Introduction to p-adic q-difference equations

(weak Frobenius structure and transfer theorems)

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Abstract. Inspired by the theory of p-adic differential equations, this paper introduces an analogous theory for q-difference equations over a local field, when $|q| = 1$. We define some basic concepts, for instance the *generic radius of convergence*, introduce technical tools, such as a *twisted Taylor formula* for analytic functions, and prove some fundamental statements, such as an *effective bound theorem*, the *existence of a weak Frobenius structure* and a *transfer theorem* in regular singular disks.

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Introduction

Since the late 1940's, q-difference equations have been almost forgotten. In the last ten years the field has recovered its original vitality and the theory has witnessed substantial advances. Authors have also considered these functional equations both from an arithmetical and a p-adic point of view (cf. for instance [BB92], [And00b] or [DV02]). This paper seeks to fill a gap in the literature, offering a systematic introduction to p-adic q-difference equations *when* $|q|=1$.

Our motivation for this work was Sauloy's result on q-deformations of the local monodromy of fuchsian differential systems over $\mathbb{P}^1_{\mathbb{C}}$ (cf. [Sau00a] and, for a survey on the topic, see [And02b], [DVRSZ03]). Sauloy considers a fuchsian differential system

$$(\mathcal{S}) \qquad x\frac{dY}{dx}=G(x)Y(x)\ ,$$

such that $G(x)\in M_\mu(\mathbb{C}(x))$. More precisely he supposes that the matrix $G(x)$ has no poles at 0 and ∞, that the difference of any two eigenvalues of $G(0)$ (resp. $G(\infty)$) is not a non-zero integer and that all the poles $x_1,\dots,x_s$ of $G(x)$ are simple. Under these assumptions one can choose $q\in\mathbb{C}$, with $|q|\neq 1$, such that $x_iq^{\mathbb{R}}\cap x_jq^{\mathbb{R}}=\emptyset$ for every $i\neq j$ and construct for any $0<\varepsilon<<1$ a q^ε-difference system of the form

$$(\mathcal{S})_{q^\varepsilon} \qquad Y(q^\varepsilon x)=\Big[\mathbb{I}_\mu+(q^\varepsilon-1)G_\varepsilon(x)\Big]Y(x)\ ,$$

such that the matrix $G_\varepsilon(x) \in M_\mu(\mathbb{C}(x))$ converges uniformly to $G(x)$ over $\Omega = \mathbb{C}^* \smallsetminus \left(\cup_i x_i q^{\mathbb{R}} \cup q^{\mathbb{R}}\right)$. One can suppose, for instance by taking $G_\varepsilon(x) = G(x)$ for all ε, that there exists a matrix $Q_\varepsilon \in Gl_\mu(\mathbb{C})$ such that $Q = \lim_{\varepsilon\to\infty} Q_\varepsilon$ and both $Q_\varepsilon G_\varepsilon(0) Q_\varepsilon^{-1}$ and $QG(0)Q^{-1}$ are upper triangular matrices, and that an analogous hypothesis is verified at ∞. Then, for ε sufficiently close to 0, Sauloy constructs two canonical solutions $Y_{\varepsilon,0}(x)$ and $Y_{\varepsilon,\infty}(x)$ of $(\mathcal{S})_q$, respectively in a neighborhood of 0 and ∞: they both turn out to be meromorphic on $\mathbb{C}^*$. Therefore the Birkhoff matrix $P_\varepsilon(x) = Y_{\varepsilon,\infty}(x)^{-1} Y_{\varepsilon,0}(x)$ is also meromorphic over $\mathbb{C}^*$ and moreover it is elliptic: $P_\varepsilon(x) = P_\varepsilon(q^\varepsilon x)$.

Theorem 0.1. [Sau00a, §4] *The matrix $P_\varepsilon(x)$ tends to a locally constant matrix $P(x)$ over Ω, when $\varepsilon \to 0$. Let P' and P'' be the values of $P(x)$ over the two connected components of Ω whose closure contains the pole x_i of $G(x)$. Then the local monodromy of $(\mathcal{S})$ around x_i is generated by $P'P''^{-1}$.*

André, Kedlaya and Mebkhout have independently proved a *Crew's p-adic monodromy conjecture*: due to the lack of analytic continuation the problem of p-adic monodromy is much more complicated than the complex theory, therefore we think it would be interesting to study the properties of q-deformations in the p-adic setting .

There is a fundamental difference between complex and p-adic q-deformations: while in $\mathbb{C}$ one can let $q \to 1$ avoiding the unitary circle, this is not possible in the p-adic world. In other words, to study p-adic q-deformations of differential equations one has to deal with the case $|q| = 1$.

We are emphasizing the "$|q| = 1$", since the literature is almost entirely devoted to the case $|q| \neq 1$, to avoid the small divisor problem. In fact, consider a q-difference equation

$$(\mathcal{E})_q \qquad y(q^\mu x) + a_{\mu-1}(x) y(q^{\mu-1}x) + \cdots + a_0(x) y(x) = 0 \ ,$$

where q is an element of a normed field, archimedean or not, such that $|q| \neq 1$, and the $a_i(x)$'s are rational functions in $K(x)$. Then any convergent solution $\sum_{n\geq 0} a_n x^n$ of $(\mathcal{E})_q$ is the expansion at 0 of a meromorphic function over $\mathbb{A}^1_K$, since the equation itself allows for a meromorphic continuation of the solution. Moreover, the Adams lemma states that even the most irregular q-difference equations always have at least one solution whose uniform part is analytic at zero (cf. [Ada29] and [Sau02b, 1.2.6]). On the other hand, when $|q| = 1$, one has to deal with the problem of estimating terms of the form $1 - q^n$ (the so called small divisors problem), which can make the prediction of the existence of a convergent solution of $(\mathcal{E})_q$ very difficult.

We point out that we are not distinguishing between the archimedean and the ultrametric case: while the differential equation theory gives rise to two substantially different theories in the p-adic and the complex framework, the dichotomy in the q-difference world is given by the cases $|q| = 1$ and $|q| \neq 1$.

cf. Y. André and L. Di Vizio, "q-difference equations and p-adic local monodromy", preprint.

Nowadays, the complex theory of q-difference equations for $|q| \neq 1$ has reached a "degree of completeness" comparable to differential equation theory, as Birkhoff and Guenther hoped [BG41]. Moreover it seems that those results should also be true in the p-adic case with similar proofs (cf. for instance [Béz92] versus [BB92] and [Sau00b, I, 2.2.4]). In the meantime very few pages are devoted to the study of the case $|q| = 1$, which, apart from the small divisor problem, is characterized by essentially two difficulties:

1) The meromorphic continuation of solutions does not work any longer, so one needs a good notion of solution at a point $\xi \neq 0$. This is a key-point of the p-adic approach, since we cannot imagine a p-adic theory of q-difference equations without an analogue of the notion of Dwork-Robba generic point and generic radius of convergence.

2) Sauloy constructs his canonical solutions using the classical Theta function

$$\Theta(x) = \sum_{n \in \mathbb{Z}} q^{-\frac{n(n-1)}{2}} x^n \ .$$

Of course it does not converge for $|q| = 1$. Moreover all the infinite products, which play such an important role in the theory of q-series (cf. for instance [GR90]) and would be very useful to write meromorphic solutions, do not converge either.

The lesson of the results contained in this paper is that p-adic q-difference equations with $|q| = 1$ present the same pathologies as p-adic differential equations, namely the uniform part of solutions at a regular singular points can be divergent, according to the type of the exponents. In fact, the q-difference theory is precisely a q-deformation of the differential situation. Since p-adic differential equation theory has geometrical implications, this allows one to imagine that p-adic q-difference equations should have some "non commutative p-adic geometric implications" (cf. [And02a]).

* * *

Concerning the content of the present work, we have chosen to introduce the notions of q-difference algebra as required by the paper. Anyway a systematic presentation with highly compatible notation can be found in [DV02, §1]. The paper is organized as follows:

Chapter I is an introduction to basic tools. First of all we study the properties of the q-difference algebra of analytic functions over a disk and the properties of q-difference operators with respect to the so called Gauss norms. We also state a result, proved in Appendix A, about the existence of a q-expansion of analytic functions, which we use to establish a good notion of solutions of a q-difference equation at a point $\xi \neq 0, \infty$. In particular this allows for the definition of generic radius of convergence at a Dwork-Robba generic point.

In Chapter II we prove an effective bound theorem in the wake of the Dwork-Robba theorem, from which we deduce a transfer theorem in ordinary q-orbits and a corollary on the q-deformations of p-adic differential equations.

Following Christol [Chr84], in Chapter III we construct a weak Frobenius structure for q-difference systems having a regular singularity at 0.

In Chapter IV we prove a q-analogue of the Christol-André-Baldassarri-Chiarellotto transfer theorem in a regular singular disk, relying on the result of Chapter III.

The Appendix is divided in three independent parts.

Appendix A contains a proof the twisted Taylor formula stated in §1 for analytic functions over a disk and a generalization to analytic functions over non connected analytic domains. The proof is completely elementary and the techniques used in it do not play any role in the paper.

In Appendix B we quickly recall some basic facts about regular singular q-difference systems that we use in Chapters III and IV.

Finally in Appendix C we have grouped some technical estimates of the q-type, which are used in Chapter IV.

I. Basic definitions and properties of p-adic q-difference systems

Until the end of the paper K will be an algebraically closed field of characteristic zero, complete with respect to a non-archimedean norm $|\ |$, inducing a p-adic norm over $\mathbb{Q} \hookrightarrow K$. We fix a normalization of $|\ |$ by setting $|p| = p^{-1}$. Moreover, we fix an element $q \in K$, such that

1. $|q| = 1$;
2. q is not a root of unity;
3. the image of q in the residue field of K generates a finite cyclic group (i.e. it is algebraic over $\mathbb{F}_p$).

Let us consider the ring

$$\mathcal{A}_D = \left\{ \sum_{n\geq 0} a_n (x-\xi)^n : a_n \in K, \liminf_{n\to\infty} |a_n|^{-1/n} \geq \rho \right\} ,$$

of analytic functions (with coefficients in K) over the open disk

$$D = D(\xi, \rho^-) = \{x \in K : |\xi - x| < \rho\}$$

of center $\xi \in \mathbb{A}^1_K$ and radius $\rho \in \mathbb{R}$, $\rho > 0$, and the field of meromorphic functions $\mathcal{M}_D = Frac(\mathcal{A}_D)$ over D. Sometimes we will write $\mathcal{A}_{D,K}$ (resp. $\mathcal{M}_{D,K}$) to stress the fact that we are considering analytic (resp. meromorphic) functions with coefficients in K.

If D is q-invariant (i.e. if D is invariant for the isometry $x \longmapsto qx$, or, equivalently, if $|(q-1)\xi| < \rho$), it makes sense to consider a *q-difference systems (of order μ, with meromorphic coefficients over D, defined over K)*:

$$(\mathcal{S})_q \qquad Y(qx) = A(x)Y(x)\ ,\ \text{with } A(x) \in Gl_\mu(\mathcal{M}_{D,K}).$$

The main purpose of this chapter is to introduce the basic properties of q-difference systems. First we study the properties of the q-difference algebra of analytic functions and of q-difference operators with respect to Gauss norms. Then we construct analytic solutions of q-difference systems, when they exist. Finally, we define the notions of generic point and generic radius of convergence.

1. The q-difference algebra of analytic functions over an open disk.

Let the disk $D = D(\xi, \rho^-)$ be q-invariant. Then the *q-difference operator*

$$\sigma_q : f(x) \longmapsto f(qx)$$

is a K-algebra isomorphism of $\mathcal{A}_D$: we say that $\mathcal{A}_D$ is a *q-difference algebra*.

One can also define a *q-derivation*

$$d_q(f)(x) = \frac{f(qx) - f(x)}{(q-1)x}\ ,$$

satisfying the twisted Leibniz rule

$$d_q(fg) = \sigma_q(f)d_q(g) + d_q(f)g\ .$$

Lemma 1.1. *The operator d_q acts over $\mathcal{A}_D$.*

Proof. Let $0 \notin D$. Then by definition $d_q(f) \in \mathcal{A}_D$ for all $f \in \mathcal{A}_D$.

If $0 \in D$, we can suppose $\xi = 0$. Observe that

$$(1.1.1) \qquad d_q x^n = \left(1 + q + \cdots + q^{n-1}\right) x^{n-1}\ \forall n \geq 1.$$

Then for any $\sum_{n\geq 0} a_n x^n \in \mathcal{A}_D$ we have:

$$d_q\left(\sum_{n\geq 0} a_n x^n\right) = \sum_{n\geq 1}\left(1 + q + \cdots + q^{n-1}\right) a_n x^{n-1} \in \mathcal{A}_D\ .$$

□

Motivated by (1.1.1), we recall the classical definition of *q-factorials* and *q-binomial coefficients*, namely for any pair of integers $n \geq i \geq 0$ we set:

$$[0]_q = 0 \ , \ [n]_q = 1 + q + \cdots + q^{n-1} = \tfrac{1-q^n}{1-q} \ ,$$

$$[0]_q^! = 1 \ , [n]_q^! = 1_q \cdots [n]_q \ ,$$

$$\binom{n}{0}_q = \binom{n}{n}_q = 1 \ , \ \binom{n}{i}_q = \frac{[n]_q^!}{[n-i]_q^![i]_q^!} = \frac{[n]_q[n-1]_q \cdots [n-i+1]_q}{[i]_q^!} \ .$$

They satisfy the relation

$$\binom{n}{i}_q = \binom{n-1}{i-1}_q + \binom{n-1}{i}_q q^i = \binom{n-1}{i-1}_q q^{n-i} + \binom{n-1}{i}_q \ , \ \text{for } n \geq i \geq 1,$$

and

$$(1.1.2) \qquad (1-x)(1-qx)\cdots(1-q^{n-1}x) = \sum_{i=0}^{n} (-1)^i \binom{n}{i}_q q^{\frac{i(i-1)}{2}} x^i \ .$$

One verifies directly the following basic properties of σ_q and d_q:

Lemma 1.2. *For any pair integers $n, i \geq 1$ and any $f, g \in \mathcal{A}_D$ we have:*

$$(1.2.1) \ \frac{d_q^i}{[i]_q^!} x^n = \begin{cases} \binom{n}{i}_q x^{n-i}, & \textit{if } n \geq i, \\ 0, & \textit{otherwise;} \end{cases}$$

$$(1.2.2) \ d_q^n(fg)(x) = \sum_{j=0}^{n} \binom{n}{j}_q d_q^{n-j}(f)(q^j x) d_q^j(g)(x);$$

$$(1.2.3) \ \sigma_q^n = \sum_{j=0}^{n} \binom{n}{j}_q (q-1)^j q^{\frac{j(j-1)}{2}} x^j d_q^j = \sum_{j=0}^{n} \prod_{i=0}^{j-1} (q^n - q^i) x^j \frac{d_q^j}{[j]_q^!};$$

$$(1.2.4) \ d_q^n = \frac{\prod_{i=0}^{j-1}(\sigma_q - q^i)}{(q-1)^n q^{\frac{n(n-1)}{2}} x^n} = \frac{(-1)^n}{(q-1)^n x^n} \sum_{j=0}^{n} (-1)^j \binom{n}{j}_{q^{-1}} q^{-\frac{j(j-1)}{2}} \sigma_q^j;$$

$$(1.2.5) \ (x d_q)^n = \frac{1}{(q-1)^n} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \sigma_q^j.$$

