2xy$.

This kernel curve has genus zero, and it can be rationally parametrized by

$$x(s) = -s^2, \quad y(s) = -\frac{s^2}{s+1}$$

and by

$$\tilde{x}(s) = -\frac{s^2}{(s+1)^2} = x\left(\frac{s}{s+1}\right), \quad y(s) = -\frac{s^2}{s+1}.$$

(Notice that these parametrizations are different from the one used in the proof of Theorem 17.)

From (5.5) with (x, y) replaced by $(x(s), y(s))$, respectively by $(\tilde{x}(s), y(s))$, we deduce by subtraction that the power series $W(x) := \frac{x^2}{2} Q_{\mathcal{D}}(-x^2, 0, \frac{1}{2})$ is the unique solution in $\mathbb{Q}[[x]]$ of the functional equation

$$W(x) - \frac{1}{(x+1)^2} \cdot W\left(\frac{x}{x+1}\right) = \frac{x^3(x+2)}{(x+1)^3}. \quad (5.6)$$

Let us introduce the power series $\beta(x) := \sum_{n \geq 0} B_n x^{n+1} = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^5}{30} + \dots$. Since $B_{2k+1} = 0$ for $k > 0$, we have that $\beta'(x) = 1 - x + \sum_{n \geq 1} (2n+1) B_{2n} x^{2n}$. To prove identity (5.4) it is enough to show that the power series $V(x) := \beta'(x) + x - 1$ also satisfies equation (5.6). Indeed, the fact that $V(x) = W(x)$ implies that

$$Q_{\mathcal{D}}(-x^2, 0, \frac{1}{2}) = (\beta'(x) + x - 1) \cdot \frac{2}{x^2} = 2 \cdot \sum_{n \geq 1} (2n+1) B_{2n} x^{2n-2},$$

which is equivalent to (5.4). Finally, the fact that $V(x)$ satisfies (5.6) is a direct consequence of the functional equation $\beta'(x) - \frac{1}{(x+1)^2} \cdot \beta'(x/(x+1)) = 2x/(x+1)^3$, which is implied by differentiating the equality $\beta(x) - \beta(x/(x+1)) = x^2/(x+1)^2$, itself equivalent to the first entry in Table 2 in [BDVR24] (see also [AIK14, Eq. (A.7), p. 241]).

Finally, assertion (a) follows from the strong D-transcendence of $\beta(x)$ and assertion (b) follows from Euler's quadratic recurrence $(2n+1)B_{2n} + \sum_{i=1}^{n-1} \binom{2n}{2i} B_{2i} B_{2n-2i} = 0$ for all $n > 1$. $\square$

Remark 27. The last part of the proof can be used to show that for any $r \in \mathbb{N}$, the power series $\sum_{n \geq 0} \frac{d_n}{(2n+r)!} x^n$ is D-algebraic. For instance, $F_D(x) := \sum_{n \geq 0} \frac{d_n}{(2n+3)!} x^n$ satisfies

$$xF_D(x)^2 - 4xF_D'(x) - 6F_D(x) + 1 = 0.$$

In fact, $F_D(x) = \frac{2 - \sqrt{x} \cot\left(\frac{\sqrt{x}}{2}\right)}{x}$.

5.3 Model $\mathcal{B}$

Let $(M_n)_{n \geq 0} = (1, 2, 8, 56, 608, \dots)$ be the sequence of *median Genocchi numbers* (A005439), implicitly defined in the identity below in terms of Bernoulli numbers.

Theorem 28. *For model $\mathcal{B}$ we have*

$$Q_{\mathcal{B}}(x, 0, \frac{1}{2}) = \sum_{n \geq 0} \frac{M_n}{2^n} x^n = 2 \cdot \sum_{n=0}^{\infty} \left(-\frac{x}{2}\right)^n \sum_{k=0}^{n+1} \binom{n+1}{k} \left(2^{n+k+2} - 1\right) B_{n+k+2}. \quad (5.7)$$

Proof. The kernel curve defined by the polynomial $K(x, y, \frac{1}{2}) = xy - \frac{1}{2}(x^2y + xy^2 + x^2 + y^2)$ can be parametrized by

$$x(s) = -\frac{2s^2}{s+1} \quad \text{and} \quad y(s) = -\frac{2s^2}{1-s}.$$

(Notice that, again, the parametrization used below is not the one used in the proof of Theorem 17.)

