

Descent for differential Galois theory of difference equations Confluence and q -dependency

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Abstract

The present paper essentially contains two results that generalize and improve some of the constructions of [HS08]. First of all, in the case of one derivation, we prove that the parameterized Galois theory for difference equations constructed in [HS08] can be descended from a differentially closed to an algebraically closed field. In the second part of the paper, we show that the theory can be applied to deformations of q -series to study the differential dependency with respect to $x \frac{d}{dx}$ and $q \frac{d}{dq}$. We show that the parameterized difference Galois group (with respect to a convenient derivation defined in the text) of the Jacobi Theta function can be considered as the Galoisian counterpart of the heat equation.

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Introduction

The present paper essentially contains two results that generalize and improve some of the constructions of [HS08]. First of all, in the case of one derivation, we perform a descent of the constructions in [HS08] from a differentially closed to an algebraically closed field. The latter being much smaller, this is a good help in the applications. In the second part of the paper, we show that the theory can be applied to deformations of q -series, which appear in many settings like quantum invariants, modular forms, ..., to study the differential dependency with respect to $x \frac{d}{dx}$ and $q \frac{d}{dq}$.

In [HS08], the authors construct a specific Galois theory to study the differential relations among solutions of a difference linear system. To do so, they attach to a linear difference system a differential Picard-Vessiot ring, *i.e.* a differential splitting ring, and, therefore, a differential algebraic group, in the sense of Kolchin, which we will call the Galois Δ -group. Roughly, this is a matrix group defined as the zero set of algebraic differential equations. In [HS08], both the differential Picard-Vessiot ring and the Galois Δ -group are proved to be well defined under the assumption that the difference operator and the derivations commute with each other and that the field of constants for the difference operator is differentially closed. The differential closure

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of a differential field K is an enormous field that contains a solution of any system of algebraic differential equations with coefficients in K that has a solution in some differential extension of K . When one works with q -difference equations, the subfield of the field of meromorphic functions over \mathbb{C}^* of constants for the homothety $x \mapsto qx$ is the field of elliptic functions: its differential closure is a very big field. The same happens for the shift $x \mapsto x + 1$, whose field of constants are periodic functions. In the applications, C. Hardouin and M. Singer prove that one can always descend the Galois Δ -group with an *ad hoc* argument. Here we prove that, in the case of one derivation, we can actually suppose that the field of constants is algebraically closed and that the Galois Δ -group descends from a differentially closed field to an algebraically closed one (see Proposition 1.20). We also obtain that the properties and the results used in the applications descend to an algebraically closed field, namely (see §1.2.1):

- the differential transcendence degree of an extension generated by a fundamental solution matrix of the difference equation is equal to the differential dimension of the Galois Δ -group (see Proposition 1.8);
- the sufficient and necessary condition for solutions of rank 1 difference equations to be differentially transcendental (see Proposition 1.9);
- the sufficient and necessary condition for a difference system to admit a linear differential system totally integrable with the difference system (see Corollary 1.26).

The proof of the descent (see Proposition 1.16) is based on an idea of M. Wibmer, which he used in a parallel difference setting (see [Wib10], [Coh65b]). The differential counterpart of his method is the differential prolongation of ideals, which goes back at least to E. Cartan, E. Vessiot, E.R. Kolchin and more recently to B. Malgrange in the nonlinear theory, but has never been exploited in this framework. Differential prolongations have been used in [Gra11] to adapt the ideas of B. Malgrange to build a Galois theory for nonlinear q -difference equations, and in [DVH10a] to prove a comparison theorem between Granier's groupoid and the parameterized Galois group of [HS08]. So it appears that the differential prolongations are a common denominator of several differential and difference Galois theories.

In the second part of the paper, we show that the theory applies to investigate the differential relations with respect to the derivations $\frac{d}{dx}$ and $\frac{d}{dq}$ among solutions of q -difference equations over the field $k(q, x)$ of rational functions with coefficients in a field k of zero characteristic. For instance, one can consider the Jacobi theta function

$$\theta_q(x) = \sum_{n \in \mathbb{Z}} q^{n(n-1)/2} x^n,$$

which is a solution of the q -difference equation $y(qx) = qxy(x)$. Let

$$\ell_q = x \frac{\theta'_q(x)}{\theta_q(x)} \text{ and } \delta = \ell_q(x) x \frac{d}{dx} + q \frac{d}{dq}.$$

In this specific case, we show below that the Galois δ -group can be considered as the Galoisian counterpart of the heat equation. We show that, for solutions of rank 1 equations $y(qx) = a(x)y(x)$ with $a(x) \in k(q, x)$, the following facts are equivalent (see Proposition 2.6):

- $a(x) = \mu x^r \frac{g(qx)}{g(x)}$, for some $r \in \mathbb{Z}$, $g \in k(q, x)$ and $\mu \in k(q)$;
- satisfying an algebraic $\frac{d}{dx}$ -relation (over some field we are going to specify in the text below);
- satisfying an algebraic δ -relation.

In the higher rank case, we deduce from a general statement a necessary and sufficient condition on the Galois group so that a system $Y(qx) = A(x)Y(x)$ can be completed in a compatible (*i.e.* integrable) way, by two linear differential systems in $\frac{d}{dx}$ and $x\ell_q \frac{d}{dx} + q \frac{d}{dq}$ (see Proposition 1.24 and Corollary 2.9). The condition consists in the property of the group of being, up to a conjugation, contained in the subgroup of constants of GL , that is the differential subgroup of GL whose points have coordinates that are annihilated by the derivations. A result of P. Cassidy [Cas72] says that, for a proper Zariski dense differential subgroup of a simple linear algebraic group, this is always the case.

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Breviary of difference-differential algebra

For the reader convenience, we briefly recall here some basic definitions of differential and difference algebra, that we will use all along the text. See [Coh65a] and [Lev08] for the general theory.

By (σ, Δ) -field we will mean a field F of zero characteristic, equipped with an automorphism σ and a set of commuting derivations $\Delta = \{\partial_1, \dots, \partial_n\}$, such that $\sigma\partial_i = \partial_i\sigma$ for any $i = 1, \dots, n$. We will use the terms (σ, Δ) -ring, Δ -ring, Δ -field, ∂ -ring, ∂ -field, for $\partial \in \Delta$, σ -ring, σ -field, ..., with the evident meaning, analogous to the definition of (σ, Δ) -field. Notice that, if F is a (σ, Δ) -field, the subfield $K = F^\sigma$ of F of the invariant elements with respect to σ is a Δ -field. We will say that a Δ -field, and in particular K , is Δ -closed if any system of algebraic differential equations in $\partial_1, \dots, \partial_n$ with coefficients in K having a solution in an extension of K has a solution in K (see [McG00]).

A (σ, Δ) -extension of F is a ring extension \mathcal{R} of F equipped with an extension of σ and of the derivations of Δ , such that the commutativity conditions are preserved.

A (σ, Δ) -ideal of a (σ, Δ) -ring is an ideal that is invariant by both σ and the derivations in Δ . A *maximal* (σ, Δ) -ideal is an ideal which is maximal with respect to the property of being a (σ, Δ) -ideal. Similar definitions can be given for Δ -ideals.

A ring of Δ -polynomials with coefficients in F is a ring of polynomials in infinitely many variables

$$F\{X_1, \dots, X_\nu\}_\Delta := F\left[X_i^{(\underline{\alpha})}; i = 1, \dots, \nu\right],$$

equipped with the differential structure such that

$$\partial_k(X_i^{(\underline{\alpha})}) = X_i^{(\underline{\alpha} + e_k)},$$

for any $i = 1, \dots, \nu$, $k = 1, \dots, n$, $\underline{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_{\geq 0}^n$, with $\underline{\alpha} + e_k = (\alpha_1, \dots, \alpha_k + 1, \dots, \alpha_n)$.

We will need the ring of *rational Δ -functions over $GL_\nu(F)$* . This is a localization of a ring of Δ -polynomials in the variables $X_{i,j}$, for $i, j = 1, \dots, \nu$:

$$F\left\{X_{i,j}, i, j = 1, \dots, \nu; \frac{1}{\det(X_{i,j})}\right\}_\Delta := F\{X_{i,j}; i, j = 1, \dots, \nu\}_\Delta \left[\frac{1}{\det(X_{i,j})}\right],$$

equipped with the induced differential structure. We write X for $(X_{i,j})$, $\partial^\alpha(X)$ for $\partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}(X)$ and $\det X$ for $\det(X_{i,j})$. If Δ is empty, the ring $F\{X, \det X^{-1}\}_\Delta$ is nothing else than the ring of rational functions $F[X, \det X^{-1}]$. By Δ -relation (over F) satisfied by a given matrix U with coefficients in a (σ, Δ) -extension R of F , we mean an element of $F\{X, \det X^{-1}\}_\Delta$ that vanishes at U . The Δ -relations satisfied by a chosen U form a Δ -ideal.

1 Differential Galois theory for difference equations

1.1 Introduction to differential Galois theory for difference equations

In this section, we briefly recall some results of [HS08]. Let F be a field of zero characteristic equipped with an automorphism σ . We denote the subfield of F of all σ -invariant elements by $K = F^\sigma$.