Proof. The first four formulas are proved in [DV02, (1.1.8) through (1.1.10)]. The proof of (1.2.5) is straightforward:

$$(x d_q)^n = \left(\frac{\sigma_q - 1}{q-1}\right)^n = \frac{1}{(q-1)^n} \sum_{i=0}^{n} (-1)^i \binom{n}{i} \sigma_q^i \ .$$

□

1.3. The topological basis $((x-\xi)^n)_{n\geq 0}$ of $\mathcal{A}_D$ is not adapted to study the action of the q-derivation over $\mathcal{A}_D$ as the relation

$$d_q(x-\xi)^n = \frac{(qx-\xi)^n-(x-\xi)^n}{(q-1)x}$$

clearly shows. So, rather than $(x-\xi)^n$, one classically consider the polynomials

$$\begin{array}{l}(x-\xi)_{q,0}=1\ ,\\ (x-\xi)_{q,n}=(x-\xi)(x-q\xi)\cdots(x-q^{n-1}\xi),\ \text{for any integer } n\geq 1,\end{array}$$

which satisfy the formula (cf. (1.1.1))

$$d_q(x-\xi)_{q,n}=[n]_q(x-\xi)_{q,n-1}\ .$$

Therefore we are naturally led to introduce the q-difference algebra

$$K\{x-\xi\}_{q,\rho}=\left\{\sum_{n\geq 0}a_n(x-\xi)_{q,n}: a_n\in K, \liminf_{n\to\infty}|a_n|^{-1/n}\geq\rho\right\}\ .$$

If $\xi=0$, the K-algebra $K\{x-\xi\}_{q,\rho}$ obviously coincides with the ring of analytic functions over the open disk of center 0 and radius ρ and its structure is well-known. If $\xi\neq 0$, there are different equivalent ways to define the multiplication of $K\{x-\xi\}_{q,\rho}$. Let $f(x)=\sum_{n\geq 0}f_n(x-\xi)_n, g(x)=\sum_{n\geq 0}g_n(x-\xi)_n\in K\{x-\xi\}_{q,\rho}$. Then for any nonnegative integers i,n we have

$$\frac{d_q^n}{[n]_q^!}(f)(\xi)=f_n \text{ and } f(q^i\xi)=\sum_{n=0}^{i}(q^i-1)(q^i-q)\cdots(q^i-q^{n-1})\xi^n f_n\ .$$

Moreover $f(x)$ is uniquely determined by the sequence $f(q^i\xi)$, $i\geq 0$. Analogous relations hold for $g(x)$. Hence the product $f(x)g(x)$ is the element of $K\{x-\xi\}_{q,\rho}$ associated to the sequence $f(q^i\xi)g(q^i\xi)$, namely, if $f(x)g(x)=\sum_{n\geq 0}h_n(x-\xi)_n$, we deduce from (1.2.4) that

$$h_n=\frac{d_q^n}{[n]_q^!}(fg)(\xi)=\frac{(-1)^n}{(q-1)^n[n]_q^!\xi^n}\sum_{j=0}^{n}(-1)^j\binom{n}{j}_{q^{-1}}q^{-\frac{j(j-1)}{2}}f(q^j\xi)g(q^j\xi)\ .$$

The twisted Leibniz Formula (1.2.2) gives another natural way of defining the coefficient h_n, in fact

$$h_n=\frac{d_q^n}{[n]_q^!}(fg)(\xi)=\sum_{j=0}^{n}\sum_{h=j}^{n}\binom{h}{j}_q f_h(q^{n-j}-1)_{h-j}\xi^{h-j}g_{n-j}\ .$$

In particular for any pair of positive integers l,k the formula above specializes to

$$(x-\xi)_l(x-\xi)_k=\sum_{n=0}^{l+k}\binom{l}{n-k}_q(q^k-1)_{l+k-n}\xi^{l+k-n}(x-\xi)_n\ .$$

The following proposition states that the natural map

$$T_{q,\xi}: f(x) \longmapsto \sum_{n\geq 0} \frac{d_q^n f}{[n]_q^!}(\xi)(x-\xi)_{q,n} \tag{1.3.1}$$

defines an isomorphism of q-difference algebras (i.e. an isomorphism of K-algebras commuting to the action of d_q) between $\mathcal{A}_D$ and $K\{x-\xi\}_{q,\rho}$. We will call the map $T_{q,\xi}$ *q-expansion* or *twisted Taylor formula.*

Proposition 1.4. *Let $D = D(\xi, \rho^-)$ be a q-invariant open disk. The map*

$$\begin{array}{cccc} T_{q,\xi}: & \mathcal{A}_D & \longrightarrow & K\{x-\xi\}_{q,\rho} \\ & f(x) & \longmapsto & \sum_{n\geq 0} \frac{d_q^n f}{[n]_q^!}(\xi)(x-\xi)_{q,n} \end{array}$$

is a q-difference algebras isomorphism. Moreover, for all $f \in \mathcal{A}_D$, the series $T_{q,\xi}(f)(x)$ converges uniformly to $f(x)$ over any closed disk $D(\xi, \eta^+)$, with $0 < \eta < \rho$.

The proof of (1.4) can be found in the Appendix (cf. §14). As a corollary we obtain the more useful statement:

Corollary 1.5. *Let $f(x) = \sum_{n\geq 0} a_n (x-\xi)_{q,n}$ be a series such that $a_n \in K$ and let $\rho = \liminf_{n\to\infty} |a_n|^{-1/n}$. Then $f(x)$ converges uniformly over any closed disk $D(\xi, \eta^+)$, with $0 < \eta < \rho$, to an analytic function if and only if $\rho > |(q-1)\xi|$.*

Proof. If $\rho > |(q-1)\xi|$, the series $f(x)$ converges by (1.4).

Suppose that $\rho < |(q-1)\xi|$. Let n_0 be the smallest positive integer such that $|(q^{n_0}-1)\xi| \leq \rho$ and let ε be a real positive number such that

$$\sup_{i=0,\dots,n_0-1} |(q^i-1)\xi| > \rho + \varepsilon \,.$$

Then for any $k \in \mathbb{Z}_{>0}$ we have

$$|(q^{kn_0}-1)\xi| \leq \sup_{i=0,\dots,k-1} |(q^{n_0}-1)q^{in_0}\xi| \leq \rho$$

and for any $r = 1, \dots, n_0-1$ we have

$$|(q^{kn_0+r}-1)\xi| = \sup\left(|(q^{kn_0}-1)\xi|, |(q^r-1)q^{kn_0}\xi|\right) > \rho + \varepsilon \,.$$

Therefore for $x_0 \in D(\xi, \rho^-)$ we obtain

$$\begin{aligned} |(x_0-\xi)_{n,q}| &= \prod_{i=0}^{n-1} |(x_0-\xi) + \xi(1-q^i)| \\ &> (\rho+\varepsilon)^{n-\left[\frac{n-1}{n_0}\right]-1} \prod_{0\leq i\leq \left[\frac{n-1}{n_0}\right]} |x_0 - \xi + \xi(1-q^{in_0})| \,. \end{aligned}$$

We conclude that

$$\sup_{|x_0-\xi|<\rho} |(x_0-\xi)_{n,q}| \geq (\rho+\varepsilon)^{n-\left[\frac{n-1}{n_0}\right]-1}\rho^{\left[\frac{n-1}{n_0}\right]+1} > \rho^n$$

and hence that $\sum_{n\geq 0} a_n(x_0-\xi)_{q,n}$ does not converge over $D(\xi,\rho^-)$.

Let $\rho = |(q-1)\xi|$. Suppose that $\sum_{n\geq 0} a_n(x-\xi)_{q,n}$ is convergent over $D(\xi,\rho^-)$. Then the series

$$f(qx) = a_0 + (qx-\xi)\sum_{n\geq 1} a_n q^{n-1}(x-\xi)_{q,n-1}$$

must also converge. By induction we conclude that for any $x_0 \in D(\xi,\rho^-)$ and any integer $n \geq 0$, the sum $f(q^n x_0)$ is convergent. Hence $f(x)$ converges over a bigger disk than the closed disk of center ξ and of radius ρ, which means that $\liminf_{n\to\infty} |a_n|^{-1/n} > \rho$. □

2. Gauss norms and q-difference operators.

Let $D = D(\xi,\rho^-)$. We recall that $\mathcal{A}_D$ comes equipped with a family of non archimedean norms $|\ |_\xi(R)$, the so-called Gauss norms, (cf. for instance [Rob00, §6, 1.4])

$$\left\|\sum_{n\geq 0} f_n(x-\xi)^n\right\|_\xi (R) = \sup_{n\geq 0} |f_n| R^n \ ,$$

defined for any $R \in (0,\rho)$ and any $\sum_{n\geq 0} f_n(x-\xi)^n \in \mathcal{A}_D$. It follows by Gauss lemma that they are multiplicative norms. If moreover $R \in |K|$, then (cf. [DGS94, IV, 1.1])

$$\left\|\sum_{n\geq 0} f_n(x-\xi)^n\right\|_\xi (R) = \sup_{x_0\in K,\ |x_0-\xi|<R} \left|\sum_{n\geq 0} f_n(x_0-\xi)^n\right| \ .$$

The norm $\|\ \|_\xi(R)$ plays a central role in the study of p-adic differential equation as well as of p-adic q-difference equations; therefore it is crucial to calculate the norm of the K-linear operator $\frac{d_q^k}{[k]_q^!}$ with respect to $\|\ \|_\xi(R)$ as well as to be able to calculate the norm of $f \in \mathcal{A}_D$ by looking at its q-expansion:

Proposition 2.1. *Let $f(x) \in \mathcal{A}_{D(\xi,\rho^-)}$. If $|(1-q)\xi| \leq R < \rho$, the norm $\|\ \|_\xi(R)$ satisfies*

$$\left\|\frac{d_q^k}{[k]_q^!} f(x)\right\|_\xi (R) \leq \frac{1}{R^k} \|f(x)\|_\xi (R) \quad \textit{for all } k \in \mathbb{Z}_{\geq 0}; \tag{2.1.1}$$

and

$$\|f(x)\|_\xi(R)=\sup_{n\geq 0}\left|\frac{d_q^n(f)}{[n]_q^!}(\xi)\right|R^n\ . \tag{2.1.2}$$

Remark. Inequality (2.1.1) generalizes [DV02, 4.2.1], where we considered the case $\xi=0$ and $R=1$.

Proof. Let us prove (2.1.1). Since

$$\frac{d_q^k}{[k]_q^!}f(x)=\sum_{n\geq k}f_n\frac{d_q^k}{[k]_q^!}(x-\xi)^n\ ,$$

it is enough to prove that

$$\left\|\frac{d_q^k}{[k]_q^!}(x-\xi)^n\right\|_\xi(R)\leq\frac{1}{R^k}\left\|(x-\xi)^n\right\|_\xi(R)=R^{n-k}\ ,\ \text{for any } n\geq k.$$

We proceed by double induction over k,n.

Let $k=1$. If n is an odd positive integer the inequality immediately follows from

$$d_q(x-\xi)^n=\frac{(qx-\xi)^n-(x-\xi)^n}{(q-1)x}=\sum_{i=0}^{n-1}(qx-\xi)^{n-1-i}(x-\xi)^i\ .$$

If n is an even positive integer we reduce to the previous case by observing that

$$\begin{aligned}d_q(x-\xi)^n&=\frac{(qx-\xi)^n-(x-\xi)^n}{(q-1)x}\\&=\frac{(qx-\xi)^{n/2}-(x-\xi)^{n/2}}{(q-1)x}\left((qx-\xi)^{n/2}+(x-\xi)^{n/2}\right)\\&=d_q(x-\xi)^{n/2}\left((qx-\xi)^{n/2}+(x-\xi)^{n/2}\right)\ .\end{aligned}$$

Now let $k>1$. The inequality is clear for $n=k$. It follows from the twisted Leibniz Formula (1.2.2) that for any $n\geq k$ we have

$$\begin{aligned}&\left\|\frac{d_q^k}{[k]_q^!}(x-\xi)^{n+1}\right\|_\xi(R)\\&=\left\|(q^kx-\xi)\tfrac{d_q^k}{[k]_q^!}(x-\xi)^n+\tfrac{d_q^{k-1}}{[k-1]_q^!}(x-\xi)^n\right\|_\xi(R)\\&\leq\sup\left(\left\|(q^kx-\xi)\tfrac{d_q^k}{[k]_q^!}(x-\xi)^n\right\|_\xi(R),\left\|\tfrac{d_q^{k-1}}{[k-1]_q^!}(x-\xi)^n\right\|_\xi(R)\right)\\&\leq R^{n+1-k}\ ,\end{aligned}$$

which completes the proof of (2.1.1).

Clearly (2.1.1) implies that

$$\|f(x)\|_\xi(R) \geq \sup_{n\geq 0}\left|\frac{d_q^n(f)}{[n]_q^!}(\xi)\right| R^n \ .$$

So it is enough to prove the opposite inequality to obtain (2.1.2). By (1.4) we have $f(x_0)=T_{q,\xi}(f)(x_0)$, for any $x_0\in D$. Then it is enough to remark that

$$\|(x-\xi)_{q,n}\|_\xi\,(R) = \left\|\prod_{i=0}^{n-1}\left[(x-\xi)+\xi(1-q^i)\right]\right\|_\xi (R) \leq R^n \quad \text{for any integer } n\geq 0,$$

to conclude that

$$\|f(x)\|_\xi(R) \leq \sup_{n\geq 0}\left|\frac{d_q^n(f)}{[n]_q^!}(\xi)\right| \|(x-\xi)_{q,n}\|_\xi\,(R) \leq \sup_{n\geq 0}\left|\frac{d_q^n(f)}{[n]_q^!}(\xi)\right| R^n \ .$$

This finishes the proof. □

The following lemma will be useful in (6.3), where we will consider some properties of families of q-difference systems deforming a differential system.