By plugging this parametrization into the kernel equation

$$K\left(x, y, \frac{1}{2}\right) Q\left(x, y, \frac{1}{2}\right) = xy - \frac{x^2}{2} Q\left(x, 0, \frac{1}{2}\right) - \frac{y^2}{2} Q\left(0, y, \frac{1}{2}\right)$$

and by using the equality $Q(x, 0, t) = Q(0, x, t)$, we obtain that the power series $f(s) := Q(s, 0; \frac{1}{2})$ satisfies the equation

$$\frac{1}{(s+1)^2} f\left(-\frac{2s^2}{s+1}\right) + \frac{1}{(1-s)^2} f\left(-\frac{2s^2}{1-s}\right) = \frac{2}{1-s^2}. \quad (5.8)$$

Since the above equation (5.8) admits a unique solution in $\mathbb{C}[[s]]$, it is enough to prove that

$$f(s) := \sum_{n \geq 0} \frac{M_n}{2^n} s^n = 1 + s + 2s^2 + 7s^3 + 38s^4 + \dots$$

satisfies (5.8) as well. We deduce from [DZ94, Cor. 2] that the power series

$$h(s) = s - s^2 \sum_{n \geq 0} M_n (-s)^n = s - s^2 + 2s^3 - 8s^4 + \dots$$

satisfies the equation

$$h\left(\frac{s^2}{1+s}\right) + h\left(\frac{s^2}{1-s}\right) = 2s^2.$$

From there, using $h(s) = s - s^2 f(-2s)$, it follows that $f(s)$ does indeed satisfy (5.8).

Finally, the expression in terms of Bernoulli numbers is a consequence of the identity

$$M_{n-1} = 2(-1)^{n+1} \sum_{k=0}^n \binom{n}{k} (2^{n+k+1} - 1) B_{n+k+1} \quad \text{for all } n \geq 1,$$

itself a consequence of [DZ94, Cor. 3],

$$M_{n-1} = \frac{1}{4^n} \cdot \sum_{k=0}^n \binom{n}{k} (2k+1) E_{2k} \quad \text{for all } n \geq 1,$$

where $(E_{2n})_{n \geq 0} = (1, 1, 5, 61, 1385, \dots)$ (A000364) is the sequence of Euler (“secant”) numbers. $\square$

Remark 29. Contrary to (the proofs of) Theorems 24 and 26, we cannot infer from the proof of Theorem 28 that some exponential version of $Q_{\mathcal{B}}(x, 0, \frac{1}{2})$ is D-algebraic. Similarly, we are unable to deduce from Eq. (5.7) a nonlinear quadratic recurrence similar to those in part (b) of Theorems 24 and 26. However, from (5.8) we can deduce the following linear recurrence (of unbounded length):

$$\sum_{\ell=0}^n (-2)^\ell \binom{2n-\ell+1}{\ell+1} b_\ell = 1, \quad \text{for all } \ell \geq 0.$$

By Theorem 5.7, this identity is equivalent to the second identity in [DZ94, Cor. 1].

Remark 30. Contrary the models $\mathcal{A}$ and $\mathcal{D}$, the power series $Q_{\mathcal{B}}(x, 0, \frac{1}{2})$ has *integer* coefficients. This is a consequence of Theorem 28 and of a result due to Barsky and Dumont [BD81, Thm 4]. The coefficient sequence $(1, 1, 2, 7, 38, 295, 3098, \dots)$ of $Q_{\mathcal{B}}(x, 0, \frac{1}{2})$ appears to coincide with the so-called *Dellac sequence* (A000366). This sequence has several interesting combinatorial interpretations; for instance Dellac proved in [Del01] that its n -th term counts the number of terms in the resultant of two degree- n polynomials. Our interpretation in terms of walks for model $Q_{\mathcal{B}}$ seems to be new. It would be interesting to have a direct understanding of this new interpretation.

Remark 31. It can be shown that $Q_{\mathcal{B}}(x, 0, -\frac{1}{2}) = -2 \cdot \sum_{n \geq 0} B_{n+1} x^n$. Indeed, the kernel at $t = -1/2$ factors $(x+y)(xy+x+y)/2$, and the unique power series solution of $x^2 f(x) + (-x/(1+x))^2 f(-x/(1+x)) = 2x^2/(x+1)$ is $f(x) = -2 \cdot \sum_{n \geq 0} B_{n+1} x^n$.

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