Definition 1.1. A (σ) -difference module $\mathcal{M} = (M, \Sigma)$ over F (also called σ -module over F or a F - σ -module, for short) is a finite-dimensional F -vector space M together with a σ -semilinear bijection $\Sigma : M \rightarrow M$ i.e. a bijection Σ such that $\Sigma(\lambda m) = \sigma(\lambda)\Sigma(m)$ for all $(\lambda, m) \in F \times M$.

One can attach a σ -difference module $\mathcal{M}_A := (F^\nu, \Sigma_A)$, with $\Sigma_A : F^\nu \rightarrow F^\nu$, $Y \mapsto A^{-1}\sigma(Y)$, to a σ -difference system

$$(1.1) \quad \sigma(Y) = AY, \text{ with } A \in GL_\nu(F), \text{ for some } \nu \in \mathbb{Z}_{>0},$$

so that the horizontal (i.e. invariant) vectors with respect to Σ_A correspond to the solutions of $\sigma(Y) = AY$. Conversely, the choice of an F -basis \underline{e} of a difference module \mathcal{M} leads to a σ -difference system $\sigma(Y) = AY$, with $A \in GL_\nu(F)$, which corresponds to the equation for the horizontal vectors of \mathcal{M} with respect to Σ_A in the chosen basis.

We define a *morphism of (σ) -difference modules over F* to be an F -linear map between the underlying F -vector spaces, commuting with the σ -semilinear operators. As defined above, the (σ) -difference modules over

F form a Tannakian category (see [Del90]), *i.e.* a category equivalent over the algebraic closure \overline{K} of K to the category of finite-dimensional representations of an affine group scheme. The affine group scheme corresponding to the sub-Tannakian category generated by the (σ) -difference module \mathcal{M}_A , whose non-Tannakian construction we are going to sketch below, is called Picard-Vessiot group of (1.1). Its structure measures the algebraic relations satisfied by the solutions of (1.1).

Let $\Delta := \{\partial_1, \dots, \partial_n\}$ be a set of commuting derivations of F such that, for all $i = 1, \dots, n$, we have $\sigma \circ \partial_i = \partial_i \circ \sigma$. In [HS08], the authors proved, among other things, that the category of σ -modules carries also a Δ -structure *i.e.* it is a differential Tannakian category as defined by A. Ovchinnikov in [Ovc09]. The latter is equivalent to a category of finite-dimensional representations of a differential group scheme (see [Kol73]), whose structure measures the differential relations satisfied by the solutions of the σ -difference modules. In the next section, we describe the Picard-Vessiot approach to the theory in [HS08] (in opposition to the differential Tannakian approach), *i.e.* the construction of minimal rings containing the solutions of $\sigma(Y) = AY$ and their derivatives with respect to Δ , whose automorphism group is a concrete incarnation of the differential group scheme defined by the differential Tannakian equivalence. We will implicitly consider the usual Galois theory of (σ) -difference equations by allowing Δ to be the empty set (see for instance [vdPS97]): we will informally refer to this theory and the objects considered in it as classical.

1.1.1 Δ -Picard-Vessiot rings

Let F be a (σ, Δ) -field as above, with $K = F^\sigma$. Let us consider a σ -difference system

$$(1.2) \quad \sigma(Y) = AY,$$

with $A \in GL_\nu(F)$, as in (1.1).

Definition 1.2 (Def. 6.10 in [HS08]). A (σ, Δ) -extension \mathcal{R} of F is a Δ -Picard-Vessiot extension for (1.2) if

1. \mathcal{R} is a simple (σ, Δ) -ring *i.e.* it has no nontrivial ideal stable under both σ and Δ ;
2. \mathcal{R} is generated as a Δ -ring by $Z \in GL_\nu(\mathcal{R})$ and $\frac{1}{\det(Z)}$, where Z is a fundamental solution matrix of (1.2).

One can formally construct such an object as follows. We consider the ring of rational Δ -functions $F\{X, \det X^{-1}\}_\Delta$. We want to equip it with a structure of a (σ, Δ) -algebra, respecting the commutativity conditions for σ and the ∂_i 's. Therefore, we set $\sigma(X) = AX$ and

$$(1.3) \quad \begin{aligned} \sigma(X^\alpha) &= \sigma(\partial^\alpha X) = \partial^\alpha(\sigma(X)) = \partial^\alpha(AX) \\ &= \sum_{i+j=\alpha} \binom{\alpha_1}{i_1} \cdots \binom{\alpha_n}{i_n} \partial^i(A) X^j, \end{aligned}$$

for each multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$. Then the quotient \mathcal{R} of $F\{X, \det X^{-1}\}_\Delta$ by a maximal (σ, Δ) -ideal obviously satisfies the conditions of the definition above and hence is a Δ -Picard-Vessiot ring. Moreover, it has the following properties:

Proposition 1.3 (Propositions 6.14 and 6.16 in [HS08]). *If the field K is Δ -closed, then:*

1. *The ring of constants \mathcal{R}^σ of a Δ -Picard-Vessiot ring \mathcal{R} for (1.2) is equal to K , *i.e.* there are no new constants with respect to σ , compared to F .*
2. *Two Δ -Picard-Vessiot rings for (1.2) are isomorphic as (σ, Δ) -rings.*

1.1.2 Δ -Picard-Vessiot groups

*Until the end of §1.1, *i.e.* all along §1.1.2 and §1.1.3, we assume that K is a Δ -closed field.*

Let \mathcal{R} be a Δ -Picard-Vessiot ring for (1.2). Notice that, as in classical Galois theory for (σ) -difference equations, the ring \mathcal{R} does not need to be a domain. One can show that it is in fact the direct sum of a finite number of copies of an integral domain, therefore one can consider the ring L of total fractions of \mathcal{R} which is isomorphic to the product of a finite number of copies of one field (see [HS08]).

Definition 1.4. The group $Gal^\Delta(\mathcal{M}_A)$ (also denoted $Aut^{\sigma,\Delta}(L/F)$) of the automorphisms of L that fix F and commute with σ and Δ is called the Δ -Picard-Vessiot group of (1.2). We will also call it the Galois Δ -group of (1.2).

Remark 1.5. The group $Gal^\Delta(\mathcal{M}_A)$ consists in the K -points of a linear algebraic Δ -subgroup of $GL_\nu(K)$ in the sense of Kolchin. That is a subgroup of $GL_\nu(K)$ defined by a Δ -ideal of $K\{X, \det X^{-1}\}_\Delta$.

Below, we recall some fundamental properties of the Galois Δ -group, which are the starting point of proving the Galois correspondence:

Proposition 1.6 (Lemma 6.19 in [HS08]).

1. The ring $L^{Gal^\Delta(\mathcal{M}_A)}$ of elements of L fixed by $Gal^\Delta(\mathcal{M}_A)$ is F .
2. Let H be an algebraic Δ -subgroup of $Gal^\Delta(\mathcal{M}_A)$. If $L^H = F$, then $H = Gal^\Delta(\mathcal{M}_A)$.

As we have already pointed out, the Galois \emptyset -group is an algebraic group defined over K and corresponds to the classical Picard-Vessiot group attached to the σ -difference system (1.2) (see [vdPS97, Sau04]):

Proposition 1.7 (Proposition 6.21 in [HS08]). *The algebraic Δ -group $Gal^\Delta(\mathcal{M}_A)$ is a Zariski dense subset of $Gal^\emptyset(\mathcal{M}_A)$.*

1.1.3 Differential dependency and total integrability

The Δ -Picard-Vessiot ring \mathcal{R} of (1.2) is a $Gal^\Delta(\mathcal{M}_A)$ -torsor in the sense of Kolchin. This implies in particular that the Δ -relations satisfied by a fundamental solution of the σ -difference system (1.2) are entirely determined by $Gal^\Delta(\mathcal{M}_A)$:

Proposition 1.8 (Proposition 6.29 in [HS08]). *The Δ -transcendence degree of \mathcal{R} over F is equal to the Δ -dimension of $Gal(\mathcal{M}_A)$.*

Since the Δ -subgroups of \mathbb{G}_a^n coincide with the zero set of a homogeneous linear Δ -polynomial $L(Y_1, \dots, Y_n)$ (see [Cas72]), we have:

Proposition 1.9. *Let $a_1, \dots, a_n \in F$ and let S be a (σ, Δ) -extension of F such that $S^\sigma = K$. If $z_1, \dots, z_n \in S$ satisfy $\sigma(z_i) - z_i = a_i$ for $i = 1, \dots, n$, then $z_1, \dots, z_n \in S$ satisfy a nontrivial Δ -relation over F if and only if there exists a nonzero homogeneous linear differential polynomial $L(Y_1, \dots, Y_n)$ with coefficients in K and an element $f \in F$ such that $L(a_1, \dots, a_n) = \sigma(f) - f$.*

Proof. If $\Delta = \{\partial_1\}$, the proposition coincide with Proposition 3.1 in [HS08]. The proof in the case of many derivations is a straightforward generalization of their argument. \square

The following proposition relates the structural properties of the Galois Δ -group (see the remark immediately below for the definition of constant Δ -group) with the holonomy of a σ -difference system:

Proposition 1.10. *The following statements are equivalent:*

1. The Δ -Galois group $Gal^\Delta(\mathcal{M}_A)$ is conjugate over K to a constant Δ -group.
2. For all $i = 1, \dots, n$, there exists a $B_i \in M_n(F)$ such that the set of linear systems

$$\begin{cases} \sigma(Y) = AY \\ \partial_1 Y = B_1 Y \\ \dots\dots\dots \\ \partial_n Y = B_n Y \end{cases}$$

is integrable, i.e. the matrices B_i and A satisfy the functional equations deduced from the commutativity of the operators:

$$\partial_i(B_j) + B_j B_i = \partial_j(B_i) + B_i B_j \text{ and } \sigma(B_j)A = \partial_j(A) + AB_j, \text{ for any } i, j = 1, \dots, n.$$

Proof. The proof is a straightforward generalization of Proposition 2.9 in [HS08] to the case of several derivations. \square

Remark 1.11. Let K be a Δ -field and C its subfield of Δ -constants. A linear Δ -group $G \subset GL_\nu$ defined over K is said to be a constant Δ -group (or Δ -constant, for short) if one of the following equivalent statements hold:

- the set of differential polynomials $\partial_h(X_{i,j})$, for $h = 1, \dots, n$ and $i, j = 1, \dots, \nu$, belong to the ideal of definition of G in the differential Hopf algebra $K\{X_{i,j}, \frac{1}{\det(X_{i,j})}\}_\Delta$ of GL_ν over K ;
- the differential Hopf algebra of G over K is an extension of scalars of a finitely generated Hopf algebra over C ;
- the points of G in K (which is Δ -closed!) coincide the C -points of an algebraic group defined over C .