Lemma 2.2. *Let $f(x)=\sum_{n\geq 0}a_n(x-\xi)^n$ and $g(x)=\sum_{n\geq 0}b_n(x-\xi)_{q,n}$ be two analytic bounded functions over $D(\xi,1^-)$, with $|\xi|\leq 1$, and let $0<|1-q|\leq\varepsilon$. Suppose $\|f-g\|_\xi\,(1)\leq\varepsilon$. Then $\left\|\frac{df}{dx}-d_q(g)\right\|_\xi(1)\leq\varepsilon$*

Proof. Notice that for any positive integer n we have

$$|[n]_q-n| = |(q-1)+\cdots+(q^{n-1}-1)| \leq \sup_{i=1,\dots,n-1}|q^i-1| \leq |q-1| < \varepsilon \ ,$$

$$\|(x-\xi)^n\|_\xi\,(1) = \|(x-\xi)_{q,n}\|_\xi\,(1) = 1$$

and

$$\|(x-\xi)^n-(x-\xi)_{q,n}\|_\xi\,(1) \leq |1-q| \ .$$

We can assume that both $\|f\|_\xi(1)\leq 1$ and $\|g\|_\xi(1)\leq 1$. Therefore one conclude that

$$\begin{aligned}
\left\|\tfrac{df}{dx}-d_q(g)\right\|_\xi(1) &= \left\|\textstyle\sum_{n\geq 1}a_n n!(x-\xi)^{n-1}-b_n[n]_q^!(x-\xi)_{q,n-1}\right\|_\xi(1)\\
&\leq \sup\nolimits_{n\geq 1}\Big(|a_n n!-b_n n!|\,,\left|b_n n!-b_n[n]_q^!\right|,\\
&\qquad \left|b_n[n]_q^!\right|\left\|(x-\xi)^{n-1}-(x-\xi)_{q,n-1}\right\|_\xi(1)\Big)\\
&\leq \sup\left(|n!|\varepsilon, |(1-q)b_n|, \left|b_n[n]_q^!(1-q)\right|\right)\\
&\leq \sup(\varepsilon,|1-q|)\leq\varepsilon \ .
\end{aligned}$$

□

3. Analytic solutions of q-difference systems.

Let us consider a q-difference system of order μ

$$(\mathcal{S})_q \qquad Y(qx) = A(x)Y(x) \ ,$$

whose coefficients are meromorphic functions over a q-invariant open disk $D(\xi,\rho^-)$. We will assume that $A(x) \in Gl_\mu(\mathcal{M}_{D(\xi,\rho^-)})$.

The system $(\mathcal{S})_q$ can be rewritten in the form

$$(\mathcal{S})'_q \qquad d_q(Y)(x) = G(x)Y(x), \text{ with } G(x) = \frac{A(x)-\mathbb{I}_\mu}{(q-1)x} \ ,$$

where $\mathbb{I}_\mu$ is the identity matrix of order μ. One can iterate $(\mathcal{S})'_q$ obtaining

$$d_q^n(Y)(x) = G_n(x)Y(x) \ ,$$

with

$$(3.0.1) \qquad \begin{array}{l} G_0(x) = \mathbb{I}_\mu,\ G_1(x) = G(x) \text{ and} \\ G_{n+1}(x) = G_n(qx)G(x) + d_q(G_n)(x), \text{ for any integer } n \geq 1. \end{array}$$

If $G(x)$ does not have any pole in $q^{\mathbb{N}}\xi = \{\xi, q\xi, q^2\xi, \dots\}$, it can be identified, by considering its q-expansion, with a matrix with entries in the ring

$$K\,[\![x-\xi]\!]_q = \left\{ \sum_{n\geq 0} a_n(x-\xi)_{q,n} : a_n \in K \right\} \ ,$$

whose multiplication is defined as in (1.3). Then a formal solution matrix of $(\mathcal{S})_q$, or equivalently of $(\mathcal{S})'_q$, at ξ is given by

$$(3.0.2) \qquad Y(\xi,x) = \sum_{n\geq 0} \frac{G_n(\xi)}{[n]_q^!}(x-\xi)_{q,n} \ .$$

The fact that $Y(\xi,\xi) = \mathbb{I}_\mu$ does not allow us to conclude that $Y(\xi,x)$ is an invertible matrix, since $K\,[\![x-\xi]\!]_q$ is not a local ring (cf. Appendix A, §15). Actually we need a stronger assumption:

Lemma 3.1. *The system $(\mathcal{S})_q$ has a formal solution matrix in $Gl_\mu(K\,[\![x-\xi]\!]_q)$ if and only if the matrix $A(x)$ does not have any poles in $q^{\mathbb{N}}\xi$ and $\det A(x)$ does not have any zeros in $q^{\mathbb{N}}\xi$.*

Remark 3.2.

1) If the conditions of the lemma above are verified, then (3.0.2) is the only solution of $(\mathcal{S})_q$ in $Gl_\mu(K\,[\![x-\xi]\!]_q)$ such that $Y(\xi,\xi) = \mathbb{I}_\mu$ and all other solution matrices of $(\mathcal{S})_q$ in $Gl_\mu(K\,[\![x-\xi]\!]_q)$ are obtained by multiplying $Y(\xi,x)$ on the right by an element of $Gl_\mu(K)$.

2) Observe that, if $(\mathcal{S})_q$ has a solution matrix $Y(x) \in Gl_\mu(\mathcal{A}_D)$ over a q-invariant disk D, the matrix $A(x) = Y(qx)Y(x)^{-1}$ is an element of $Gl_\mu(\mathcal{A}_D)$. Hence neither $A(x)$ has a pole in $q^{\mathbb{Z}}\xi$ nor $\det A(x)$ has a zero in $q^{\mathbb{Z}}\xi$. It follows by the statement

above that $(\mathcal{S})_q$ can have a formal solution in $Gl_\mu(K[\![x-\xi]\!]_q)$ which is not the q-expansion of an analytic solution.

3) Suppose that $Y(\xi,x)\in Gl_\mu(K[\![x-\xi]\!]_q)$ is the q-expansion of an analytic solution of $(\mathcal{S})_q$ converging over $D(\xi,\rho^-)$ and let $|\zeta-\xi|<\rho$. Then necessarily we have

$$Y(\zeta,x)=Y(\xi,x)Y(\xi,\zeta)^{-1}\in Gl_\mu(K[\![x-\zeta]\!]_q)\ , \tag{3.2.1}$$

since both matrices are analytic solution matrix of $(\mathcal{S})_q$ at ζ, of maximal rank, having value $\mathbb{I}_\mu$ at ζ.

Proof. By the remark above, the system $(\mathcal{S})_q$ has a formal solution matrix in the ring $Gl_\mu(K[\![x-\xi]\!]_q)$ if and only if $Y(\xi,x)$ is in $Gl_\mu(K[\![x-\xi]\!]_q)$.

For any non negative integer k it makes sense to evaluate $Y(\xi,x)$ at $q^k\xi$:

$$Y(\xi,q^k\xi)=\sum_{n\geq 0}\frac{G_n(\xi)}{[n]_q^!}\xi^n(q^k-1)_{q,n}\ ,$$

since the sum on the right hand side is actually finite. Of course there are precise relations between the sequences $\left(\frac{G_n(\xi)}{[n]_q^!}\right)_{k\geq 0}$ and $(Y(\xi,q^k\xi))_{k\geq 0}$ that one can easily deduce by (1.2.3) and (1.2.4). It turns out that an element $Y(x)\in M_{\mu\times\mu}(K[\![x-\xi]\!]_q)$ is uniquely determined by $(Y(q^k\xi))_{k\geq 0}$. Therefore $Y(\xi,x)\in Gl_\mu(K[\![x-\xi]\!]_q)$ if and only if $Y(\xi,q^k\xi)\in Gl_\mu(K)$ for all $k\geq 0$, the inverse of $Y(\xi,x)$ being the element of $Gl_\mu(K[\![x-\xi]\!]_q)$ associated to the data $\left(Y(\xi,q^k\xi)^{-1}\right)_{k\geq 0}$.

To conclude it is enough to observe that

$$\begin{array}{rcl}Y(\xi,q^k\xi) & = & A(q^{k-1}\xi)A(q^{k-2}\xi)\cdots A(\xi)Y(\xi,\xi)\\ & = & A(q^{k-1}\xi)A(q^{k-2}\xi)\cdots A(\xi)\ .\end{array}$$

□

In the next corollaries we give some sufficient conditions for having a fundamental analytic solution matrix, i.e. an invertible solution matrix $Y(x)$ such that $Y(x)$ and $Y(x)^{-1}$ have analytic coefficients over a convenient q-invariant open disk. They are just partial results and we will reconsider the problem of the existence of analytic solutions for $(\mathcal{S})_q$ in the next sections.

From (1.4) one immediately obtains:

Corollary 3.3. *The system $(\mathcal{S})_q$ has a fundamental analytic solution matrix at ξ if and only if*

- *the matrix $A(x)$ does not have any poles in $q^{\mathbb{N}}\xi$,*

- $\det A(x)$ *does not have any zeros in $q^{\mathbb{N}}\xi$,*

- $\displaystyle\limsup_{n\to\infty}\left|\frac{G_n(\xi)}{[n]_q^!}\right|^{1/n}<|(q-1)\xi|^{-1}$.

Before stating the following result we need to introduce the number π_q, which plays a role analogous to the π of Dwork for p-adic differential equations. We recall that π is an element of an extension of K such that $\pi^{p-1} = -p$

Notation 3.4. We suppose that there exists an element π_q of K such that $\lim_{n\to\infty} |[n]_q^!|^{1/n} = |\pi_q|$.

Remark 3.5. The condition above is somehow a minimal require for π_q, and actually it does not define it uniquely. Further developments of the theory point out extra conditions that will determine more precisely the choice of π_q (cf. [And02a]).

Corollary 3.6. *Let $\rho \leq 1$, $\rho|\pi_q| > |(1-q)\xi|$ and $D = D(\xi, \rho^-)$. Suppose we are given a square matrix $G(x)$ analytic over D such that*

$$\sup_{x\in D} |G(x)| \leq \frac{1}{\rho}$$

and that the determinant of $A(x) = (q-1)xG(x) + \mathbb{I}_\mu$ does not have any zeros in $q^{\mathbb{N}}\xi$. Then $Y(qx) = A(x)Y(x)$ has an analytic fundamental solution at ξ.

Proof. It follows by (2.1.1) and (3.0.1) that

$$|G_n(\xi)| \leq \frac{1}{\rho} \sup(1, \|G_{n-1}(\xi)\|) \leq \frac{1}{\rho^n} \text{ , for any } n \geq 1,$$

which implies that

$$\limsup_{n\to\infty} \left|\frac{G_n(\xi)}{[n]_q^!}\right|^{1/n} = \frac{1}{|\pi_q|} \limsup_{n\to\infty} |G_n(\xi)|^{1/n} \leq \frac{1}{\rho|\pi_q|} \text{ .}$$

Since $\rho|\pi_q| > |(1-q)\xi|$, the matrix $Y(x, \xi)$ in (3.0.2) is the q-expansion of an analytic fundamental solution. □

3.7. Iteration of $(\mathcal{S})_q$ and existence of analytic solutions.

In the rest of the paper we will often assume that $(\mathcal{S})_q$ has an analytic fundamental solution at some point ζ or that $|1-q|$ is smaller than some constant: this is not always true.

Sometimes one can easily reduce to the case of having a fundamental analytic solution by iterating the q-difference system. In the same way one can reduce to the case of a $q \in K$ such that $|1-q| << 1$.

Let us analyze the situation in detail. Suppose that $(\mathcal{S})_q$ does not have an analytic fundamental solution at ζ. Then it may happen that there exists $n_0 > 1$ such that the system

$$(\mathcal{S})_{q^{n_0}} \quad Y(q^{n_0}x) = A_{n_0}(x)Y(x) \text{ , with } A_{n_0}(x) = A(q^{n_0-1}x)A(q^{n_0-2}x)\cdots A(x),$$

obtained from $(\mathcal{S})_q$ by iteration, has a fundamental analytic solution $Y(x)$ over a q^{n_0}-invariant open disk $D(\zeta, \eta^-)$. If n_0 is the smallest positive integer having this property, then one can construct a fundamental solution $F(x)$ of $(\mathcal{S})_q$, analytic over the non-connected q-invariant analytic domain

$$D(\zeta,\eta^-)\cup D(q\zeta,\eta^-)\cup\cdots\cup D(q^{n_0-1}\zeta,\eta^-) \tag{3.7.1}$$

by setting

$$\begin{array}{l} F(q^ix)=A_i(x)Y(x)=A(q^{i-1}x)A(q^{i-2}x)A(q^{i-1}x)\cdots A(x)Y(x)\ ,\\ \text{for any } x\in D(\zeta,\eta^-) \text{ and any } i=0,\dots,n_0-1. \end{array}$$

Observe that the restriction of $F(x)$ to $D(q^i\zeta,\eta^-)$, for any $i\in\mathbb{Z}$, is an analytic fundamental solution of $(\mathcal{S})_{q^{n_0}}$.

So, if the q-difference system has an analytic solution over a non-connected analytic domain as above, it is enough to consider a system obtained by iteration to reduce to the case of a system having an analytic solution over a q-invariant open disk. In the appendix we will consider the q-expansion of analytic functions over non connected domain of the form (3.7.1).

The same trick allows us to reduce to the case of a small $|1-q|$, knowing that $\liminf\limits_{n\to\infty}|1-q^n|=0$.

3.8. Removing apparent and trivial singularities.

In this subsection we will consider q-difference system having meromorphic solutions or analytic solutions with meromorphic inverse. Our purpose is to explain how to reduce by gauge transformation to the assumption of having an analytic fundamental solution.

Once again we consider a q-difference system

$$Y(qx)=A(x)Y(x) \tag*{$(\mathcal{S})_q$}$$

with meromorphic coefficients over a q-invariant disk $D=D(\xi,\rho^-)$, defined over K. For any matrix $F(x)\in Gl_\mu(\mathcal{M}_D)$, the matrix $Z(x)=F(x)Y(x)$ is a solution to

$$Z(qx)=A_{[F]}(x)Z(x)\ ,\ \text{with } A_{[F]}(x)=F(qx)A(x)F(x)^{-1}. \tag{3.8.1}$$

The matrix $F(x)$ is usually called a *meromorphic gauge transformation matrix*. Notice that

$$G_{[F]}(x)=\frac{A_{[F]}(x)-\mathbb{I}_\mu}{(q-1)x}=F(qx)G(x)F(x)^{-1}+d_q(F)(x)F(x)^{-1}\ . \tag{3.8.2}$$

Following the classical terminology of differential equations (cf. for instance [DGS94, page 172]) we give the definition:

Definition 3.9. *We say that $q^{\mathbb{Z}}\xi\subset D$ is an* ordinary q-orbit (resp. trivial singularity, apparent singularity) *for $(\mathcal{S})_q$, if $(\mathcal{S})_q$ has a solution in $Gl_\mu(\mathcal{A}_{D'})$ (resp. $Gl_\mu(\mathcal{M}_{D'})\cap M_{\mu\times\mu}(\mathcal{A}_{D'})$, $Gl_\mu(\mathcal{M}_{D'})$), where $D'\subset D$ is a q-invariant analytic domain of the form (3.7.1) containing ξ.*

Remark 3.10. In the sequel we will informally say that the system $(\mathcal{S})_q$ has "*at worst a finite number of apparent singularities in* D" to mean that $(\mathcal{S})_q$ has only ordinary q-orbits in D, apart from a finite number of apparent singularities.

The following statement is a q-analogue of the Frobenius-Christol device (cf. [Chr81, II, §8]) to remove apparent and trivial singularities over

$$D^\times = \begin{cases} D & \text{if } 0 \notin D, \\ D \smallsetminus \{0\} & \text{otherwise.} \end{cases}$$

Proposition 3.11. *We assume that*
(3.11.1)
the system $(\mathcal{S})_q$ has at worst a finite number of apparent singularities in $D^\times$.

