About the proof of Proposition 2.7 in [ADV04]

Letter to Bernard Le Stum

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We need to show that there exist $M > 0$ and $\varepsilon > 0$, such that for any $i > M$ and any $|x| < 1 + \varepsilon$ we have:

$$\left|\frac{\mu_i x^i}{q^i - 1}\right| \leq |\pi| \tag{1}$$

In fact, if (1) holds, then [ADV04, Lemma 2.6] implies that

$$u(x) = \prod_{i>M} e_{q^i}\left(\frac{\mu_i x^i}{q^i - 1}\right)$$

is overconvergent. Notice that, in the notation of [ADV04, Proposition 2.7], we have

$$\frac{u(qx)}{u(x)} a(x) = \prod_{i=1}^{M} (1 + \mu_i x^i),$$

therefore this ends the proof [ADV04, Proposition 2.7]. First of all we shall prove the lemma:[1]

Lemma 1. *If the infinite product $\prod_{i=1}^{\infty}(1 + \mu_i x^i)$ is convergent for $|x| \leq 1 + \varepsilon$ then for any $0 < \varepsilon' < \varepsilon$ there exists M such that $|\mu_i| \leq (1 + \varepsilon')^i$ for any $i > M$.*

Proof. We denote $S_n = \prod_{i=1}^{n}(1 + \mu_i x^i)$. By definition of convergence of an infinite product $\lim_n S_n \neq 0$. Moreover the convergence is uniform on any smaller disk, therefore for any $0 < \varepsilon' < \varepsilon$ we have:

$$0 = \lim_{n\to\infty} \sup_{|x|\leq 1+\varepsilon'} |S_{n+1} - S_n| = \lim_{n\to\infty} \sup_{|x|\leq 1+\varepsilon'} |\mu_{n+1} x^{n+1} S_n| = \lim_{n\to\infty} \sup_{|x|\leq 1+\varepsilon'} |\mu_{n+1} S_n| (1 + \varepsilon')^{n+1}.$$

Since $\lim_{n\to\infty} \sup_{|x|\leq 1+\varepsilon'} |S_n|$ tends to a constant, we must have $|\mu_n| \leq |(1 + \varepsilon')^{-n}$ for $n >> 0$. □

Proof of Formula(1). Let us consider ε' and ε'' such that $0 < \varepsilon'' < \varepsilon' < \varepsilon$. Since the infinite product $\prod_{i=1}^{\infty}(1 + \mu_i x^i)$ is overconvergent, there exists $M > 0$, such that $|\mu_i| < (1 + \varepsilon')^{-i}$ for any $i > M$.

As far as the term $|q^i - 1|$ is concerned, the properties of the p-adic logarithm (see for instance [DGS94, II, Proposition 1.1]) imply that:

$$|q^i - 1| \left\{ \begin{array}{l} = |i \log q| \text{ , if } |q^i - 1| < |\pi|, \\ \geq |\pi|. \end{array} \right.$$

We deduce that

$$|q^i - 1| \geq \inf(|\pi|, |i \log q|).$$

Hence (*cf.* the radius of convergence of the log for $\limsup |i|^{-1/i}$)

$$\limsup_{i\to\infty} \left|\frac{\mu_i}{q^i - 1}\right|^{1/i} \leq \frac{1}{1 + \varepsilon'} \limsup_{i\to\infty} \sup\left(\frac{1}{|\pi|}, \frac{1}{|i \log q|}\right)^{1/i} \leq \frac{1}{1 + \varepsilon'}.$$

[1] The converse is also true but we don't need it!

Then for any $i > M$ and $|x| < (1 + \varepsilon'')$ we have:

$$\limsup_{i\to\infty} \left| \frac{\mu_i x^i}{q^i - 1} \right| \le \lim_{i\to\infty} \left(\frac{1+\varepsilon''}{1+\varepsilon'} \right)^i = 0,$$

which implies the claim, maybe enlarging a bit M. □

References

[ADV04] Y. André and L. Di Vizio. q-difference equations and p-adic local monodromy. *Astérisque*, (296):55–111, 2004. Analyse complexe, systèmes dynamiques, sommabilité des séries divergentes et théories galoisiennes. I.

[DGS94] B. Dwork, Giovanni Gerotto, and Francis J. Sullivan. *An introduction to G-functions*, volume 133 of *Annals of Mathematics Studies*. Princeton University Press, 1994.