For instance, let \mathbb{G}_m be the multiplicative group defined over K . Its differential Hopf algebra is $K\{x, \frac{1}{x}\}_\Delta$, *i.e.* the Δ -ring generated by x and $\frac{1}{x}$. The constant Δ -group $\mathbb{G}_m(C)$ corresponds to the differential Hopf algebra

$$\frac{K\{x, \frac{1}{x}\}_\Delta}{(\partial_h(x); h = 1, \dots, n)} \cong C \left[x, \frac{1}{x} \right] \otimes_C K,$$

where $C[x, \frac{1}{x}]$ is a Δ -ring with the trivial action of the derivations in Δ .

According to [Cas72], if H is a Zariski dense Δ -subgroup of a simple linear algebraic group $G \subset GL_\nu(K)$, defined over a differentially closed field K , then either $H = G$ or there exists $P \in GL_\nu(K)$ such that PHP^{-1} is a constant Δ -subgroup of PGP^{-1} . Therefore:

Corollary 1.12. *If $Gal^\varnothing(\mathcal{M}_A)$ is simple, we are either in the situation of the proposition above or there are no Δ -relations among the solutions of $\sigma(Y) = AY$.*

1.2 Descent over an algebraically closed field

In this subsection, we consider a (σ, Δ) -field F , where σ is an automorphism of F and $\Delta = \{\partial\}$ is a set containing only one derivation. Moreover we suppose that σ commutes with ∂ .

We have recalled above the theory developed in [HS08], where the authors assume that the field of constants K is differentially closed. Although in most applications of [HS08] a descent argument *ad hoc* proves that one can consider smaller, nondifferentially closed, field of constants, the assumption that K is Δ -closed is quite restrictive. We show here that if the σ -constants K of F form an algebraically closed field, we can construct a ∂ -Picard-Vessiot ring, whose ring of σ -constants coincides with K , which allows us to descend the group introduced in the previous section from a ∂ -closed field to an algebraically closed field. This kind of results were first tackled in [PN11] using model theoretic arguments. Here we use an idea of M. Wibmer of developing a differential analogue of Lemma 2.16 in [Wib10]. In [Wib11], M. Wibmer has given a more general version of Proposition 1.16. A Tannakian approach to the descent of parameterized Galois groups can be found in [GGO11].

For now, we do not make any assumption on $K = F^\sigma$. Let Θ be the semigroup generated by ∂ and, for all $k \in \mathbb{Z}_{\geq 0}$, let $\Theta_{\leq k}$ be the set of elements of Θ of order less or equal to k . We endow the differential rational function ring $\mathcal{S} := K\{X, \frac{1}{\det(X)}\}_\partial$, where $X = (X_{i,j})$, with the grading associated to the usual ranking *i.e.* we consider for all $k \in \mathbb{Z}_{\geq 0}$ the rational function ring $\mathcal{S}_k := K\left[\beta(X), \frac{1}{\det(X)}; \forall \beta \in \Theta_{\leq k}\right]$. Of course, we have $\partial(\mathcal{S}_k) \subset \mathcal{S}_{k+1}$. It is convenient to set $\mathcal{S}_{-1} = K$.

Definition 1.13 (see §3 in [Lan70]). The prolongation of an ideal \mathcal{J}_k of \mathcal{S}_k is the ideal $\pi_1(\mathcal{J}_k)$ of \mathcal{S}_{k+1} generated by $\Theta_{\leq 1}(\mathcal{J}_k)$. We say that a prime ideal \mathcal{J}_k of \mathcal{S}_k , for $k \geq 0$, is a differential kernel of length k if the prime ideal $\mathcal{J}_{k-1} := \mathcal{J}_k \cap \mathcal{S}_{k-1}$ of \mathcal{S}_{k-1} is such that $\pi_1(\mathcal{J}_{k-1}) \subset \mathcal{J}_k$.

Notice that, according to the definition above, any prime ideal \mathcal{J}_0 of \mathcal{S}_0 is a differential kernel.

Remark 1.14. In [Lan70], the author defines a differential kernel as a finitely generated field extension $F(A_0, A_1, \dots, A_k)/F$, together with an extension of ∂ to a derivation of $F(A_0, A_1, \dots, A_{k-1})$ into $F(A_0, A_1, \dots, A_k)$ such that $\partial(A_i) = A_{i+1}$ for $i = 0, \dots, k-1$. We can recover \mathcal{J}_{k-1} as the kernel of the F -morphism $\mathcal{S}_k = F[X, \dots, \partial^k(X)] \rightarrow F(A_0, A_1, \dots, A_k)$, with $\partial^i(X) \mapsto A_i$. We will not use Lando's point of view here.

Notice that we have used Malgrange's prolongation π_1 (see [Mal05]) to rewrite Lando's definition. One should pay attention to the fact tha Malgrange also considers a weak prolongation $\tilde{\pi}_1$, which we won't consider here (see Chapter V in [Mal05]).

Proposition 1.15 (Proposition 1 in [Lan70]). *For $k \geq 0$, let \mathfrak{J}_k be a differential kernel of \mathcal{S}_k . There exists a differential kernel \mathfrak{J}_{k+1} of \mathcal{S}_{k+1} such that $\mathfrak{J}_k = \mathfrak{J}_{k+1} \cap \mathcal{S}_k$.*

Let $\sigma(Y) = AY$ be a σ -difference system with coefficient in F as (1.2), \mathcal{R} a ∂ -Picard-Vessiot ring for $\sigma(Y) = AY$, constructed as in §1.1 under the assumption that K is ∂ -closed, and \mathfrak{J} be the defining ideal of \mathcal{R} , i.e. the ideal such that

$$\mathcal{R} \cong \frac{F \left\{ X, \frac{1}{\det(X)} \right\}_{\partial}}{\mathfrak{J}}.$$

Then Proposition 6.21 in [HS08] says that $\mathfrak{J}_k := \mathfrak{J} \cap \mathcal{S}_k$ is a maximal σ -ideal of \mathcal{S}_k endowed with the σ -structure induced by $\sigma(X) = AX$, which implies that \mathfrak{J} itself is a σ -maximal ideal of \mathcal{S} . In order to prove the descent of the ∂ -Picard-Vessiot ring \mathcal{R} , we are going to proceed somehow in the opposite way. Without any assumption on K , we will construct a sequence $(\mathfrak{J}_k)_{k \in \mathbb{N}}$ of σ -maximal ideals of \mathcal{S}_k such that $\cup_{k \in \mathbb{N}} \mathfrak{J}_k$ is a σ -maximal ideal of \mathcal{S} stable by ∂ . Such an ideal will provide us with a ∂ -Picard-Vessiot ring $\mathcal{R}^\#$, which will be a simple σ -ring. If, moreover, K is algebraically closed, we will be able to compare its group of automorphisms and the Galois ∂ -group of the previous section.

Proposition 1.16. *Let $A \in GL_\nu(F)$. Then there exists a (σ, ∂) -extension $\mathcal{R}^\#$ of F such that:*

1. $\mathcal{R}^\#$ is generated over F as a ∂ -ring by $Z \in GL_\nu(\mathcal{R}^\#)$ and $\frac{1}{\det(Z)}$ for some matrix Z satisfying $\sigma(Z) = AZ$;
2. $\mathcal{R}^\#$ is a simple σ -ring, i.e. it has no nontrivial ideals stable under σ .

Remark 1.17. Of course, a simple σ -ring carrying a structure of ∂ -ring is a simple (σ, ∂) -ring and thus $\mathcal{R}^\#$ is a ∂ -Picard-Vessiot ring in the sense of Definition 1.2.