Then there exists $H(x) \in Gl_\mu(K(x))$ such that the q-difference system $Y(qx) = A_{[H]}(x)Y(x)$ has only ordinary orbits in $D^\times$.

The proposition immediately follows from the more precise statement:

Proposition 3.12. *Suppose that (3.11.1) is verified. Then the following propositions hold:*
1) There exists a polynomial $P(x) \in K[x]$, with $P(0) \neq 0$, such that the q-difference system $Y(qx) = A_{[P\mathbb{I}_\mu]}(x)Y(x)$ has only trivial singularities in $D^\times$.
2) Suppose that $(\mathcal{S})_q$ has only trivial singularities in $D^\times$. Then there exists $H(x) \in Gl_\mu(K(x))$ such that
- $H(x)$ does not have a pole at 0 and $H(0) \in Gl_\mu(K)$,
- $H(0) \in Gl_\mu(K)$,
- $\|H(x)\|_{0,\rho}^{-1} = \left\|H(x)^{-1}\right\|_{0,\rho} = \rho$,
- the q-difference system $Y(qx) = A_{[H]}(x)Y(x)$ has only ordinary orbits in $D^\times$.

Proof. Let $P(x) \in K[x]$ be a polynomial such that for any $\zeta \in D^\times$ and any solution matrix $U_\zeta(x)$ meromorphic on a convenient q-invariant analytic domain containing ζ, the matrix $P(x)U_\zeta(x)$ is analytic at ζ. Then $P(x)U_\zeta(x)$ is a solution matrix of the q-difference system associated to $A_{[P\mathbb{I}_\mu]}(x)$, which has only trivial singularities in $D^\times$. This completes the proof of the first part of the statement.

Now we prove *2)*. Let $\zeta \in D^\times$ and let $Y(x) \in M_{\mu\times\mu}(\mathcal{A}_{D'}) \cap Gl_\mu(\mathcal{M}_{D'})$ be a solution at ζ of $(\mathcal{S})_q$. If $(\mathcal{S})_q$ has a trivial singularity at $q^{\mathbb{Z}}\zeta$, then necessarily $\det Y(x)$ has a zero in $q^{\mathbb{Z}}\zeta$. Notice that, since any infinite subset of $q^{\mathbb{Z}}\zeta$ has a limit point in D', the analytic function $\det Y(x) \in \mathcal{A}_{D'}$ has only a finite number of zeros in $q^{\mathbb{Z}}\zeta$.

Let $q^n\zeta$ be a zero of order $k > 0$ of $\det Y(x)$ and let $\vec{B} = {}^t(B_1, \dots, B_\mu)$ be a non-zero vector in K^μ such that $\vec{B}Y(q^n\zeta) = 0$. We fix $1 \leq \iota \leq \mu$ such that $|B_\iota| = \max_{j=1,\dots,\mu} |B_j| > 0$. Of course one can suppose that $B_\iota = 1$. Let us consider the gauge transformation matrix

$$H(x)=\left(\begin{array}{ccc|c|ccc} & \mathbb{I}_{\iota-1} & & \begin{matrix}0\\ \vdots\\ 0\end{matrix} & & 0 & \\ \hline \frac{B_1}{x-q^n\zeta} & \cdots & \frac{B_{\iota-1}}{x-q^n\zeta} & \frac{1}{x-q^n\zeta} & \frac{B_{\iota+1}}{x-q^n\zeta} & \cdots & \frac{B_\mu}{x-q^n\zeta} \\ \hline & 0 & & \begin{matrix}0\\ \vdots\\ 0\end{matrix} & & \mathbb{I}_{\mu-\iota} & \end{array}\right).$$

Since $\vec{B}Y(q^n\zeta)=0$, the matrix $H(x)Y(x)$ is still analytic at $q^n\xi$ and

$$H(x)^{-1}=\left(\begin{array}{ccc|c|ccc} & \mathbb{I}_{\iota-1} & & \begin{matrix}0\\ \vdots\\ 0\end{matrix} & & 0 & \\ \hline -B_1 & \cdots & -B_{\iota-1} & x-q^n\zeta & -B_{\iota+1} & \cdots & -B_\mu \\ \hline & 0 & & \begin{matrix}0\\ \vdots\\ 0\end{matrix} & & \mathbb{I}_{\mu-\iota} & \end{array}\right).$$

Moreover $\det\left(H(x)Y(x)\right)$ has a zero at $q^n\zeta$ of order $k-1$. By iteration, one can construct a basis change satisfying all the conditions in *2)*. □

4. Generic points.

We consider an extension Ω/K of ultrametric fields with the following properties (for the construction of such a field see for instance [Rob00, §3, 2]):

1. the field Ω is complete and algebraically closed;
2. the set of values of Ω is $\mathbb{R}_{\geq 0}$;
3. the residue field of Ω is transcendental over the residue field of K.

Then for any $R\in\mathbb{R}_{\geq 0}$ the field Ω contains an element t_R, transcendent over K, such that $|t_R|=R$ and that the norm induced by Ω over $K(t_R)$ is defined by

$$\left|\frac{\sum a_i t_R^i}{\sum b_j t_R^j}\right|=\frac{\sup_i |a_i|R^i}{\sup_j |b_j|R^j}\ .$$

Observe that, if $f(x)$ is an analytic function over a disk of center 0 and radius $\rho > R$, we have

$$|f(t_R)| = \|f(x)\|_0 (R) = \sup_{x\in\Omega, |x|<R} |f(x)| \ .$$

Definition 4.1. *We call* t_R a generic point with respect to K (at distance R from 0). *A* generic point $t_{\xi,R}$ with respect to K (at distance R from $\xi \in K$) *is defined by shifting.*

Dwork-Robba's generic points play a fundamental role in p-adic differential equation theory: historically, their introduction has been the first attempt to fill the gap left by the absence of a p-adic analytic continuation. In fact, the radius of solutions at $t_{\xi,R}$ is a sort of global invariant for the differential equation that is equal to the radius of convergence of solutions at almost any point of the disk $D(\xi, R^-)$ and allows for an estimate in the other points. From a more recent point of view, one should think of generic points as points of a Berkovich analytic space.

Consider a q-difference system of order μ with meromorphic coefficients over a q-invariant disk $D(\xi, \rho^-)$:

$$(\mathcal{S})_q \qquad Y(qx) = A(x)Y(x) \ , \text{ with } A(x) \in Gl_\mu(\mathcal{M}_{D,K}).$$

As in the previous section (cf. (3.0.2)), for any $R < \rho$ one can consider the formal solution of $(\mathcal{S})_q$ at $t_{\xi,R}$:

$$(4.1.1) \qquad Y(t_{\xi,R}, x) = \sum_{n\geq 0} \frac{G_n(t_{\xi,R})}{[n]^!_q}(x - t_{\xi,R})_{q,n} \in Gl_\mu(\Omega\, [\![x - t_{\xi,R}]\!]_q) \ .$$

Notice that since the system $(\mathcal{S})_q$ is defined over K, the matrix $A(x)$ cannot have any poles in $q^{\mathbb{N}} t_{\xi,R}$ and $\det A(x)$ cannot have any zeros in $q^{\mathbb{N}}\xi$: it follows that $Y(t_{\xi,R}, x)$ is necessarily in $Gl_\mu(\Omega\, [\![x - t_{\xi,R}]\!]_q)$.

Definition 4.2. *We call* generic radius of convergence of $(\mathcal{S})_q$ at $t_{\xi,R}$ *the number*

$$\chi_{\xi,R}(A, q) = \inf \left(R, \liminf_{n\to\infty} \left| \frac{G_n(t_{\xi,R})}{[n]^!_q} \right|^{-1/n} \right) \ .$$

Lemma 4.3. *Let* $1 \geq R \geq |(1-q)\xi|$*. The generic radius of convergence* $\chi_{\xi,R}(A, q)$ *is invariant under meromorphic gauge transformation and*

$$\chi_{\xi,R}(A, q) \geq \frac{R|\pi_q|}{\sup\left(1, |G_1(t_{\xi,R})|\right)} \ .$$

Proof. The first assertion is proved in [DV02, 4.2.3] in the case $\xi = 0$ and $R = 1$, but the same proof applies to this case. The second assertion is immediately deduced from the recursive relation satisfied by the $G_n(x)$'s and (2.1.1), since $\lim_{n\to\infty} |[n]^!_q|^{1/n} = |\pi_q|$. □

Lemma 4.4. *If $A(x)$ is analytic over the disk $D(\xi,\rho^-)$ and $\rho \geq \chi_{\xi,R}(A,q) > |(1-q)\xi|$, then $(\mathcal{S})_q$ has a fundamental analytic solution over $D(\xi,\chi_{\xi,R}(A,q)^-)$.*

Proof. It is enough to remark that

$$\left|\frac{G_n(\xi)}{[n]_q^!}\right| \leq \left|\frac{G_n(t_{\xi,R})}{[n]_q^!}\right| .$$

□

The theorems estimating the radius of convergence of $Y(\xi,x)$ with respect to $\chi_{\xi,R}(A,q)$ are usually called *transfer theorems*: in the next chapter we will prove a transfer theorem from a disk where an analytic solution exists to a contiguous disk, where the system has only ordinary orbits. This result is a consequence of the effective bound theorem. Chapters III and IV are devoted to the proof of a transfer theorem for regular singular disks.

4.5. The cyclic vector lemma and the q-analogue of the Dwork-Frobenius theorem.

It may seem that calculating a generic radius of convergence is as difficult as calculating a radius of convergence at points which are rational over K. This is not completely true; in fact the generic radius of convergence is very easy to calculate when it is small, using a q-difference equation associated to $(\mathcal{S})_q$: it is the q-analogue of the Dwork-Frobenius theorem [DGS94, VI, 2.1].

As in the differential world, a q-difference equation associated to $Y(qx) = A(x)Y(x)$, with $A(x) \in Gl_\mu(\mathcal{M}_{D,K})$, is constructed using a cyclic vector lemma (cf. for instance [Sau00a, Annexe B] or [DV02, 1.3]), which states the existence of a meromorphic matrix $H(x) \in Gl_\mu(\mathcal{M}_{D,K})$ such that

$$A_{[H]} = \left(\begin{array}{c|c} 0 & \\ \vdots & \mathbb{I}_{\mu-1} \\ 0 & \\ \hline a_0(x) & a_1(x)\dots a_{\mu-1}(x) \end{array}\right) . \tag{4.5.1}$$

Then $y(x)$ is a solution of the q-difference equations

$$y(q^\mu x) - a_{\mu-1}(x)y(q^{\mu-1}x) - \cdots - a_0(x)y(x) = 0 \tag{4.5.2}$$

if and only if

$$\begin{pmatrix} y(qx) \\ y(q^2x) \\ \vdots \\ y(q^\mu x) \end{pmatrix} = A_{[H]} \begin{pmatrix} y(x) \\ y(qx) \\ \vdots \\ y(q^{\mu-1}x) \end{pmatrix} .$$

Consider the lower triangular gauge transformation matrix

(4.5.3)

$$\widetilde{H}=(a_{i,j})_{i,j=0,\dots,\mu-1}\ ,\ \text{with } a_{i,j}=\begin{cases}\dfrac{1}{x^i}\dfrac{(-1)^{i+j}}{(q-1)^i}\dbinom{i}{j}_{q^{-1}}q^{-\frac{j(j-1)}{2}} & \text{if } j\leq i;\\[2ex] 0 & \text{otherwise.}\end{cases}$$

It follows from (1.2.4) that $y(x)$ is a solution of the q-difference equations (4.5.2) if and only if

$$\begin{pmatrix} d_q y(x)\\ d_q^2 y(x)\\ \vdots\\ d_q^\mu y(x)\end{pmatrix}=G_{[\tilde{H}H]}\begin{pmatrix} y(x)\\ d_q y(x)\\ \vdots\\ d_q^{\mu-1}y(x)\end{pmatrix}.$$

Moreover

$$G_{[\tilde{H}H]}=\left(\begin{array}{c|c} 0 & \\ \vdots & \mathbb{I}_{\mu-1}\\ 0 & \\ \hline b_0(x) & b_1(x)\dots b_{\mu-1}(x)\end{array}\right).$$

Proposition 4.6. *Let $|1-q|<1$. If $\sup_{i=0,\dots,\mu-1}|b_i(t_{0,R})|>R^{i-\mu}$ then*

$$\chi_{0,R}(A,q)=\frac{|\pi_q|}{\sup_{i=0,\dots,\mu-1}|b_i(t_{0,R})|^{1/(\mu-i)}}.$$

Proof. Notice that it is enough to prove the statement for $R=1$. In fact, the general statement can be deduced by rescaling. Moreover, the proposition is proved in [DV02, 4.3] in the case $R=1$, $|1-q|<|\pi|$, so we are only sketching the proof.

Let $\gamma\in K$ be such that

$$|\gamma|=\sup_{i=0,\dots,\mu-1}|b_i(t_{0,1})|^{1/(\mu-i)}$$

and let

$$G=\left(\begin{array}{c|c} 0 & \\ \vdots & \mathbb{I}_{\mu-1}\\ 0 & \\ \hline b_0(x) & b_1(x)\dots b_{\mu-1}(x)\end{array}\right)\ \text{and } H=\begin{pmatrix}\gamma^{\mu-1} & & & \\ & \gamma^{\mu-2} & & \\ & & \ddots & \\ & & & 1\end{pmatrix}.$$

Then

$$H^{-1}GH = \gamma W(x) \ , \text{ with } W(x) = \left(\begin{array}{c|c} 0 & \\ \vdots & \mathbb{I}_{\mu-1} \\ 0 & \\ \hline \frac{b_0(x)}{\gamma^\mu} & \frac{b_1(x)}{\gamma^{\mu-1}} \cdots \frac{b_{\mu-1}(x)}{\gamma} \end{array} \right) .$$

Notice that $\chi(A, q) = \chi(\gamma W(x), q)$, hence we can calculate $\chi(A, q)$ iterating the system $d_q Y = \gamma W(x)Y$. So we set $H_1(x) = \gamma W(x)$ and we define recursively the family of q-difference systems $d_q^n Y = H_n(x)Y$. By induction over n one can prove that $|H_n(t_{0,1})| \leq |\gamma|^n$ (cf. [DV02, 4.3]) and hence that $\chi(A, q) \geq |\pi_q \gamma^{-1}|$.

The opposite inequality follows from the fact that the reduction of $\gamma^{-n} H_n(t_{0,1})$ in the residue field is equal to the reduction of $W(x)^n$, which has a non zero eigenvalue in positive characteristic (cf. [DV02, 4.3]). □

II. Effective bounds for q-difference systems

In this chapter we prove an effective bound theorem for q-difference systems: it is the analogue of a theorem of Dwork and Robba (cf. [DR80] for the proof in the case of differential equations. The statement concerning differential systems is proved for instance in [Bom81], [And89] and [DGS94]).

Let us explain the effective bound theorem for an analytic differential equations of order one: the theorem is actually almost trivial in this case, but we can already point out the differences with the q-difference version.

Let $\xi \in \mathbb{A}^1_K$, $\xi \neq 0$, and let $u(x)$ be a meromorphic function over an open disk of center ξ and radius $\rho > 0$. For all $R \in (0, \rho)$, the multiplicative norm $\| \ \|_\xi(R)$ (cf. §2) induces a norm over the field of meromorphic functions over $D(\xi, \rho^-)$. Let

$$g_n(x) = \frac{1}{n!} \frac{d^n u}{dx^n}(x) u(x)^{-1} \ , \ \forall n \geq 0 \ .$$

The effective bound theorem for differential equation of order 1 states that for any $R \in (0, \rho)$

$$\|g_n(x)\|_\xi(R) \leq R^{-n} \ .$$

Of course this inequality is easy to prove, in fact

$$\begin{array}{rcl} \|g_n(x)\|_\xi(R) & \leq & \left\| \frac{1}{n!} \frac{d^n u}{dx^n}(x) \right\|_\xi (R) \left\| u^{-1}(x) \right\|_\xi (R) \\ & \leq & R^{-n} \|u(x)\|_\xi(R) \|u(x)\|_\xi(R)^{-1} \\ & \leq & R^{-n} \ . \end{array}$$

The multiplicativity of $\| \ \|_\xi(R)$ is the key point of the inequalities above.

Let us consider a meromorphic q-difference system: its solutions may be meromorphic over a non-connected analytic domain (as the one considered in (3.7.1)). Therefore a natural analogue of $\| \ \|_\xi(R)$ would be a sup-norm over a non-connected domain, which is necessarily non-multiplicative. In particular $\|u(x)^{-1}\|_\xi(R)$ could be greater than $\|u(x)\|_\xi(R)^{-1}$. In other words, the assumption of having a solution matrix (analytic or meromorphic) over an q-invariant disk cannot be avoided.

5. Effective bound theorem for q-difference systems.

Let $D = D(\xi, \rho^-)$ be an open disk of center $\xi \in \mathbb{A}^1_K$ and $\mathcal{M}_D$ be the field of meromorphic functions over D. For any $R \in (0, \rho)$ the norm $\| \ \|_\xi(R)$ extends from $\mathcal{A}_D$ to $\mathcal{M}_D$ by multiplicativity. For any $f(x) \in \mathcal{M}_D$ one usually sets:

$$\|f(x)\|_{\xi,\rho} = \lim_{R\to\rho} \|f(x)\|_\xi(R) \ .$$

This limit may be not bounded, but respects multiplication of functions, as well as $\| \ \|_\xi(R)$.

We suppose that D is q-invariant and we consider $Y(x) \in Gl_\mu(\mathcal{M}_D)$. Then $Y(x)$ is solution of the q-difference systems defined by

$$G_n(x) = d_q^n Y(x) Y(x)^{-1} \ , \ \forall n \geq 0.$$

Obviously the entries of $G_n(x)$ are meromorphic functions over D.

Theorem 5.1.

$$\left\| \frac{G_n(x)}{[n]_q^!} \right\|_{\xi,\rho} \leq \{n, \mu-1\}_p^q \left(\sup_{i=0,\dots,\mu-1} \|G_i(x)\|_{\xi,\rho}\, \rho^i \right) \frac{1}{\rho^n} \ ,$$

where

$$\{n,\mu-1\}_p^q = \begin{cases} 1, & \text{if } n=0, \\ \displaystyle\sup_{1\leq\lambda_1<\cdots<\lambda_{\mu-1}\leq n,\lambda_i\in\mathbb{Z}} \frac{1}{|[\lambda_1]_q \cdots [\lambda_{\mu-1}]_q|}, & \text{otherwise.} \end{cases}$$

One can also consider a matrix $Y(x)$ analytic over a q-invariant disk around $t_{\xi,R}$, with coefficients in Ω (cf. §4), and assume that it is solution of a q-difference system $(\mathcal{S})_q$ defined over K. Then:

Corollary 5.2.

$$\left| \frac{G_n(t_{\xi,R})}{[n]_q^!} \right| \leq \{n,\mu-1\}_p^q \left(\sup_{i=0,\dots,\mu-1} |G_i(t_{\xi,R})|\, \chi_{\xi,R}(A,q)^i \right) \frac{1}{\chi_{\xi,R}(A,q)^n} \ .$$

The proof of theorem 5.1 (cf. (5.5) below) follows the proof of the Dwork-Robba theorem concerning the effective bounds for p-adic differential systems. As in the differential case, it relies on the analogous result for q-difference equations.

5.3. Effective bounds for q-difference equations.

We consider $\vec{u} = (u_1, \dots, u_\mu) \in \mathcal{M}_D^\mu$, with $\mu \in \mathbb{Z}$, $\mu \geq 1$, such that $u_1, \dots, u_\mu$ are linearly independent over the field of constants K of $\mathcal{M}_D$. By the q-analogue of the Wronskian lemma (cf. [Sau00a, Appendice] or [DV02, §1.2]) this is equivalent to supposing that the q-Wronskian matrix

$$W_q(\vec{u}) = \begin{pmatrix} u_1(x) & \dots & u_\mu(x) \\ d_q u_1(x) & \dots & d_q u_\mu(x) \\ \vdots & \ddots & \vdots \\ d_q^{\mu-1} u_1(x) & \dots & d_q^{\mu-1} u_\mu(x) \end{pmatrix}$$

is in $Gl_\mu(\mathcal{M}_D)$.

We consider the q-difference equationsdefined by

$$d_q^n \vec{u} = \vec{g}_n(x) W_q(\vec{u}) \ , \ \text{where } \vec{g}_n \in \mathcal{M}_D^\mu \text{ and } n \geq 0.$$

Lemma 5.4.

$$\left\| \frac{\vec{g}_n}{[n]_q^!} \right\|_{\xi,\rho} \leq \{n, \mu - 1\}_p^q \frac{1}{\rho^n} \ . \tag{5.4.1}$$

Proof. By extending K, we can find $\alpha \in K$ such that $|\alpha| = \rho$. If we set $y = \alpha x$ and $\zeta = \alpha\xi$, we are reduced to proving the lemma for $\rho = 1$.

So we suppose $\rho = 1$ and we prove the lemma by induction on μ. For $\mu = 1$ and any $R \in ((q-1)\xi, 1)$ we have (cf. (2.1.1))

$$\left\| \frac{\vec{g}_n}{[n]_q^!} \right\|_\xi (R) \leq \left\| \frac{d_q^n u_1}{[n]_q^!}(x) \right\|_\xi (R) \|u_1(x)^{-1}\|_\xi(R) \leq \frac{1}{R^n} \ .$$

Letting $R \to 1$, we obtain (5.4.1).

Let $\mu > 1$. For $n = 0$ the inequality is trivial, so let $n > 0$. We set:

$$\vec{u} = (u_1, \dots, u_{\mu+1}) = u(1, \vec{\tau}) \in \mathcal{M}_D^{\mu+1} \ , \ \text{where } u = u_1 \text{ and } \vec{\tau} = (\tau_1, \dots, \tau_\mu) \in \mathcal{M}_D^\mu.$$

The idea of the proof is to apply the inductive hypothesis to $d_q\vec{\tau}$. We know that the vector $\vec{h}_n \in \mathcal{M}_D^\mu$, defined for any $n \geq 0$ by

$$d_q^n (d_q\vec{\tau}) = \vec{h}_n(x) W_q (d_q\vec{\tau}) \ ,$$

satisfies the inequality

$$\left\| \frac{\vec{h}_n}{[n]_q^!} \right\|_{\xi,1} \leq \{n, \mu - 1\}_p^q \ . \tag{5.4.2}$$

Notice that $\vec{h}_n$ and $\vec{g}_n$ verify the relation

(5.4.3)
$$\begin{array}{rcl}\dfrac{\vec{g}_n}{[n]_q^!}W_q(\vec{u}) = \dfrac{d_q^n}{[n]_q^!}\vec{u} = \dfrac{d_q^n}{[n]_q^!}\left(u(1,\vec{\tau})\right) \\ = \dfrac{d_q^n}{[n]_q^!}(u)(1,\vec{\tau}) + \left(0, \displaystyle\sum_{j=1}^{n}\frac{d_q^{n-j}}{[n-j]_q^!}(u)(q^jx)\frac{1}{[j]_q}\frac{\vec{h}_{j-1}}{[j-1]_q^!}\Big(0, W_q(d_q\vec{\tau})\Big)\right) ,\end{array}$$

while $W_q(\vec{u})$ and $W_q(d_q\vec{\tau})$ satisfy

$$W_q(\vec{u}) = W_q\left(u(1,\vec{\tau})\right) = uP\begin{pmatrix}1 & \vec{\tau}\\ 0 & W_q(d_q\vec{\tau})\end{pmatrix} ,$$

where

$$P = \frac{1}{u(x)}\begin{pmatrix} u(x) & 0 & 0 & \cdots & 0\\ d_qu(x) & u(qx) & 0 & \ddots & \vdots\\ \binom{2}{2}_q d_q^2u(x) & \binom{2}{1}_q d_qu(qx) & u(q^2x) & \ddots & 0\\ \vdots & \vdots & \vdots & \ddots & 0\\ \binom{\mu}{\mu}_q d_q^{\mu}u(x) & \binom{\mu}{\mu-1}_q d_q^{\mu-1}u(qx) & \binom{\mu}{\mu-2}_q d_q^{\mu-2}u(q^2x) & \cdots & u(q^{\mu}x)\end{pmatrix} .$$

Since $\left\|\binom{\mu}{\mu-i}_q d_q^{\mu-i}u(q^ix)u(x)^{-1}\right\|_{\xi,1} \leq 1$ and $\left\|u(q^ix)u(x)^{-1}\right\|_{\xi,1} = 1$, we have $\|P\|_{\xi,1} = \|\det P\|_{\xi,1} = 1$ and hence $\|P^{-1}\|_{\xi,1} \leq 1$. This implies that

$$\text{(5.4.4)}\qquad \left\|\begin{pmatrix}1 & \vec{\tau}\\ 0 & W_q(d_q\vec{\tau})\end{pmatrix}W_q(\vec{u})^{-1}\right\|_{\xi,1} = \|u^{-1}P^{-1}\|_{\xi,1} \leq \|u^{-1}\|_{\xi,1} .$$

We obtain the desired inequality by combining (5.4.2), (5.4.3) and (5.4.4):

$$\begin{array}{rcl}\left\|\dfrac{\vec{g}_n}{[n]_q^!}\right\|_{\xi,1} & \leq & \displaystyle\sup_{j=1,\dots,n}\left(1, \left\|\frac{\vec{h}_{j-1}}{[j]_q^!}\right\|_{\xi,1}\right)\\ & \leq & \displaystyle\sup_{j=1,\dots,n}\left(1, \left\|\frac{1}{[j]_q}\right\|_{\xi,1}\{j-1,\mu-1\}_p^q\right)\\ & \leq & \{j,\mu-1\}_p^q .\end{array}$$

□

5.5. Proof of theorem 5.1.

We recall that we are given a matrix $Y(x) \in Gl_{\mu}(\mathcal{M}_D)$ meromorphic over a q-invariant open disk D and that we have set

$$G_n(x) = d_q^nY(x)Y(x)^{-1} \in M_{\mu\times\mu}(\mathcal{M}_D) .$$

We want to prove that

$$\left\|\frac{G_n(x)}{[n]^!_q}\right\|_{\xi,\rho} \le \{n,\mu-1\}^q_p \left(\sup_{i=0,\dots,\mu-1} \|G_i(x)\|_{\xi,\rho}\,\rho^i\right)\frac{1}{\rho^n}\ .$$

First of all we observe that:
1) as in the case of q-difference equations it is enough to prove the inequality for $\rho=1$;
2) it is enough to prove the inequality above for the first row of $G_n(x)$.
So we suppose $\rho=1$ and we call $\vec u=(u_1,\dots,u_\mu)\in\mathcal{M}_D^\mu$ the first row of $Y(x)$. Let $k\le\mu$ be the rank of $W_q(\vec u)$ and let $E\in Gl_\mu(K)$ be such that

$$\vec uE=(\vec z,\underline{0})\ ,\ \text{with } \vec z\in\mathcal{M}_D^k \text{ and } \underline{0}=(\underbrace{\underline{0,\dots,0}}_{\times(n-k)})\ .$$

By (5.4) the vectors $\vec h_n$, defined by

$$d_q^n(\vec z)=\vec h_n W_q(\vec z)\ ,\ \text{for any } n\in\mathbb{Z}_{\ge0},$$

satisfy the inequality

$$\left|\frac{\vec h_n}{[n]^!_q}\right|_{\xi,1} \le \{n,\mu-1\}^q_p\ .$$

Moreover the q-wronskian $W_q(\vec z)$ satisfies

$$\begin{pmatrix}\vec u\\ d_q\vec u\\ \vdots\\ d_q^{k-1}\vec u\end{pmatrix}E=(W_q(\vec z),\underline{0})\in M_{k\times\mu}(\mathcal{M}_D)\ .$$

In order to deduce an estimate of $\frac{G_n(x)}{[n]^!_q}$ from the above estimate of $\frac{\vec h_n}{[n]^!_q}$ we need to express $\frac{d_q^n}{[n]^!_q}(\vec u)$ in terms of $\frac{\vec h_n}{[n]^!_q}$:

$$\frac{d_q^n}{[n]^!_q}(\vec u)E=\frac{d_q^n}{[n]^!_q}(\vec uE)=\frac{d_q^n}{[n]^!_q}(\vec z,\underline{0})=\frac{\vec h_n}{[n]^!_q}(W_q(\vec z),\underline{0})=\frac{\vec h_n}{[n]^!_q}\begin{pmatrix}\vec u\\ d_q\vec u\\ \vdots\\ d_q^{k-1}\vec u\end{pmatrix}E\ .$$

Since $E\in Gl_\mu(K)$, we obtain

$$\left(\text{first row of }\frac{G_n(x)}{[n]^!_q}\right)Y(x)=\frac{d_q^n}{[n]^!_q}(\vec u)=\frac{\vec h_n}{[n]^!_q}\begin{pmatrix}\vec u\\ d_q\vec u\\ \vdots\\ d_q^{k-1}\vec u\end{pmatrix}$$

and hence we deduce that

$$\begin{aligned}\left\|\text{first row of } \frac{G_n(x)}{[n]_q^!}\right\|_{\xi,1} &\leq \left\|\tfrac{\vec{h}_n}{[n]_q^!}\right\|_{\xi,1} \left\| \begin{pmatrix} \vec{u} \\ d_q\vec{u} \\ \vdots \\ d_q^{k-1}\vec{u} \end{pmatrix} Y(x)^{-1} \right\|_{\xi,1} \\ &\leq \{n,\mu-1\}_p^q \left(\sup_{i=0,\dots,k-1} \|\text{first row of } G_i(x)\|_{\xi,1} \right) .\end{aligned}$$

This completes the proof. □

6. Some consequences: a transfer theorem in ordinary disks and a corollary about q-deformations.

6.1. Transfer theorem in ordinary disks

As in p-adic differential equation theory, the following transfer result follows from the effective bound theorem:

Corollary 6.2. *Let $G(x) \in M_{\mu\times\mu}(\mathcal{M}_{D(\xi,\rho^-)})$. We suppose that the q-difference system $d_qY(x) = G(x)Y(x)$ has a meromorphic fundamental solution over the q-invariant disk $D(\xi,\eta^-)$, with $\eta < \rho$, and that there exists a point ζ such that*
- $|\xi-\zeta| = \eta$,
- $G(x)$ is analytic over $D(\zeta,\eta^-)$.
Then $d_qY(x) = G(x)Y(x)$ has an analytic solution in $D(\zeta,\eta^-)$.