Proof. Let $\mathcal{S} = F \left\{ X, \frac{1}{\det(X)} \right\}_{\partial}$ be the differential rational function ring in the variables $X = (X_{i,j})$ and let \mathcal{S}_k , $k \in \mathbb{N}$ be as above. We define a σ -ring structure on \mathcal{S} as in (1.3), so that, in particular, $\sigma(X) = AX$. We will prove by induction on $k \geq 0$ that there exists a maximal σ -ideal \mathfrak{J}_k of \mathcal{S}_k such that \mathfrak{J}_k is a differential kernel of length k and $\mathfrak{J}_{k-1} = \mathfrak{J}_k \cap \mathcal{S}_{k-1}$. For $k = 0$, we can take \mathfrak{J}_0 to be any σ -maximal ideal of \mathcal{S}_0 . Then, the σ -ring $\mathcal{S}_0/\mathfrak{J}_0$ is a classical Picard-Vessiot ring for $\sigma(Y) = AY$ in the sense of [vdPS97]. It follows from Proposition 1.15 that there exists a differential kernel \mathfrak{J}_1 of \mathcal{S}_1 such that $\mathfrak{J}_1 \cap \mathcal{S}_0 = \mathfrak{J}_0$ and $\pi_1(\mathfrak{J}_0) \subset \mathfrak{J}_1$.

Now, let us construct \mathfrak{J}_{k+1} starting from \mathfrak{J}_{k-1} and \mathfrak{J}_k . By Corollary 1.16 in [vdPS97], both \mathfrak{J}_{k-1} and \mathfrak{J}_k can be written as intersections of the form:

$$\mathfrak{J}_{k-1} = \bigcap_{i=0}^{t_{k-1}} \mathfrak{J}_i^{(k-1)} \left(\text{resp. } \mathfrak{J}_k = \bigcap_{i=0}^{t_k} \mathfrak{J}_i^{(k)} \right),$$

where the $\mathfrak{J}_i^{(k-1)}$ (resp. $\mathfrak{J}_i^{(k)}$) are prime ideals of \mathcal{S}_{k-1} (resp. \mathcal{S}_k)¹. We shall assume that these representations are minimal and, so, unique. Then,

1. the prime ideals $\mathfrak{J}_i^{(k-1)}$ (resp. $\mathfrak{J}_i^{(k)}$) are permuted by σ ;
2. for any $i = 1, \dots, t_k$, there exists $j \in \{1, \dots, t_{k-1}\}$ such that $\mathfrak{J}_i^{(k)} \cap \mathcal{S}_{k-1} = \mathfrak{J}_j^{(k-1)}$.

The last assertion means that, for all $i = 0, \dots, t_k$, the prime ideal $\mathfrak{J}_i^{(k)}$ is a differential kernel of \mathcal{S}_k . Proposition 1.15 implies that $\pi_1(\mathfrak{J}_i^{(k)})$, and hence $\cap_{i=0}^{t_k} \pi_1(\mathfrak{J}_i^{(k)})$ is a proper σ -ideal of \mathcal{S}_{k+1} . Therefore there exists a σ -maximal ideal \mathfrak{J}_{k+1} of \mathcal{S}_{k+1} containing $\cap_{i=0}^{t_k} \pi_1(\mathfrak{J}_i^{(k)})$. Moreover, the σ -maximality of \mathfrak{J}_k and the inclusion $\mathfrak{J}_k \subset \mathfrak{J}_{k+1} \cap \mathcal{S}_k$ imply that $\mathfrak{J}_k = \mathfrak{J}_{k+1} \cap \mathcal{S}_k$, which ends the recursive argument. The ideal $\mathfrak{J} = \cup_{k \in \mathbb{N}} \mathfrak{J}_k$ is clearly σ -maximal in \mathcal{S} and ∂ -stable. Then, $\mathcal{R}^\# := \mathcal{S}/\mathfrak{J}$ satisfies the requirements. \square

Remark 1.18. By Lemma 6.8 in [HS08], where the assumption that the field K is ∂ -closed plays no role, we have that there exists a set of idempotents $e_0, \dots, e_r \in \mathcal{R}^\#$ such that $\mathcal{R}^\# = \mathcal{R}_0 \oplus \dots \oplus \mathcal{R}_r$, where $\mathcal{R}_i = e_i \mathcal{R}^\#$ is an integral domain and σ permutes the set $\{\mathcal{R}_0, \dots, \mathcal{R}_r\}$. Let L_i be the fraction field of \mathcal{R}_i . The total field of fraction $L^\#$ of $\mathcal{R}^\#$ is equal to $L_0 \oplus \dots \oplus L_r$.

If \tilde{K} denotes a ∂ -closure of K and \tilde{F} is the fraction field of $\tilde{K} \otimes_K F$ equipped with an extension of σ that acts as the identity on \tilde{K} , then the construction of $\mathcal{R}^\#$ over \tilde{F} gives a ring which is isomorphic to a ∂ -Picard-Vessiot

¹We point out, although the remark plays no role in the proof, that the Ritt-Raudenbush theorem on the ∂ -Noetherianity of \mathcal{S} implies that the sequence of integers $(t_k)_{k \in \mathbb{N}}$ becomes stationary.

extension of the σ -difference system $\sigma(Y) = AY$ viewed as a σ -difference system with coefficients in \tilde{F} , in the sense of §1.1.1.

Notice that two Picard-Vessiot rings as in the proposition above may require a finitely generated extension of K to become isomorphic (see [Wib11]).

Corollary 1.19. *If $K = F^\sigma$ is an algebraically closed field, the set of σ -constants of $\mathcal{R}^\#$ is equal to K .*

Proof. Let $c \in \mathcal{R}^\#$ be a σ -constant. In the notation of the previous proof, there exists $k \in \mathbb{N}$ such that $c \in \mathcal{R}_k := \mathcal{S}_k/\mathcal{I}_k$. By construction, \mathcal{R}_k is a simple σ -ring and a finitely generated F -algebra. By Lemma 1.8 in [vdPS97], the σ -constants of \mathcal{R}_k coincide with K . \square

Proposition 1.20. *Let $\sigma(Y) = AY$ be a linear σ -difference system with coefficients in F and let $\mathcal{R}^\#$ be the ∂ -Picard-Vessiot ring constructed in Proposition 1.16.*

If $K = F^\sigma$ is algebraically closed, the functor

$$(1.4) \quad \begin{array}{ccc} \text{Aut}^{\sigma, \partial} & : & K\text{-}\partial\text{-algebras} \quad \longrightarrow \quad \text{Groups} \\ & & S \quad \longmapsto \quad \text{Aut}^{\sigma, \partial}(\mathcal{R}^\# \otimes_K S/F \otimes_K S) \end{array}$$

is representable by a linear algebraic ∂ -group scheme G_A defined over K . Moreover, G_A becomes isomorphic to $\text{Gal}^\partial(\mathcal{M}_A)$ over a differential closure of K (see Definition 1.4 above).

Remark 1.21. Without getting into too many details, the representability of the functor (1.4) is precisely the definition of a linear algebraic ∂ -group scheme G_A defined over K .

Proof. The first assertion is proved exactly as in [HS08], p. 368, where the authors only use the fact that the constants of the ∂ -Picard-Vessiot ring do not increase with respect to the base field F , the assumption that K is ∂ -closed being used to prove this property of ∂ -Picard-Vessiot rings. The second assertions is just a consequence of the theory of differential Tannakian category (see [GGO11]), which asserts that two differential fiber functors become isomorphic on a common ∂ -closure of their fields of definition. \square

Definition 1.22. We say that G_A is the ∂ -group scheme attached to $\sigma(Y) = AY$.

Since we are working with schemes and not with the points of linear algebraic ∂ -groups in a ∂ -closure of K , we need to consider functorial definitions of ∂ -subgroup scheme and invariants (see [Mau10] in the case of iterative differential equations). A ∂ -subgroup functor H of the functor G_A is a ∂ -group functor

$$H : \{K\text{-}\partial\text{-algebras}\} \rightarrow \{\text{Groups}\}$$

such that for all K - ∂ -algebra S , the group $H(S)$ is a subgroup of $G_A(S)$. So, let $L^\#$ be the total ring of fractions of $\mathcal{R}^\#$ and let H be a ∂ -subgroup functor of G_A . We say that $r = \frac{a}{b} \in L^\#$, with $a, b \in \mathcal{R}^\#$, b not a zero divisor, is an invariant of H if for all K - ∂ -algebra S and all $h \in H(S)$, we have

$$h(a \otimes 1).(b \otimes 1) = (a \otimes 1).h(b \otimes 1).$$

We denote the ring of invariant of $L^\#$ under the action of H by $(L^\#)^H$.

The Galois correspondence is proved by classical arguments starting from the following theorem.

Theorem 1.23. *Let H be a ∂ -subgroup functor H of G_A . Then $(L^\#)^H = K$ if and only if $H = G_A$.*

Proof. The proof relies on the same arguments as Theorem 11.4 in [Mau10] and Lemma 6.19 in [HS08]. \square

1.2.1 Descent of the criteria for hypertranscendancy

The main consequence of Proposition 1.20 and Theorem 1.23 is that Propositions 1.8 and 1.9 remain valid if one only assumes that the σ -constants K of the base field F form an algebraically closed field and replaces the differential Picard-Vessiot group $\text{Gal}^\Delta(\mathcal{M}_A)$ defined over the differential closure of K by the Δ -group scheme G_A defined over K . In [GO], the authors exhibit a linear differential algebraic group defined over K which can be conjugated to a constant ∂ -group only over a transcendental extension of K . Thus, one has to be careful when trying to prove a descended version of Proposition 1.10. We go back to the multiple derivation case to state the following proposition, which is a generalization of Corollary 3.12 in [HS08]:

Proposition 1.24. We suppose that F is a (σ, Δ) -field², with $\Delta = \{\partial_1, \dots, \partial_n\}$, such that $K = F^\sigma$ is an algebraically closed field and $F = K(x_1, \dots, x_n)$ a purely transcendental extension of transcendence basis x_1, \dots, x_n . We suppose also that σ induces an automorphisms of $K(x_i)$ for any $i = 1, \dots, n$.

Let \tilde{K} be a Δ -closure of K and \tilde{F} be the fraction field of $\tilde{K} \otimes_K F$ equipped with an extension of σ acting as the identity on \tilde{K} , so that one can consider $\text{Gal}^\Delta(\mathcal{M}_A)$ as in §1.1.2.

We consider a σ -difference system $\sigma(Y) = AY$ with coefficients in F . The following statements are equivalent:

1. $\text{Gal}^\Delta(\mathcal{M}_A)$ is conjugate over \tilde{K} to a constant Δ -group.
2. There exist $B_1, \dots, B_n \in M_\nu(F)$ such that the system

$$\begin{cases} \sigma(Y) = AY \\ \partial_i Y = B_i Y, \quad i = 1, \dots, n \end{cases}$$

is integrable.

Remark 1.25. Since σ induces an automorphism of $K(x_i)$ for any $i = 1, \dots, n$, it acts on x_i through a Moebius transformation. By choosing another transcendence basis of F/K , we can suppose that either $\sigma(x_i) = q_i x$ for some $q_i \in K$ or $\sigma(x_i) = x_i + h_i$ for some $h_i \in K$. We are therefore in the most classical situation.

Proof. First of all we prove that the field \tilde{F} is a purely algebraic extension of \tilde{K} . Suppose that σ acts periodically on x_1 , so that r is the minimal positive integer such that $\sigma^r(x_1) = x_1$. Then the polynomial

$$(T - x_1)(T - \sigma(x_1)) \cdots (T - \sigma^{r-1}(x_1))$$

has coefficients in K and vanishes at x_1 . This is impossible since x_1 is transcendental over K . Therefore we deduce that σ does not acts periodically on x_1 , or on any element of the transcendence basis.

Suppose now that x_1, \dots, x_n , considered as elements of \tilde{F} , satisfy an algebraic relation over \tilde{K} . This means that there exists a nonzero polynomial $P \in \tilde{K}[T_1, \dots, T_n]$ such that $P(x_1, \dots, x_n) = 0$. We can suppose that there exists $i_0 = 1, \dots, n$ such that $\sigma(x_i) = q_i x$ for $i \leq i_0$ and $\sigma(x_i) = x_i + h_i$ for $i > i_0$ (see remark above) and choose P so that the number of monomials appearing in its expression is minimal. Let $T_1^{\alpha_1} \cdots T_n^{\alpha_n}$ be a monomial of maximal degree appearing in P . We can of course suppose that its coefficient is equal to 1. Then $P(q_1 T_1, \dots, q_{i_0} T_{i_0}, T_{i_0+1} + h_{i_0}, \dots, T_n + h_n)$ is another polynomial which annihilates at x_1, \dots, x_n . It contains a monomial of higher degree of the form $q_1^{\alpha_1} \cdots q_{i_0}^{\alpha_{i_0}} T_1^{\alpha_1} \cdots T_n^{\alpha_n}$. Therefore

$$P(T_1, \dots, T_n) - q_1^{-\alpha_1} \cdots q_{i_0}^{-\alpha_{i_0}} P(q_1 T_1, \dots, q_{i_0} T_{i_0}, T_{i_0+1} + h_{i_0}, \dots, T_n + h_n)$$

also vanishes at x_1, \dots, x_n and contains less terms than P , against our assumptions on P . This proves that x_1, \dots, x_n are algebraically independent over \tilde{K} and hence that $\tilde{F} = \tilde{K}(x_1, \dots, x_n)$.

Now let us prove the proposition. The group $\text{Gal}^\Delta(\mathcal{M}_A)$ is the Δ -Galois group of $\sigma(Y) = AY$ considered as a σ -difference system with coefficients in \tilde{F} . Then by Proposition 1.10, there exists $\tilde{B}_i \in M_\nu(\tilde{F})$ such that

$$(1.5) \quad \begin{cases} \sigma(\tilde{B}_i) = A\tilde{B}_i A^{-1} + \partial_i(A)A^{-1} \\ \partial_i(\tilde{B}_j) + \tilde{B}_j \tilde{B}_i = \partial_j(\tilde{B}_i) + \tilde{B}_i \tilde{B}_j \end{cases} .$$

Now, we can replace the coefficients in \tilde{B}_i before the monomials in \underline{x} by indeterminates. From (1.5), we obtain the system

$$\begin{cases} \sigma(\bar{B}_i) = A\bar{B}_i A^{-1} + \partial_i(A)A^{-1} \\ \partial_i(\bar{B}_j) + \bar{B}_j \bar{B}_i = \partial_j(\bar{B}_i) + \bar{B}_i \bar{B}_j \end{cases} .$$

with indeterminate coefficients, which possess a specialization in \tilde{K} . By clearing the denominators and identifying the coefficients of the monomials in the transcendence basis \underline{x} , we see that this system is equivalent to a finite set of polynomial equations with coefficient in K . Since K is algebraically closed and this set of equations has a solution in \tilde{K} , it must have a solution in K . Thus, there exists $B_i \in M_\nu(F)$ with the required properties.

On the other hand, if there exists $B_i \in M_\nu(F)$ with the required commutativity properties, then it follows from Proposition 1.10 that $\text{Gal}^\Delta(\mathcal{M}_A)$ is conjugate over \tilde{K} to a constant Δ -group. \square

²We recall that we assume that the derivation of Δ and σ commute with each other.

Going back to the one derivative situation, we have:

Corollary 1.26. *We suppose that F/K a purely transcendental extension as in the previous proposition. Let $\sigma(Y) = AY$ be a σ -difference system with coefficients in F and let G_A be its ∂ -group scheme over K . The following statements are equivalent:*

1. G_A is conjugate over \tilde{K} to a constant ∂ -group.
2. There exists $B \in M_\nu(F)$ such that $\sigma(B) = ABA^{-1} + \partial(A)A^{-1}$, i.e. such that the system

$$\begin{cases} \sigma(Y) = AY \\ \partial Y = BY \end{cases}$$

is integrable.

Proof. Let us assume that G_A is conjugate over \tilde{K} to a constant ∂ -group. Since G_A becomes isomorphic to $\text{Gal}^{\partial}(\mathcal{M}_A)$ over \tilde{K} , the latter is conjugate over \tilde{K} to a constant ∂ -group and we can conclude by the proposition above. On the other hand, if there exists $B \in M_\nu(F)$ such that $\sigma(B) = ABA^{-1} + \partial(A)A^{-1}$, then it follows from Proposition 1.10 that $\text{Gal}^{\partial}(\mathcal{M}_A)$ is conjugate over \tilde{K} to a constant ∂ -group. Therefore the same holds for G_A . So we can apply the previous proposition. \square

2 Confluence and q -dependency

2.1 Differential Galois theory for q -dependency

Let k be a characteristic zero field, $k(q)$ the field of rational functions in q with coefficients in k and K a finite extension of $k(q)$. We fix an extension $|\cdot|$ to K of the q^{-1} -adic valuation on $k(q)$. This means that $|\cdot|$ is defined on $k[q]$ in the following way: there exists $d \in \mathbb{R}$, $d > 1$, such that $|f(q)| = d^{\deg_q(f)}$ for any $f \in k[q]$. It extends by multiplicativity to $k(q)$. By definition, we have $|q| > 1$ and therefore it makes sense to consider elliptic functions with respect to $|\cdot|$. So let $(C, |\cdot|)$ be the smallest valuated extension of $(K, |\cdot|)$ which is both complete and algebraically closed, $\text{Mer}(C^*)$ the field of meromorphic functions over $C^* := C \setminus \{0\}$ with respect to $|\cdot|$, i.e. the field of fractions of the analytic functions over C^* , and C_E the field of elliptic functions on the torus $C^*/q^{\mathbb{Z}}$, i.e. the subfield of $\text{Mer}(C^*)$ invariant with respect to the q -difference operators $\sigma_q : f(x) \mapsto f(qx)$.

Since the derivation $\delta_q = q \frac{d}{dq}$ is continuous on $k(q)$ with respect to $|\cdot|$, it naturally acts on the completion of K with respect to $|\cdot|$, and therefore on the completion of its algebraic closure, which coincides with C (see Chapter 3 in [Rob00]). It extends to $\text{Mer}(C^*)$ by setting $\delta_q x = 0$. The fact that $\delta_x = x \frac{d}{dx}$ acts on $\text{Mer}(C^*)$ is straightforward. We notice that

$$\begin{cases} \delta_x \circ \sigma_q = \sigma_q \circ \delta_x; \\ \delta_q \circ \sigma_q = \sigma_q \circ (\delta_x + \delta_q). \end{cases}$$

This choice of the derivations is not optimal, in the sense that we would like to have two derivations commuting each other and, more important, commuting to σ_q . We are going to reduce to this assumption in two steps. First of all, we consider the logarithmic derivative $\ell_q(x) = \frac{\delta_x(\theta_q)}{\theta_q}$ of the Jacobi Theta function:

$$\theta_q(x) = \sum_{n \in \mathbb{Z}} q^{-n(n-1)/2} x^n.$$

We recall that, if $|q| > 1$, the formal series θ_q naturally defines a meromorphic function on C^* and satisfies the the q -difference equation

$$\theta_q(qx) = qx\theta_q(x),$$

so that $\ell_q(qx) = \ell_q(x) + 1$. This implies that $\sigma_q \delta_x(\ell_q) = \delta_x(\ell_q)$ and hence that $\delta_x(\ell_q)$ is an elliptic function.