Proof. Let $d_q^nY(x) = G_n(x)Y(x)$ for any non-negative integer n. By the previous theorem we have

$$\left|\frac{G_n(t_{\xi,\eta})}{[n]_q^!}\right| = \left\|\frac{G_n(x)}{[n]_q^!}\right\|_{\xi,\eta} \leq C\{n,\mu-1\}_p^q \frac{1}{\eta^n} ,$$

where C is a constant depending only on $G(x)$ and η. The point $t_{\xi,\eta}$ is also a generic point at distance η from ζ, hence

$$\left|\frac{G_n(\zeta)}{[n]_q^!}\right| \leq \left|\frac{G_n(t_{\xi,\eta})}{[n]_q^!}\right| .$$

Let κ be the smallest positive integer such that $|1-q^\kappa| < |\pi|$ and let p^{l_n} be the greatest integer power of p smaller or equal to n. Then we have

$$\{n,\mu-1\}_p^q \leq \left|\left[p^{l_n}\kappa\right]_q\right|^{1-\mu} \leq \left(|p^{l_n}[\kappa]_q|\right)^{1-\mu} = \left(p^{-l_n}|[\kappa]_q|\right)^{1-\mu} \leq n^{\mu-1}|[\kappa]_q|^{1-\mu} .$$

We conclude by applying (1.4), since

$$\limsup_{n\to\infty}\left|\frac{G_n(\zeta)}{[n]_q^!}\right|^{1/n} \leq \eta^{-1}\limsup_{n\to\infty} n^{(\mu-1)/n}|[\kappa]_q|^{-(\mu-1)/n} = \eta^{-1} .$$

□

6.3. Effective bounds and q-deformation of p-adic differential equations

Let $q_k \in K$, $k \in \mathbb{N}$, be a sequence such that $q_k \to 1$ when $k \to \infty$ and $G^{(k)}(x)$ a sequence of square matrix of order μ whose entries are *analytic bounded functions* over an open disk D of center ξ and radius ρ, with $|\xi| \leq \rho \leq 1$. By rescaling, we can assume that $D = D(\xi, 1^-)$ and $|\xi| \leq 1$. Suppose that the sequence of matrices $G^{(k)}(x)$ tends to a matrix $G(x)$ uniformly over any $D(\xi, \eta^+) \subset D$. Then we say that the family of systems

$$(\mathcal{S})'_{q_k} \qquad d_{q_k} Y(x) = G^{(k)}(x) Y(x)$$

is a *q-deformation of the differential system*

$$(\mathcal{S}) \qquad \frac{dY}{dx}(x) = G(x) Y(x) \ .$$

Let $G_n^{(k)}(x)$ and $G_n(x)$, for $n \geq 0$, be the matrices respectively defined by

$$d_{q_k}^n Y(x) = G_n^{(k)}(x) Y(x) \text{ and } \frac{d^n Y}{dx^n}(x) = G_n(x) Y(x) \ .$$

Lemma 6.4. *Let $\varepsilon \in \mathbb{R}_{>0}$ and $\bar{n}$ be a positive integer. There exists $\bar{k} >> 0$, depending on ε and $\bar{n}$, such that, if*

$$\|G(x) - G^{(k)}(x)\|_{\xi,1} < \varepsilon, \text{ for any } k \geq \bar{k},$$

then

$$\|G_n(x) - G_n^{(k)}(x)\|_{\xi,1} < \varepsilon, \text{ for any } k \geq \bar{k} \text{ and any } 1 \leq n \leq \bar{n}.$$

Proof. Let $\bar{k}$ be a positive integer such that

$$|1 - q_k| \sup \Big(1, \|G_1(x)\|_{\xi,1}\Big)^{\bar{n}} < \varepsilon$$

for all $k \geq \bar{k}$. By assumption, $\|G(x) - G^{(k)}(x)\|_{\xi,1} < \varepsilon$. Suppose

$$\|G_n^{(k)}(x) - G_n(x)\|_{\xi,1} < \varepsilon \ ,$$

for $\bar{n} > n > 1$. Then by (2.2), for any $k \geq \bar{k}$, we have

$$\left\| d_{q_k} G_n^{(k)}(x) - \frac{d}{dx} G_n(x) \right\|_{\xi,1} < \varepsilon \ .$$

Moreover $G_n^{(k)}(x)$ are analytic bounded function over $D(\xi, 1^-)$, hence

$$\begin{array}{rcl} \left\| G_n^{(k)}(q_k x) - G_n^{(k)}(x) \right\|_{\xi,1} & \leq & |1 - q_k| \left\| G_n^{(k)}(x) \right\|_{\xi,1} \\ & \leq & |1 - q_k| \sup \Big(\varepsilon, \|G_n(x)\|_{\xi,1}\Big) \\ & \leq & |1 - q_k| \sup \Big(1, \|G_1(x)\|_{\xi,1}\Big)^n \\ & < & \varepsilon \ . \end{array}$$

Finally

$$
\begin{aligned}
&\|G_{n+1}^{(k)}(x)-G_{n+1}(x)\|_{\xi,1}\\
&\quad = \left\|G_1^{(k)}(x)G_n^{(k)}(q_kx)-d_{q_k}G_n^{(k)}(x)-G_1(x)G_n(x)+\tfrac{d}{dx}G_n(x)\right\|_{\xi,1}\\
&\quad \leq \sup\Big(\left\|(G_1^{(k)}(x)-G_1(x))G_n^{(k)}(q_kx)\right\|_{\xi,1},\left\|G_1(x)(G_n^{(k)}(q_kx)-G_n^{(k)}(x))\right\|_{\xi,1},\\
&\qquad \left\|G_1(x)(G_n^{(k)}(q_kx)-G_n(x))\right\|_{\xi,1},\left\|d_{q_k}G_n^{(k)}(x)-\tfrac{d}{dx}G_n(x)\right\|_{\xi,1}\Big)\\
&\quad < \varepsilon\ .
\end{aligned}
$$

□

Proposition 6.5. *Under the assumption above we have:*
1) For any $k>>0$ the q_k-difference system $d_{q_k}Y(x)=G^{(k)}(x)Y(x)$ has an analytic fundamental solution $Y^{(k)}(x)$ over a disk $D(\xi,\eta_k^-)\subset D$, verifying $Y^{(k)}(\xi)=\mathbb{I}_\mu$. Moreover, $\liminf_{k\to\infty}\eta_k>0$.
2) Let $\eta=\liminf_{k\to\infty}\eta_k$. Then $Y^{(k)}(x)$ tends pointwise over $D(\xi,\eta^-)$ to a fundamental solution $Y(x)$ of $\frac{dY(x)}{dx}=G(x)Y(x)$. Moreover $Y^{(k)}(x)$ tends uniformly to $Y(x)$ over $D(\xi,(\eta')^-)$, for any $0<\eta'<\eta$.

Proof.

1) By (3.1) the system $(\mathcal{S})'_{q_k}$ has a formal solution of the form

$$Y^{(k)}(\xi,x)=\sum_{n\geq 0}\frac{G_n^{(k)}(\xi)}{[n]^!_{q_k}}(x-\xi)_{q_k,n}\ .$$

We set

$$\eta_k^{-1}=\limsup_{n\to\infty}\left|\frac{G_n^{(k)}(\xi)}{[n]^!_{q_k}}\right|^{1/n}\ .$$

Since $G^{(k)}(x)$ converges uniformly to $G(x)$ over any closed disk contained in D, there exists a real positive constant C such that

$$\left\|G_n^{(k)}(x)\right\|_\xi(1)\leq\sup\left(1,\|\,G^{(k)}(x)\,\|_\xi(1)\right)^n\leq C^n$$

for any $k>>0$. Therefore for $k>>0$ we have $\eta_k\geq|\pi_q|C^{-1}\geq|\pi|C^{-1}$. It follows that, for $k>>0$, $\liminf_{k\to\infty}\eta_k\geq|\pi|C^{-1}>|(1-q_k)\xi|$. Hence the matrix $Y^{(k)}(\xi,x)$ is the q-expansion of an analytic fundamental solution of $d_{q_k}Y(x)=G^{(k)}(x)Y(x)$.

2) The proof of the second assertion follows faithfully the proof of [DGS94, IV, 5.4]. Let η_k be the radius of convergence of $Y^{(k)}(\xi,x)$. Letting $k\to\infty$ in the effective bound estimate

$$\left|\frac{G_n^{(k)}(\xi)}{[n]^!_{q_k}}\right| \leq \{n,\mu-1\}_p^{q_k}\, C_k \frac{1}{\eta_k^n}\ ,\ \text{where } C_k = \sup_{i=0,\dots,\mu-1} \left\|G_i^{(k)}(x)\right\|_{\xi,\eta_k} \eta_k^i\ ,$$

we obtain (cf. [DGS94, IV, 3.1])

$$\left|\frac{G_n(\xi)}{n!}\right| \leq \{n,\mu-1\}_p\, C\frac{1}{\eta^n}\ ,\ \text{where } C = \sup_{i=0,\dots,\mu-1} \|G_i(x)\|_{\xi,\eta}\, \eta^i\ ,$$

$$\text{and } \{n,\mu-1\}_p = \begin{cases} 1, & \text{if n=0,}\\ \displaystyle\sup_{1\leq\lambda_1<\dots<\lambda_{\mu-1}\leq n} \frac{1}{|\lambda_1\cdots\lambda_{\mu-1}|}, & \text{otherwise.}\end{cases}$$

This proves that the solution $Y(\xi,x)$ of $(\mathcal{S})_q$ at ξ converges over $D(\xi,\eta^-)$.

As far as the uniform convergence is concerned, notice that for any $R\in(0,\eta)$ and $\varepsilon>0$ there exist $\bar{n},\bar{k}\in\mathbb{N}$ such that for any $n\geq\bar{n}$ and any $k\geq\bar{k}$ we have:

$$\left\|\frac{G_n^{(k)}(\xi)}{[n]^!_{q_k}}(x-\xi)_{q_k,n}\right\|_\xi (R) \leq C_k\{n,\mu-1\}_p^{q_k}\left(\frac{R}{\eta_k}\right)^n \leq C_k n^{\mu-1}\left(\frac{R}{\eta_k}\right)^n < \varepsilon$$

and

$$\left\|\frac{G_n(\xi)}{n!}(x-\xi)^n\right\|_\xi (R) \leq C\{n,\mu-1\}_p\left(\frac{R}{\eta}\right)^n \leq C n^{\mu-1}\left(\frac{R}{\eta}\right)^n < \varepsilon\ .$$

By considering a bigger $\bar{k}$, we deduce from (6.4) that for any $k\geq\bar{k}$ we have also

$$\begin{aligned}&\left\|\sum_{n=0}^{\bar{n}}\frac{G_n^{(k)}(\xi)}{[n]^!_{q_k}}(x-\xi)_{q_k,n} - \sum_{n=0}^{\bar{n}}\frac{G_n(\xi)}{n!}(x-\xi)^n\right\|_\xi (R)\\ &\quad\leq \sup_{n=0,\dots,\bar{n}}\left\|\frac{G_n^{(k)}(\xi)}{[n]^!_{q_k}}(x-\xi)_{q_k,n} - \frac{G_n(\xi)}{n!}(x-\xi)^n\right\|_\xi (R) < \varepsilon\ .\end{aligned}$$

Finally we obtain the uniform convergence over any $D(\xi,R^-)$:

$$\begin{aligned}&\left\|Y(\xi,x)-Y^{(k)}(\xi,x)\right\|_\xi(R)\\ &\quad\leq \sup\left(\left\|\sum_{n=0}^{\bar{n}}\frac{G_n^{(k)}(\xi)}{[n]^!_{q_k}}(x-\xi)_{q_k,n} - \sum_{n=0}^{\bar{n}}\frac{G_n(\xi)}{n!}(x-\xi)^n\right\|_\xi (R),\right.\\ &\qquad\left.\sup_{n\geq\bar{n}}\left\|\frac{G_n^{(k)}(\xi)}{[n]^!_{q_k}}(x-\xi)_{q_k,n} - \frac{G_n(\xi)}{n!}(x-\xi)^n\right\|_\xi (R)\right) < \varepsilon\ .\end{aligned}$$

$\square$

III. Weak Frobenius structure over a disk

Warning. *In chapters III and IV we will assume that $|1-q|<|\pi|=|\pi_q|$: this implies that for any integer n we have*

$$|1-q^n| = |\log q^n| = |n||\log q| = |n||1-q| \ .$$

Moreover throughout all of chapters III and IV we will consider only q-difference systems over $D=D(0,1^-)$, therefore we will suppress everywhere the index "${}_{0,1}$": so we will write t for $t_{0,1}$, $\| \ \|$ for $\| \ \|_{0,1}$, $\chi(A,q)$ for $\chi_{0,1}(A,q)$ and so on.

In this chapter we study the action of the Frobenius map $x \longmapsto x^{p^\ell}$, $\ell \in \mathbb{Z}_{>0}$, on q-difference systems. Namely, following [Chr84], under convenient assumptions, we will construct a matrix $H(x)$, depending on the q-difference system $Y(qx)=A(x)Y(x)$, such that $A_{[H]}(x)$ is a function of x^{p^ℓ}. This means that

$$Y(qx) = A_{[H]}(x)Y(x)$$

is actually a q^{p^ℓ}-difference system in the variable x^{p^ℓ}. The interest of such a construction is that it changes the generic radius of convergence. In fact,

$$\chi(A_{[H]}(x^{p^\ell}), q^{p^\ell}) = \chi(A(x),q)^{p^\ell} \ .$$

The original Christol theorem in [Chr84] for p-adic differential equation, as well as its generalization to differential modules over an annulus, due to Christol-Dwork [CD94], is an irreplaceable tool: we think that the theorem above is destined to play an analogous role for q-difference equations.

7. Frobenius action on q-difference systems.

Until now we have worked with analytic and meromorphic functions: in the next sections we will restrict our attention to analytic elements. An introduction to the theory of analytic elements can be found for instance in [Rob00, §6, 4]. We briefly recall the definition:

Definition 7.1. *The ring E_0 of the* analytic elements on the disk $D(0,1^-)$ (defined over K) *is the completion of the subring of $K(x)$ of all rational functions not having poles in $D(0,1^-)$ with respect to the norm $\| \ \|$.*

We denote by E_0' its quotient field.

Remark 7.2.

1) Of course one can define analytic elements over any disk, but in this chapter we will deal with disks centered at 0. Therefore, by rescaling, we will always consider analytic elements over $D(0,1^-)$.

2) In the sequel we will essentially use two properties of analytic elements, namely:

- By taking Taylor expansion at zero, one can identify analytic elements with bounded analytic functions over $D(0,1^-)$: this ring embedding is an isometry with respect to $\| \ \|$.
- Any element of E_0 has a finite number of zeros in $D(0,1^-)$ (cf. [DGS94, IV, 5.2]), hence any element of E_0' has a finite number of zeros and poles in $D(0,1^-)$.