Lemma 2.1. *The following derivations of $\text{Mer}(C^*)$*

$$\begin{cases} \delta_x, \\ \delta = \ell_q(x)\delta_x + \delta_q. \end{cases}$$

commute with σ_q .

Proof. For δ_x , it is clear. For δ , we have:

$$\begin{aligned}
\delta \circ \sigma_q(f(q, x)) &= [\ell_q(x)\delta_x + \delta_q] \circ \sigma_q(f(q, x)) \\
&= \sigma_q \circ [(\ell_q(q^{-1}x) + 1)\delta_x + \delta_q] f(q, x) \\
&= \sigma_q \circ [\ell_q(x)\delta_x + \delta_q] f(q, x) \\
&= \sigma_q \circ \delta(f(q, x)).
\end{aligned}$$

□

Corollary 2.2.

1. The derivations δ_x, δ of $\text{Mer}(C^*)$ stabilize C_E in $\text{Mer}(C^*)$.
2. The field of constants $\text{Mer}(C^*)^{\delta_x, \delta}$ of $\text{Mer}(C^*)$ with respect to δ_x, δ is equal to the algebraic closure \bar{k} of k in C .

Proof. The first part of the proof immediately follows from the lemma above. The constants of $\text{Mer}(C^*)$ with respect to δ_x coincide with C . As far as the constants of C with respect to δ is concerned, we are reduced to determining the constants C^{δ_q} of C with respect to δ_q . Since the topology induced by $|\cdot|$ on k is trivial, one concludes that C^{δ_q} is the algebraic closure of k in C . □

Since

$$[\delta_x, \delta] = \delta_x \circ \delta - \delta \circ \delta_x = \delta_x(\ell_q(x))\delta_x,$$

we can consider a (δ_x, δ) -closure \tilde{C}_E of C_E (see [Yaf01], [Pie03] and [Sin07]). We extend σ_q to the identity of \tilde{C}_E . The (δ_x, δ) -field of σ_q -constants \tilde{C}_E almost satisfies the hypothesis of [HS08], apart from the fact that δ_x, δ do not commute with each other:

Lemma 2.3. *There exists $h \in \tilde{C}_E$ verifying the differential equation*

$$\delta(h) = \delta_x(\ell_q(x))h,$$

such that the derivations

$$\begin{cases} \partial_1 = h\delta_x; \\ \partial_2 = \delta = \ell_q(x)\delta_x + \delta_q. \end{cases}$$

commute with each other and with σ_q .

Proof. Since $\ell_q(qx) = \ell_q(x) + 1$, we have $\delta_x(\ell_q(x)) \in C_E$. Therefore, we are looking for a solution h of a linear differential equation of order 1, with coefficients in C_E . Let us suppose that $h \in \tilde{C}_E$ exists. Then since $\sigma_q h = h$, the identity $\sigma_q \circ \partial_i = \partial_i \circ \sigma_q$ follows from Lemma 2.1 for $i = 1, 2$. The verification of the fact that $\partial_1 \circ \partial_2 = \partial_2 \circ \partial_1$ is straightforward and, therefore, left to the reader.

We now prove the existence of h . Consider the differential rational function ring $\mathcal{S} := C_E\{y, \frac{1}{y}\}_{\delta_x}$. We endow \mathcal{S} with an extension of δ as follows:

$$\delta(\delta_x^n(y)) = \delta_x^{n+1}(\ell_q)y$$

for all $n \geq 0$. Since $\delta_x \circ \delta = \delta_x(\ell_q(x))\delta_x + \delta \circ \delta_x$, the definition of δ over \mathcal{S} is consistent and the commutativity relation between δ_x and δ extends from C_E to \mathcal{S} . Now let \mathfrak{M} be a maximal (δ_x, δ) -maximal ideal of \mathcal{S} . Then, the ring \mathcal{S}/\mathfrak{M} is a simple (δ_x, δ) - C_E -algebra. By Lemma 1.17 in [vdPS03], it is also an integral domain. Let L be the quotient field of \mathcal{S}/\mathfrak{M} . The field L is a (δ_x, δ) -field extension of C_E which contains a solution of the equation $\delta(y) = \delta_x(\ell_q(x))y$. Since \tilde{C}_E is the (δ_x, δ) -closure of C_E , there exists $h \in \tilde{C}_E$ satisfying the differential equation $\delta(h) = \delta_x(\ell_q(x))h$. □

Let $\Delta = \{\partial_1, \partial_2\}$. Notice that, since $\ell_q(qx) = \ell_q(x) + 1$ and σ_q commutes with Δ , we have:

$$\sigma_q \partial_1(\ell_q(x)) = \partial_1(\ell_q(x)), \quad \sigma_q(\partial_2(\ell_q(x))) = \partial_2(\ell_q(x)),$$

and therefore $\partial_i(\ell_q(x)) \in C_E$ for $i = 1, 2$. We conclude that the subfield $C_E(x, \ell_q(x))$ of $\text{Mer}(C^*)$ is actually a (σ_q, Δ) -field. Moreover, extending the action of σ_q trivially to \tilde{C}_E , we can consider the (σ_q, Δ) -field $\tilde{C}_E(x, \ell_q(x))$. Since the fields $C_E(x, \ell_q)$ and \tilde{C}_E are linearly disjoint over C_E (see Lemma 6.11 in [HS08]), $\tilde{C}_E(x, \ell_q(x))$ has a Δ -closed field of constants, which coincide with \tilde{C}_E .

2.2 Galois Δ -group and q -dependency

The subsection above shows that one can attach two linear differential algebraic groups to a q -difference system $\sigma_q(Y) = A(x)Y$ with $A \in GL_n(C(x))$:

1. The group $Gal^\Delta(\mathcal{M}_A)$ which corresponds to Definition 1.4 applied to the (σ_q, Δ) -field $\tilde{C}_E(x, \ell_q)$. This group is defined over \tilde{C}_E and measures all differential relations satisfied by the solutions of the q -difference equation with respect to δ_x and δ_q . However its computation may be a little difficult. Indeed, since the derivations of Δ are themselves defined above \tilde{C}_E , there is no hope of a general descent argument. Nonetheless, in some special cases, one can use the linear disjunction of the field \tilde{C}_E^Δ of Δ -constants and $C_E(x, \ell_q)$ above C_E to simplify the computations.
2. The Galois ∂_2 -group $Gal^{\partial_2}(\mathcal{M}_A)$ which corresponds to Definition 1.4 applied to the (σ_q, ∂_2) -field $\tilde{C}_E(x, \ell_q)$.

Let us consider the q -difference system

$$(2.1) \quad Y(qx) = A(x)Y(x), \text{ with } A \in GL_\nu(C_E(x, \ell_q)).$$

In view of Proposition 1.20, the Galois ∂_2 -group $Gal^{\partial_2}(\mathcal{M}_A)$ attached to (2.1) is defined above the algebraic closure \overline{C}_E of C_E . We will prove below that, in fact, it descends to C_E and thus reduce all the computations to calculus over the field of elliptic functions.

The field $C_E(x, \ell_q)$ is a subfield of the field of meromorphic functions over C^* , therefore (2.1) has a fundamental solution matrix $U \in GL_\nu(\text{Mer}(C^*))$. In fact, the existence of such a fundamental solution U is actually equivalent to the triviality of the pull-back on C^* of vector bundles over the torus $C^*/q^\mathbb{Z}$ (see [Pra86] for an analytic argument). The (σ_q, ∂_2) -ring $\mathcal{R}_{\text{Mer}} := C_E(x, \ell_q)\{U, \det U^{-1}\}_{\partial_2} \subset \text{Mer}(C^*)$ is generated as a (σ_q, ∂_2) -ring by a fundamental solution U of (2.1) and by $\det U^{-1}$ and has the property that $\mathcal{R}_{\text{Mer}}^{\sigma_q} = C_E$, in fact:

$$C_E \subset \mathcal{R}^{\sigma_q} \subset \text{Mer}(C^*)^{\sigma_q} = C_E.$$

Notice that \mathcal{R}_{Mer} does not need to be a simple (σ_q, ∂_2) -ring. For this reason we call it a weak ∂_2 -Picard-Vessiot ring. We have:

Proposition 2.4. *$Aut^{\sigma_q, \partial_2}(\mathcal{R}_{\text{Mer}}/C_E(x, \ell_q))$ consists of the C_E -points of a linear algebraic ∂_2 -group G_{C_E} defined over C_E such that $G_{C_E} \otimes_{C_E} \overline{C}_E \cong Gal^{\partial_2}(\mathcal{M}_A)$.*

Proof. See the proof of Theorem 9.5 in [DVH10a], which gives an analogous statement for the derivation $x \frac{d}{dx}$. \square

Therefore, as in Proposition 1.9, one can prove:

Corollary 2.5. *Let $a_1, \dots, a_n \in C_E(x, \ell_q)$ and let $S \subset \text{Mer}(C^*)$ be a (σ_q, ∂_2) -extension of $C_E(x, \ell_q)$ such that $S^{\sigma_q} = C_E$. If $z_1, \dots, z_n \in S$ satisfy $\sigma(z_i) - z_i = a_i$ for $i = 1, \dots, n$, then $z_1, \dots, z_n \in S$ satisfy a nontrivial ∂_2 -relation over $C_E(x, \ell_q)$ if and only if there exists a nonzero homogeneous linear differential polynomial $L(Y_1, \dots, Y_n)$ with coefficients in C_E and an element $f \in C_E(x, \ell_q)$ such that $L(a_1, \dots, a_n) = \sigma(f) - f$.*

Proof. The proof strictly follows Proposition 3.1 in [HS08], so we only sketch it. First of all, if such an L exists, $L(z_1, \dots, z_n) - f$ is σ_q -invariant and therefore belongs to C_E , which gives a nontrivial ∂_2 -relation among the z_1, \dots, z_n . On the other hand, we can suppose that S coincides with the ring \mathcal{R}_{Mer} introduced above for the system $\sigma_q Y = AY$, where A is the diagonal block matrix

$$A = \text{diag} \left(\begin{pmatrix} 1 & a_1 \\ 0 & 1 \end{pmatrix}, \dots, \begin{pmatrix} 1 & a_n \\ 0 & 1 \end{pmatrix} \right).$$

In fact, a solution matrix of $\sigma_q Y = AY$ is given by

$$U = \text{diag} \left(\begin{pmatrix} 1 & z_1 \\ 0 & 1 \end{pmatrix}, \dots, \begin{pmatrix} 1 & z_n \\ 0 & 1 \end{pmatrix} \right) \in GL_{2n}(\text{Mer}(C^*)).$$

This implies that $Aut^{\sigma_q, \partial_2}(\mathcal{R}_{\text{Mer}}/C_E(x, \ell_q))$ is a ∂_2 -subgroup of the group $(C_E, +)^n$, therefore there exists a nonzero homogeneous linear differential polynomial $L(Y_1, \dots, Y_n)$ with coefficients in C_E such that $Aut^{\sigma_q, \partial_2}(\mathcal{R}_{\text{Mer}}/C_E(x, \ell_q))$ is contained in the set of zeros of L in $(C_E, +)^n$. We set $f = L(z_1, \dots, z_n)$. A Galoisian argument shows that $f \in C_E(x, \ell_q)$ and that $L(a_1, \dots, a_n) = \sigma_q(f) - f$. \square

2.3 Galoisian approach to heat equation

We want to show how the computation of the Galois ∂_2 -group of the q -difference equation $y(qx) = qxy(x)$ leads to the heat equation. We recall that the Jacobi theta function verifies $\theta_q(qx) = qx\theta_q(x)$. Corollary 2.5 applied to this equation becomes: the function θ_q satisfies a ∂_2 -relation with coefficients in $C_E(x, \ell_q)$ if and only if there exist $a_1, \dots, a_m \in C_E$ and $f \in C_E(x, \ell_q)$ such that

$$\sum_{i=0}^m a_i \partial_2^i \left(\frac{\partial_2(qx)}{qx} \right) = \sigma_q(f) - f.$$

A simple computation leads to

$$\frac{\partial_2(qx)}{qx} = \ell_q + 1 = \sigma_q \left(\frac{1}{2} (\ell_q^2 + \ell_q) \right) - \left(\frac{1}{2} (\ell_q^2 + \ell_q) \right)$$

and therefore to

$$\sigma_q \left(2 \frac{\partial_2(\theta_q)}{\theta_q} - (\ell_q^2 + \ell_q) \right) = 2 \frac{\partial_2(\theta_q)}{\theta_q} - (\ell_q^2 + \ell_q).$$

The last identity is equivalent to the fact that

$$(2.2) \quad 2 \frac{\partial_2(\theta_q)}{\theta_q} - (\ell_q^2 + \ell_q) = 2 \frac{\delta_q(\theta_q)}{\theta_q} + \ell_q^2 - \ell_q$$

is an elliptic function and implies that

$$(2.3) \quad \text{Gal}^{\partial_2}(\mathcal{M}_{qx}) \subset \left\{ \frac{\partial_2(c)}{c} = 0 \right\}.$$

The heat equation³

$$(2.4) \quad 2\delta_q\theta_q = -\delta_x^2\theta_q + \delta_x\theta_q$$

can be rewritten as

$$2 \frac{\delta_q\theta_q}{\theta_q} + \ell_q^2 - \ell_q = -\delta_x(\ell_q).$$

Since $\delta_x(\ell_q)$ is an elliptic function, taking into account (2.2), we see that the Galois ∂_2 -group of θ_q , as described in (2.3), can somehow be viewed as a Galoisian counterpart of the heat equation.

2.4 q -hypertranscendancy of rank 1 q -difference equations

We want to study the q -dependency of the solutions of a q -difference equation of the form $y(qx) = a(x)y(x)$, where $a(x) \in k(q, x)$, $a(x) \neq 0$.

Theorem 2.6. *Let u be a nonzero meromorphic solution of $y(qx) = a(x)y(x)$, $a(x) \in k(q, x)$, in the sense of the previous subsections. The following statements are equivalent:*

1. $a(x) = \mu x^r \frac{g(qx)}{g(x)}$ for some $r \in \mathbb{Z}$, $g \in k(q, x)$ and $\mu \in k(q)$.
2. u is a solution of a nontrivial algebraic δ_x -relation with coefficients in $C_E(x, \ell_q)$ (and therefore in $C(x)$).
3. u is a solution of a nontrivial algebraic ∂_2 -relation with coefficients in $C_E(x, \ell_q)$.

First of all, we remark that the equivalence between 1. and 2. follows from Theorem 1.1 in [Har08] (replacing \mathbb{C} by $k(q)$ is not an obstacle in the proof). Moreover being δ_x -algebraic over $C_E(x, \ell_q)$ or over $C(x)$ is equivalent, since $C_E(\ell_q)$ is δ_x -algebraic over $C(x)$. The equivalence between 1. and 3. is proved in the proposition below:

Proposition 2.7. *Let u be a nonzero meromorphic solution of $y(qx) = a(x)y(x)$, with $a(x) \in k(q, x)$. Then u satisfies an algebraic differential equation with respect to ∂_2 with coefficients in $C_E(x, \ell_q)$ if and only if $a(x) = \mu x^r \frac{g(qx)}{g(x)}$ for some $r \in \mathbb{Z}$, $g \in k(q, x)$ and $\mu \in k(q)$.*

³One deduces from (2.4) that $\tilde{\theta}(q, x) := \theta_q(q^{1/2}x)$ satisfies the equation $2\delta_q\tilde{\theta} = \delta_x^2\tilde{\theta}$. Then, to recover the classical form of the heat equation, it is enough make the change of variables $q = \exp(-2i\pi\tau)$ and $x = \exp(2i\pi z)$.

Proof. By Lemma 3.3 in [Har08], there exists $f(x) \in k(q, x)$ such that $a(x) = \tilde{a}(x) \frac{f(qx)}{f(x)}$ and $\tilde{a}(x)$ has the property that if α is a zero (resp. a pole) of $\tilde{a}(x)$, then $q^n \alpha$ is neither a zero nor a pole of $\tilde{a}(x)$ for any $n \in \mathbb{Z} \setminus \{0\}$. Replacing u by $\frac{u}{f(x)}$, we can suppose that $a(x) = \tilde{a}(x)$ and we can write $a(x)$ in the form:

$$a(x) = \mu x^r \prod_{i=1}^s (x - \alpha_i)^{l_i},$$

where $\mu \in k(q)$, $r \in \mathbb{Z}$, $l_1, \dots, l_s \in \mathbb{Z}$ and, for all $i = 1, \dots, s$, the α_i 's are nonzero elements of a fixed algebraic closure of $k(q)$, such that $q^{\mathbb{Z}} \alpha_i \cap q^{\mathbb{Z}} \alpha_j = \emptyset$ if $i \neq j$. By Corollary 2.5, the solutions of $y(qx) = a(x)y(x)$ will satisfy a nontrivial algebraic differential equation in ∂_2 if and only if there exists $f \in C_E(x, \ell_q)$, $a_1, \dots, a_m \in C_E$ such that

$$\sum_{i=0}^m a_i \partial_2^i \left(\frac{\partial_2(a(x))}{a(x)} \right) = f(qx) - f(x).$$

We can suppose that $a_m = 1$. Notice that

$$\frac{\partial_2(a(x))}{a(x)} = \frac{\partial_2(x^r)}{x^r} + \frac{\delta_q(\mu)}{\mu} + \sum_{i=1}^s \frac{\partial_2(x - \alpha_i)^{l_i}}{(x - \alpha_i)^{l_i}},$$

where

$$\frac{\partial_2(x^r)}{x^r} = r \ell_q(x) = \left(\frac{r}{2} (\ell_q^2 - \ell_q) \right) (qx) - \left(\frac{r}{2} (\ell_q^2 - \ell_q) \right) (x), \text{ with } \ell_q^2 - \ell_q \in C_E(x, \ell_q),$$

and

$$\frac{\delta_q(\mu)}{\mu} = \left(\frac{\delta_q(\mu)}{2\mu} \ell_q \right) (qx) - \left(\frac{\delta_q(\mu)}{2\mu} \ell_q \right) (x), \text{ with } \frac{\delta_q(\mu)}{2\mu} \ell_q \in C_E(x, \ell_q).$$