The following theorem concerns a class $\mathcal{H}_q^\ell$, $\ell \in \mathbb{Z}_{>0}$, of q-difference systems $Y(qx) = A(x)Y(x)$ satisfying the properties:

1. $A(x) \in Gl_\mu(E_0')$;
2. $A(x)$ is analytic at 0 and $A(0) \in Gl_\mu(K)$;
3. $Y(qx) = A(x)Y(x)$ has at worst apparent singularities in $D(0,1^-) \smallsetminus \{0\}$;
4. $\chi(A,q) > |\pi|^{\frac{1}{p^{\ell-1}}}$.

We will say that a system in $\mathcal{H}_q^\ell$ is in *normal form* if moreover it satisfies the conditions 1 through 3 of the following proposition (cf. §8 for the proof):

Proposition 7.3. *Let $Y(qx) = A(x)Y(x)$ be in $\mathcal{H}_q^\ell$. Then there exists $U \in Gl_\mu(K(x))$ such that the q-difference system associated to $A_{[U]}(x)$ is in $\mathcal{H}_q^\ell$ and moreover:*

1. *$A_{[U]}(x) \in Gl_\mu(E_0)$;*
2. *any two eigenvalues λ_1, λ_2 of $A_{[U]}(0)$ satisfy either $\lambda_1 = \lambda_2$ or $\lambda_1 \notin \lambda_2 q^{\mathbb{Z}}$;*
3. *any eigenvalue λ of $A_{[U]}(0)$ satisfies $\left|\frac{\lambda-1}{q-1}\right| < p^{1-\ell}$.*

Finally we are able to state the main theorem:

Theorem 7.4. *Let us consider a q-difference system in $\mathcal{H}_q^\ell$ in its normal form*

$$Y(qx) = A(x)Y(x) \ .$$

Then there exists a matrix $H(x) \in Gl_\mu(E_0') \cap M_{\mu\times\mu}(E_0)$ such that

(7.4.1) *$H(0) \in Gl_\mu(K)$ and $|H(0)| = \left|H(0)^{-1}\right| = 1$;*

(7.4.2) *$A_{[H]}(x) = F(x^{p^\ell})$;*

(7.4.3) *the eigenvalues of $A(0)$ and $F(0)$ coincide.*

Moreover the q^{p^ℓ}-difference system $V(q^{p^\ell}X) = F(X)V(X)$, with $X = x^{p^\ell}$, has the following properties:

(7.4.4) *$\chi(F, q^{p^\ell}) = \chi(A,q)^{p^\ell}$;*

(7.4.5) *$F(X) \in Gl_\mu(E_0)$.*

Remark 7.5. It follows from §3 that the matrix $A(x) \in Gl_\mu(E_0')$ of a q-difference system $Y(qx) = A(x)Y(x)$, having only ordinary q-orbits in $D(0,1^-) \smallsetminus \{0\}$ and a regular singularity at 0 (cf. Appendix B), is necessarily an element of $Gl_\mu(E_0)$.

The proof of the above theorem, which follows the proof of Christol [Chr84], is the goal of §9.

8. Normal form for a system in $\mathcal{H}_q^\ell$.

Proposition 7.3 is a consequence of the following lemma:

Lemma 8.1. *If $\chi(A,q) > |\pi|^{\frac{1}{p^{\ell-1}}}$, then for any eigenvalue λ of $A(0)$ we have:*

$$\mathrm{dist}(\lambda, q^{\mathbb{Z}_p}) = \inf_{\alpha\in\mathbb{Z}_p} |\lambda - q^\alpha| \leq p^{1-\ell}|q-1| \ ,$$

where q^α, for $\alpha\in\mathbb{Z}_p$, is defined as the sum of the binomial series

$$q^\alpha = \sum_{n\geq 0} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}(q-1)^n \ .$$

In fact:

Proof of (7.3). There exists a *shearing transformation* $U_1(x)$ with coefficients in $K\left[x,\frac{1}{x}\right]$, (cf. Appendix B, in particular (17.3)) such that the eigenvalues of $A_{[U_1]}(0)$ are the eigenvalues of $A(0)$, multiplied by chosen powers of q. Hence by the previous lemma we can assume that any two eigenvalues λ_1, λ_2 of $A_{[U_1]}(0)$ satisfy the conditions:

1) either $\lambda_1=\lambda_2$ or $\lambda_1\notin\lambda_2 q^{\mathbb{Z}}$;

2) $\left|\dfrac{\lambda_1-1}{q-1}\right| < p^{1-\ell}$.

Then by (3.11) there exists a matrix $U_2(x)$, with coefficients in $K(x)$ and analytic at 0, such that $A_{[U_2U_1]}(x)$ has only ordinary orbits in $D\smallsetminus\{0\}$.

Since $A_{[U_2U_1]}(x)$ is analytic at 0 and $Y(qx)=A_{[U_2U_1]}(x)Y(x)$ has only ordinary orbits in $D\smallsetminus\{0\}$, we conclude that $A_{[U_2U_1]}(x)\in Gl_\mu(E_0)$. Moreover the eigenvalues of $A_{[U_1]}(0)$ and $A_{[U_2U_1]}(0)$ coincide, since $U_2(0)\in Gl_\mu(K)$. Therefore it is enough to set $U(x)=U_2(x)U_1(x)$ to finish the proof. □

Let us prove lemma 8.1:

Proof of (8.1). Consider the sequence of q-difference systems

$$\frac{d_q^n}{[n]_q^!}Y(x) = \mathcal{G}_n(x)Y(x) \ , \ \text{for all } n\geq 0, \tag{8.1.1}$$

obtained iterating

$$Y(qx) = A(x)Y(x). \tag{$\mathcal{S}$}_q$$

Then

$$\mathcal{G}_1(x) = \frac{1}{x}\frac{A(0)-\mathbb{I}_\mu}{q-1} + \text{higher order terms.}$$

By (1.2.4) and (1.1.2), this formula generalizes to any $n \geq 1$ in the following way:

$$\mathcal{G}_n(x) = \frac{q^{-\frac{n(n-1)}{2}}}{x^n} \frac{(A(0)-1)(A(0)-q)\cdots\left(A(0)-q^{n-1}\right)}{(q-1)(q^2-1)\cdots(q^n-1)} + \text{h.o.t.} \tag{8.1.2}$$

We set

$$\widetilde{\mathcal{G}}_n = \frac{(A(0)-1)(A(0)-q)\cdots\left(A(0)-q^{n-1}\right)}{(q-1)(q^2-1)\cdots(q^n-1)} . \tag{8.1.3}$$

Since

$$\left|\widetilde{\mathcal{G}}_n\right| \leq \|x^n \mathcal{G}_n(x)\|$$

the series $g(x) = \sum_{n\geq 1} \widetilde{\mathcal{G}}_n x^n$ converges for $|x| < \chi(A,q)$, with $\chi(A,q) > |\pi|^{\frac{1}{p^{\ell}-1}}$.

Let $C \in Gl_\mu(K)$ be a constant matrix such that $C^{-1}A(0)C$ is a matrix in the Jordan normal form, then

$$C^{-1}g(x)C = \sum_{n\geq 1} \frac{\left(C^{-1}A(0)C-1\right)\left(C^{-1}A(0)C-q\right)\cdots\left(C^{-1}A(0)C-q^{n-1}\right)}{(q-1)(q^2-1)\cdots(q^n-1)} x^n .$$

Hence we conclude that the eigenvalues of $A(0)$ satisfy the desired inequality by applying the following result:

Lemma 8.2. *Let $\lambda \in K$ and let $\Lambda(x) = \sum_{n\geq 1} \lambda_n x^n$, with*

$$\lambda_n = \frac{(\lambda-1)(\lambda-q)\cdots\left(\lambda-q^{n-1}\right)}{(q-1)(q^2-1)\cdots(q^n-1)} .$$

Then:
1) The series $\Lambda(x)$ converges for $|x| < 1$ if and only if $\lambda \in q^{\mathbb{Z}_p}$, i.e. if and only if $\mathrm{dist}(\lambda, q^{\mathbb{Z}_p}) = 0$.
2) If $\mathrm{dist}(\lambda, q^{\mathbb{Z}_p}) = |p|^{k+\varepsilon}|q-1|$, *with $k \in \mathbb{Z}_{\geq 0}$ and $\varepsilon \in [0,1)$, then $\Lambda(x)$ converges for $|x| < |p|^{-\frac{1}{p^k}\left(\frac{\varepsilon}{p}-\frac{1}{p-1}\right)}$.*
3) If the series $\Lambda(x)$ converges for $|x| \leq |\pi|^{\frac{1}{p^{\ell}-1}}$, then $\mathrm{dist}(\lambda, q^{\mathbb{Z}_p}) \leq p^{1-\ell}|q-1|$.

Proof.

1) Let $\alpha \in \mathbb{Z}_p$ be such that $\lambda = q^\alpha$. If n_k is a sequence of integers such that $n_k \to \alpha$, then the binomial series x^{n_k} tends uniformly to x^α over any closed disk $D(1, R^+)$, with $R < 1$ (cf. [DGS94, IV, 5.4 and IV, §7]). Hence, since $|q^n - 1| \leq |q-1| < |\pi|$ for any $n \in \mathbb{Z}$, by the density of $\mathbb{Z}$ in $\mathbb{Z}_p$, we have also $|q^{\alpha-n} - 1| < |\pi|$. We deduce

from the p-adic properties of logarithm that

$$
\begin{aligned}
|\lambda_n| &= \left|\frac{(q^\alpha - 1)\left(q^\alpha q^{-1} - 1\right)\cdots\left(q^\alpha q^{1-n} - 1\right)}{(q-1)(q^2-1)\cdots(q^n-1)}\right| \\
&= \frac{|\log q^\alpha||\log(q^\alpha q^{-1})|\cdots|\log(q^\alpha q^{1-n})|}{|\log q|^n|n!|} \qquad (8.2.1)\\
&= \frac{1}{|n!|}\left|\frac{\log q^\alpha}{\log q}\right|\left|\frac{\log q^\alpha}{\log q} - 1\right|\cdots\left|\frac{\log q^\alpha}{\log q} - (n-1)\right| .
\end{aligned}
$$

Since $\log q^n = n\log q$ for any n and $\mathbb{Z}$ is dense in $\mathbb{Z}_p$, one verifies that $\frac{\log q^\alpha}{\log q} = \alpha$. We conclude that the radius of convergence of $\Lambda(x)$ is equal to the radius of convergence of the binomial series $\sum_{n\geq 0}\binom{\alpha}{n}x^n$, hence is equal to 1 [DGS94, IV, 7.5].

Let us prove the opposite implication. Remark that

$$
\lambda_n = \frac{1}{[n]_q^!}\frac{\lambda-1}{q-1}\left(\frac{\lambda-1}{q-1} - [1]_q\right)\cdots\left(\frac{\lambda-1}{q-1} - [n-1]_q\right) .
$$

Hence if $\left|\frac{\lambda-1}{q-1}\right| > 1$ we have

$$
|\lambda_n| = \left|\frac{\lambda-1}{q-1}\right|^n\frac{1}{|n!|}
$$

and the radius of convergence of $\Lambda(x)$ would be $\left|\pi\frac{q-1}{\lambda-1}\right| < 1$. We conclude that $\Lambda(x)$ convergent for $|x|<1$ implies $|\lambda-1|\leq|q-1|<|\pi|$. Suppose that $\Lambda(x)$ is convergent for $|x|<1$. As in (8.2.1) we obtain:

$$
|\lambda_n| = \frac{1}{|n!|}\left|\frac{\log\lambda}{\log q}\right|\left|\frac{\log\lambda}{\log q} - 1\right|\cdots\left|\frac{\log\lambda}{\log q} - (n-1)\right| .
$$

We conclude that $\alpha = \frac{\log\lambda}{\log q} \in \mathbb{Z}_p$ since the series $\sum_{n\geq 0}\binom{\log\lambda/\log q}{n}x^n$ converges for $|x|<1$ [DGS94, IV, 7.5]. Then $\log\lambda = \log q^\alpha$, with $|\lambda-1|, |q^\alpha - 1| < |\pi|$, and therefore

$$
\lambda = \exp\log\lambda = \exp\log q^\alpha = q^\alpha .
$$

2) Since $\mathbb{Z}_p$ is compact there exists $\alpha\in\mathbb{Z}_p$ such that $|\lambda - q^\alpha| = |q-1||p|^{k+\varepsilon}$. Let s be a nonnegative integer such that $|q^s - q^\alpha| < |q-1||p|^{k+\varepsilon}$; then $|\lambda - q^s| = |q-1||p|^{k+\varepsilon} = \mathrm{dist}(\lambda, q^{\mathbb{Z}_p})$. Hence we can choose $\alpha = s\in\mathbb{Z}_{\geq 0}$.

Let us suppose $s=0$. Then for any $m\in\mathbb{Z}$ we have:

$$
\left|\frac{\lambda - q^m}{q-1}\right| = \left|\frac{\lambda-1}{q-1} - \frac{q^m-1}{q-1}\right| \geq \left|\frac{\lambda-1}{q-1}\right|
$$

We obtain the estimate (cf. for instance [BC92b, 2.2] or [DGS94, IV, 7.3] for the analogous classical estimate):

$$\mathrm{ord}_p([n]_q^!\lambda_n)=\sum_{i=1}^{k}\left(1+\left[\frac{n-1}{p^i}\right]\right)+\varepsilon\left[\frac{n-1}{p^{k+1}}\right]=\mathrm{ord}_p\binom{\frac{\lambda-1}{q-1}}{n}\ ,$$

with

$$\binom{\frac{\lambda-1}{q-1}}{n}=\frac{1}{n!}\frac{\lambda-1}{q-1}\left(\frac{\lambda-1}{q-1}-1\right)\cdots\left(\frac{\lambda-1}{q-1}-n+1\right)\ .$$

Since $|[n]_q^!|=|n!|$, the radius of convergence of $\Lambda(x)$ is the same as the radius of convergence of the binomial series $\sum_{n\geq 0}\binom{\frac{\lambda-1}{q-1}}{n}x^n$, namely $p^{\frac{1}{p^k}\left(\frac{\varepsilon}{p}-\frac{1}{p-1}\right)}$ (see [BC92b, 2.2] or [DGS94, IV, 7.3] for the explicit calculation).

3) If $\lambda\in q^{\mathbb{Z}_p}$ the assertion follows immediately from 1), hence it is enough to consider the case $\lambda\notin q^{\mathbb{Z}_p}$. Moreover, if $\left|\frac{\lambda-1}{q-1}\right|>1$ the series $\Lambda(x)$ has radius of convergence $\left|\pi\frac{q-1}{\lambda-1}\right|<|\pi|^{\frac{1}{p^{\ell-1}}}$, therefore λ necessarily satisfies $\mathrm{dist}(\lambda,q^{\mathbb{Z}_p})\leq|q-1|$. Let

$$\frac{\mathrm{dist}(\lambda,q^{\mathbb{Z}_p})}{|q-1|}=\left|\frac{\lambda-1}{q-1}\right|=|p|^{k+\varepsilon}\ ,$$

with $k\in\mathbb{Z}_{\geq 0}$ and $\varepsilon\in[0,1)$. We finish the proof by observing that $|\pi|^{\frac{1}{p^{\ell-1}}}\leq p^{\frac{1}{p^k}\left(\frac{\varepsilon}{p}-\frac{1}{p-1}\right)}$ implies that $\ell-1<k+\varepsilon$ (cf. again [BC92b, 2.2] or [DGS94, IV, 7.3]).