It remains to show that a solution of $y(qx) = a(x)y(x)$ satisfies a nontrivial differential equation in ∂_2 if and only if there exists $h \in C_E(x, \ell_q)$ such that

$$(2.5) \quad \sum_{j=0}^m a_j \partial_2^j \left(\sum_{i=1}^s \frac{l_i (x \ell_q(x) - \delta_q(\alpha_i))}{(x - \alpha_i)} \right) = h(qx) - h(x).$$

If we prove that (2.5) never holds, we can conclude that a solution of $y(qx) = a(x)y(x)$ satisfies a nontrivial algebraic differential equation in ∂_2 with coefficients in $C_E(x, \ell_q)$ if and only if $a(x) = \mu x^r$ (modulo the reduction done at the beginning of the proof). For all $i = 1, \dots, s$ and $j = 0, \dots, m$, the fact that $\partial_2 \ell_q(x) \in C_E$ allows to prove inductively that:

$$(2.6) \quad \partial_2^j \left(\frac{l_i (x \ell_q(x) - \delta_q(\alpha_i))}{(x - \alpha_i)} \right) = \frac{l_i (-1)^j j! (x \ell_q(x) - \delta_q(\alpha_i))^{j+1}}{(x - \alpha_i)^{j+1}} + \frac{h_{i,j}}{(x - \alpha_i)^j},$$

for some $h_{i,j} \in C_E[\ell_q]$.

Since x is transcendental over $C_E(\ell_q)$, we can consider $f(x) = \sum_{j=0}^m a_j \partial_2^j \left(\sum_{i=1}^s \frac{l_i (x \ell_q(x) - \delta_q(\alpha_i))}{(x - \alpha_i)} \right)$ as a rational function in x with coefficients in $C_E(\ell_q)$. In the partial fraction decomposition of $f(x)$, the polar term in $\frac{1}{(x - \alpha_i)}$ of highest order is $\frac{l_i (-1)^m m! (x \ell_q(x) - \delta_q(\alpha_i))^{m+1}}{(x - \alpha_i)^{m+1}}$. By the partial fraction decomposition theorem, identities (2.5) and (2.6) imply that this last term appears either in the decomposition of $h(x)$ or in the decomposition of $h(qx)$. In both cases, there exists $s \in \mathbb{Z}^*$ such that the term $\frac{1}{(q^s x - \alpha_i)^{m+1}}$ appears in the partial fraction decomposition of $h(qx) - h(x)$. This is in contradiction with the assumption that the poles α_i of $a(x)$ satisfy $q^{\mathbb{Z}} \alpha_i \cap q^{\mathbb{Z}} \alpha_j = \emptyset$ if $i \neq j$. \square

Remark 2.8. The Jacobi theta function is an illustration of the theorem above. Notice that a meromorphic solution of the q -difference equations of the form $y(qx) = a(x)y(x)$, with $a(x) = \mu x^r \frac{g(qx)}{g(x)} \in k(q, x)$ is given by $y(x) = \frac{\theta_q(\mu x/q^r)}{\theta_q(x)} \theta(x)^r g(x) \in \text{Mer}(C^*)$.

2.5 Integrability in x and q

Let $Y(qx) = AY(x)$ be a q -difference system with $A \in GL_\nu(C(x, \ell_q))$ and $\Delta = \{\partial_1, \partial_2\}$ as in Lemma 2.3. We deduce the following from Proposition 1.24.

Corollary 2.9. *The Galois Δ -group $Gal^\Delta(\mathcal{M}_A)$ of $Y(qx) = AY(x)$ is Δ -constant (see Remark 1.11) if and only if there exist two square matrices $B_1, B_2 \in M_\nu(\overline{C}_E(x, \ell_q))$ such that the system*

$$\begin{cases} \sigma_q(Y) = AY(x) \\ \delta_x Y = B_1 Y \\ \partial_2 Y = B_2 Y \end{cases}$$

is integrable, in the sense that these matrices verify:

$$(2.7) \quad \sigma_q(B_1) = AB_1 A^{-1} + \delta_x(A).A^{-1},$$

$$(2.8) \quad \sigma_q(B_2) = AB_2 A^{-1} + \partial_2(A).A^{-1},$$

$$(2.9) \quad \partial_2(B_1) + B_1 B_2 + \delta_x(\ell_q)B_1 = \delta_x(B_2) + B_2 B_1.$$

Proof. It follows from Proposition 1.10 that there exist $B_1, B_2 \in M_\nu(\tilde{C}_E(x, \ell_q))$ such that the system

$$\begin{cases} \sigma_q(Y) = AY(x) \\ \partial_1 Y = hB_1 Y \\ \partial_2 Y = B_2 Y \end{cases}$$

is integrable. This is easily seen to be equivalent to (2.7), (2.8) and (2.9). Notice that, as in Proposition 1.24, we have:

- x and ℓ_q are transcendental and algebraically independent over \overline{C}_E (see for instance Theorem 3.6 in [Har08]);
- $\sigma_q(x) = qx$ and $\sigma_q(\ell_q) = \ell_q + 1$.

It follows that $\overline{C}_E(x, \ell_q) \otimes_{\overline{C}_E} \tilde{C}_E = \tilde{C}_E(x, \ell_q)$ and that, therefore, the same argument as in the proof of Proposition 1.24 allows to conclude. \square

Example 2.10. We want to study the q -dependency of the q -difference system

$$(2.10) \quad Y(qx) = \begin{pmatrix} \lambda & \eta \\ 0 & \lambda \end{pmatrix} Y(x),$$

where $\lambda, \eta \in k(q, x)$. First of all, the solutions of the equation $y(qx) = \lambda y(x)$ will admit a Δ -relation if and only if $\lambda = \mu x^r \frac{g(qx)}{g(x)}$ for some $r \in \mathbb{Z}$, $g \in k(q, x)$ and $\mu \in k(q)$ (see Theorem 2.6). To simplify the exposition we can suppose that $\lambda = \mu x^r$. Then clearly the Galois Δ -group of $y(qx) = \lambda y(x)$ is Δ -constant if and only if $\mu \in k(q)$ is a power of q , *i.e.* if and only if $\mu \in q^{\mathbb{Z}}$. Let $\mu = q^s$ for $s \in \mathbb{Z}$. A solution of $y(qx) = \lambda y(x)$ is given by

$$y(x) = x^{s-r} \theta_q(x)^r.$$

To obtain an integrable system

$$y(qx) = \mu x^r y(x), \quad \delta_x y = b_1 y, \quad \partial_2 y = b_2 y,$$

satisfying (2.7), (2.8) and (2.9), it is enough to take:

$$b_1 = \frac{\delta_x(y)}{y}, \quad b_2 = \frac{\partial_2(y)}{y}.$$

If $\mu = q^s \in q^{\mathbb{Z}}$ then $b_1 = r\ell_q(x) + s - r \in C_E(x, \ell_q(x))$. The same hypothesis on μ implies that $b_2 \in C_E(x, \ell_q(x))$, in fact we have:

$$\sigma_q \left(\frac{\partial_2(\theta_q(x))}{\theta_q(x)} \right) = (\ell_q(x) + 1) + \frac{\partial_2(\theta_q(x))}{\theta_q(x)},$$

which implies that

$$\frac{\partial_2(\theta_q(x))}{\theta_q(x)} = \frac{\ell_q(x)(\ell_q(x) + 1)}{2} + e(x), \text{ for some } e(x) \in C_E.$$

To go back to the initial system, we have to find

$$B_1 = \begin{pmatrix} b_1 & \alpha(x) \\ 0 & b_1 \end{pmatrix}, B_2 = \begin{pmatrix} b_2 & \beta(x) \\ 0 & b_2 \end{pmatrix} \in M_2(\overline{C}_E(x, \ell_q)),$$

satisfying (2.7), (2.8) and (2.9). We find that the q -difference system (2.10) has constant Galois Δ -group if and only if there exist $\alpha(x), \beta(x) \in \overline{C}_E(x, \ell_q)$ such that

- $\delta_x(\eta) = r\eta + (\alpha(qx) - \alpha(x))\mu x^r$;
- $\delta_q(\eta) = \frac{\delta_{q\mu}}{\mu}\eta + (\beta(qx) - \beta(x) + \ell_q(\alpha(qx) - \alpha(x)))\mu x^r$.
- $\partial_2(\alpha(x)) + \delta_x(\ell_q)\alpha(x) = \delta_x(\beta(x))$.

Remark 2.11. In [Pul08], Pulita shows that given a p -adic differential equations $\frac{dY}{dx} = GY$ with coefficients in some classical algebras of functions, or a q -difference equations $\sigma_q(Y) = AY$, one can always complete it in an integrable system:

$$\begin{cases} \sigma_q(Y) = AY \\ \frac{dY}{dx} = GY \\ \frac{dY}{dq} = 0 \end{cases}.$$

This is not the case in the complex framework, even extending the coefficients from $C_E(\ell_q, x)$ to $\mathcal{M}er(C^*)$.

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