Finally if $s>0$, then for any $n>s$ we have:

$$\lambda_n=\lambda_s\frac{(q^{-s}\lambda-1)\,(q^{-s}\lambda-q)\cdots\left(q^{-s}\lambda-q^{n-s-1}\right)}{(q-1)(q^2-1)\cdots(q^{n-s-1}-1)}\ .$$

By taking $q^{-s}\lambda$ instead of λ, we reduce to the case $s=0$. □

9. Proof of (7.4).

The proof of (7.4) is divided into two steps:

Step 1: construction of the matrix $H(x)$ and the proof of (7.4.1).

Step 2: proof of (7.4.2) through (7.4.5).

They are handled respectively in (9.1) and (9.2).

9.1. Construction of the matrix $H(x)$ and proof of (7.4.1).

By assumption (cf. (8.1.1) for notation) the solution $Y(\xi,x)=\sum_{n\geq 0}\mathcal{G}_n(\xi)(x-\xi)_{q,n}$ of $Y(qx)=A(x)Y(x)$ is defined for any $\xi\in(D(0,1^-)\smallsetminus\{0\})\cup D(t,1^-)$ and converges for $|x-\xi|<\chi(A,q)$. We set:

$$\text{(9.1.1)}\qquad H(x)\ :=\ \frac{1}{p^\ell}\sum_{\zeta^{p^\ell}=1}Y(x,\zeta x)=\frac{1}{p^\ell}\sum_{n\geq 0}\mathcal{G}_n(x)x^n\sum_{\zeta^{p^\ell}=1}(\zeta-1)_{q,n}\ .$$

Since $\mathcal{G}_n(x)x^n \in M_{\mu\times\mu}(E_0)$, with

$$\liminf_{n\to\infty} \|\mathcal{G}_n(x)x^n\|^{-1/n} = \chi(A,q),$$

and

$$\begin{aligned} |(\zeta-1)_{q,n}| &= \prod_{i=0}^{n} |\zeta-1+1-q^i| \\ &\leq \prod_{i=0}^{n} \sup\left(|\zeta-1|,|i||1-q|\right) \\ &\leq |\pi|^{\frac{n}{p^{\ell-1}}} < \chi(A,q)^n \ , \end{aligned}$$

the matrix $H(x)$ is a well-defined element of $M_{\mu\times\mu}(E_0)$ and it makes sense to evaluate $H(x)$ at 0.

The constant term of $\mathcal{G}_n(x)x^n$ is equal to $q^{-\frac{n(n-1)}{2}}\widetilde{\mathcal{G}}_n$ (cf. (8.1.3)). Hence we have

$$H(0) = \frac{1}{p^\ell}\sum_{n\geq 0} q^{-\frac{n(n-1)}{2}}\widetilde{\mathcal{G}}_n \sum_{\zeta^{p^\ell}=1} (\zeta-1)_{q,n}$$

By (8.2), $H(0)$ converges in K. Since (cf. [DGS94, IV, 7.3])

$$\frac{1}{p^\ell}\sum_{n\geq 0}\sum_{\zeta^{p^\ell}=1} \binom{\widetilde{\mathcal{G}}_1}{n}(\zeta-1)^n = \mathbb{I}_\mu$$

and

$$|H(0)-\mathbb{I}_\mu| = \left|\frac{1}{p^\ell}\sum_{n\geq 0}\sum_{\zeta^{p^\ell}=1}\left(q^{-\frac{n(n-1)}{2}}\widetilde{\mathcal{G}}_n(\zeta-1)_{q,n} - \binom{\widetilde{\mathcal{G}}_1}{n}(\zeta-1)^n\right)\right| < 1 \ ,$$

we conclude that $|H(0)| = 1$. This also proves that $H(0)$ has an inverse, which has norm 1.

9.2. The matrix $H(x)$ satisfies (7.4.2) through (7.4.5).

By assumption the solution $Y(t,x)$ converges for $|x-t| < \chi(A,q)$. It follows from [DGS94, VI, §6] that the matrix

$$V(x) = \frac{1}{p^\ell}\sum_{z^{p^\ell}=x} Y(t,z)$$

is analytic over the disk $D(t^{p^\ell},\chi(A,q)^{p^\ell})$. The formula (3.2.1) implies that

$$\begin{array}{rcl} V(x^{p^\ell}) & = & \displaystyle\frac{1}{p^\ell}\sum_{z^{p^\ell}=x^{p^\ell}} Y(t,z) \\ & = & \displaystyle\frac{1}{p^\ell}\sum_{\zeta^{p^\ell}=1} Y(t,\zeta x) \\ & = & \displaystyle\frac{1}{p^\ell}\sum_{\zeta^{p^\ell}=1} Y(x,\zeta x)Y(t,x) \\ & = & H(x)Y(t,x)\ . \end{array}$$

Hence the matrix

$$F(x)=V(q^{p^\ell}x)V(x)^{-1}$$

satisfies (7.4.2). In fact,

$$\begin{array}{rcl} F(x^{p^\ell}) & = & V(q^{p^\ell}x^{p^\ell})V(x^{p^\ell})^{-1} \\ & = & H(qx)Y(t,qx)Y(t,x)^{-1}H(x)^{-1} \\ & = & H(qx)A(x)H(x)^{-1}\ . \end{array}$$

Then (7.4.3) follows from (7.4.1).

Let $X=x^{p^\ell}$. Then $V(X)$ is solution to

$$V(q^{p^\ell}X)=F(X)V(X)\ .$$

Since $V(X)$ is analytic around t^{p^ℓ} and $V(t^{p^\ell})=H(t)$, the generic radius $\chi(F,q^{p^\ell})$ at t^{p^ℓ} coincides with the radius of convergence of $H(t)^{-1}V(X)$. Hence

$$\chi(F,q^{p^\ell})=\chi(A,q)^{p^\ell}\ .$$

Finally let $\xi\in D(0,1^-)$, $\xi\neq 0$. Similar calculations to the ones we have worked out for $V(x)$ shows that an analytic fundamental solution of $V(q^{p^\ell}X)=F(X)V(X)$ at ξ^{p^ℓ} is given by

$$\frac{1}{p^\ell}\sum_{z^{p^\ell}=x^{p^\ell}} Y(\xi,z)\ .$$

Since

$$\begin{array}{rcl} D(0,1^-) & \longrightarrow & D(0,1^-) \\ \xi & \longmapsto & \xi^{p^\ell} \end{array}$$

is a bijective map, this proves (7.4.5).

IV. Transfer theorems in regular singular disks

In §6 we have seen how the generic radius of convergence is linked to the radius of convergence of analytic solutions at ordinary q-orbits. By (3.11) we are able to deduce estimates also for the radius of meromorphy of solutions at apparent and trivial singularities. In this chapter we are going to consider a q-difference system having a regular singularity at 0 (cf. Appendix B for a summary of basic properties of regular singular q-difference equations). In particular we are going to prove a q-difference version of the Christol-André-Baldassarri-Chiarellotto theorem (cf. [Chr84], [And87], [BC92a] and [BC92b]).

10. An analogue of Christol's theorem.

A q-difference system

$$Y(qx) = A(x)Y(x)$$

with coefficients in E_0' is *regular singular at* 0 if it is regular singular regarded as a q-difference system with coefficients in the field $K((x))$ via the canonical immersion $E_0' \hookrightarrow K((x))$, i.e. if there exists $U(x) \in Gl_\mu(K((x)))$ such that $A_{[U]}(x) \in Gl_\mu(K[\![x]\!])$.

Let $r(U(x))$ be the radius of convergence of the matrix $x^N U(x)$, for N large enough to have $x^N U(x) \in Gl_\mu(K[\![x]\!])$. Very similarly to the differential case, the theorem below establishes an estimate of $r(U(x))$ with respect to the generic radius of convergence and a certain number attached to the eigenvalues of $A_{[U]}(0)$:

Definition 10.1. *We call the* q-type *of* $\alpha \in K$*, and we write* $\mathrm{type}_q(\alpha)$*, the radius of convergence of*

$$\sum_{n\geq 0, \alpha\neq q^n} \frac{1-q}{1-q^n\alpha} x^n .$$

Remark 10.2. In the p-adic theory of differential equations one calls the type of a number $\alpha \in K$ the radius of convergence $\mathrm{type}(\alpha)$ of the series

$$\sum_{n\geq 0, n\neq\alpha} \frac{x^n}{n-\alpha} .$$

The definition above is an analogue of this notion (see Appendix C, §19, for an estimate of the q-type in terms of the classical type and some general properties of the q-type).

The following is a q-analogue of Christol theorem [Chr84]. It is a transfer theorem for q-difference systems $Y(qx) = A(x)Y(x)$ such that $\chi(A, q) = 1$. Statements concerning the general situation can be found in §13.

Theorem 10.3. *Let us consider a system* $Y(qx) = A(x)Y(x)$ *such that*

1) $A(x) \in Gl_\mu(E'_0)$*;*

2) the system $Y(qx) = A(x)Y(x)$ *has only apparent singularities in* $D(0,1^-)\smallsetminus\{0\}$*;*

3) the system $Y(qx) = A(x)Y(x)$ *has a regular singularity at* 0*, i.e. there exists* $U(x) \in Gl_\mu(K((x)))$ *such that* $A_{[U]} \in M_{\mu\times\mu}(K)$*;*

4) $\chi(A,q) = 1$*.*

Then

$$r(U(x)) \geq \prod \mathrm{type}_q\left(\alpha\beta^{-1}\right)\ , \tag{10.3.1}$$

where the product is taken over all the (ordered) couples of eigenvalues α,β *of* $A_{[U]}$*.*

The proof of this theorem, as for the Christol-André-Baldassarri-Chiarellotto theorem, which inspired it, relies on the existence of the weak Frobenius structure and it is quite long. Sections §11 and §12 are devoted to the proof of some preliminary estimates of $r(U(x))$. As a corollary, in §13 we prove the transfer theorem 10.3 plus some more general statements, without any assumption on $\chi(A,q)$.

11. A first rough estimate.

In this section we prove a first estimate, which is not very sharp, but crucial for the proof of (10.3):

Proposition 11.1. *Let* $Y(qx) = A(x)Y(x)$ *be a system such that*

1) $A(x) \in Gl_\mu(E_0)$*;*

2) the system $Y(qx) = A(x)Y(x)$ *has only ordinary* q*-orbits in* $D(0,1^-)\smallsetminus\{0\}$*;*

3) any two eigenvalues α,β *of* $A(0)$ *satisfy either* $\alpha=\beta$ *or* $\alpha\beta^{-1}\notin q^{\mathbb{Z}}$*.*

Then the matrix $U(x) \in Gl_\mu(K[\![x]\!])$ *satisfying* $A_{[U]}(x) = A(0)$ *is such that*

$$r(U(x)) \geq \inf\left(|\pi|,\chi(A,q)\right)^{\mu^2} |\det A(0)|^\mu \prod_{\alpha,\beta\ \text{e.v. of}\ A(0)} \mathrm{type}_q\left(\alpha\beta^{-1}\right)\ .$$

Remark 11.2. Observe that under the assumption of the proposition above, the matrix $U(x)$ always exists (cf. Appendix B, (17.2)).

Proof. The proof is divided into two steps:

Step 1. One can assume that $Y(qx) = A(x)Y(x)$ *satisfies*

$$\left\|\frac{A(x)-\mathbb{I}_\mu}{(q-1)x}\right\| \leq \sup\left(1,\frac{|\pi|}{\chi(A,q)}\right)\ . \tag{11.2.1}$$

Let us set $\chi = \chi(A, q)$ to simplify notation. By the cyclic vector lemma there exists a matrix $H_1(x) \in Gl_\mu(E_0')$ such that the matrix $A_{[H_1]}(x)$ is in the form

$$A_{[H_1]}(x) = \left(\begin{array}{c|c} 0 & \\ \vdots & \mathbb{I}_{\mu-1} \\ 0 & \\ \hline & \\ a_0(x) & a_1(x) \dots a_{\mu-1}(x) \end{array}\right) .$$

An analogue of Fuchs' theory for q-difference equation (cf. Appendix B, §18) assures that $A_{[H_1]}(x)$ is analytic at 0 and $A_{[H_1]}(0)$ is an invertible constant matrix. Consider the matrix $\widetilde{H}$:

$$\widetilde{H} = (a_{i,j})_{i,j=0,\dots,\mu-1} \ , \text{ with } a_{i,j} = \begin{cases} \dfrac{(-1)^j}{(q-1)^i} \dbinom{i}{j} & \text{if } j \leq i; \\ \\ 0 & \text{otherwise.} \end{cases}$$

Then we have (cf. (1.2.5))

$$xG_{[\tilde{H}H_1]} = \frac{A_{[\tilde{H}H_1]}(x) - \mathbb{I}_\mu}{q-1} = \left(\begin{array}{c|c} 0 & \\ \vdots & \mathbb{I}_{\mu-1} \\ 0 & \\ \hline & \\ b_0(x) & b_1(x) \dots b_{\mu-1}(x) \end{array}\right) .$$

Remark that $xG_{[\tilde{H}H_1]}$ is analytic at 0 and $A_{[\tilde{H}H_1]}(0) \in Gl_\mu(K)$, since $A_{[H_1]}(x)$ has the same properties.

By proposition 4.6, if $\chi \geq |\pi|$, then

$$\left\| G_{[\tilde{H}H_1]} \right\| \leq 1 .$$

On the other hand, if $\chi < |\pi|$, we immediately deduce from (4.3) and (4.6) that

$$\chi = \frac{|\pi|}{\sup_i \|b_i(x)\|^{1/(\mu-i)}} .$$

Since K is algebraically closed, we can find $\gamma \in K$ such that

$$|\gamma| = \sup_i \|b_i(x)\|^{1/(\mu-i)} ;$$

then an explicit calculation shows that the diagonal gauge transformation matrix

$$H_2 = \begin{pmatrix} 1 & & & \\ & 1/\gamma & & \\ & & \ddots & \\ & & & 1/\gamma^{\mu-1} \end{pmatrix}$$

satisfies

$$\left\|G_{[H_2\tilde{H}H_1]}(x)\right\| \leq \frac{|\pi|}{\chi} .$$

We set $H(x) = \widetilde{H}H_1(x)$ if $\chi \geq |\pi|$ and $H(x) = H_2\widetilde{H}H_1(x)$ otherwise. In both cases $H(x) \in Gl_\mu(E'_0)$, $A_{[H]}(x)$ is analytic at 0 with $A_{[H]}(0) \in Gl_\mu(K)$, since $A_{[\tilde{H}H_1]}(x)$ has the same properties, and

$$\left\|G_{[H]}(x)\right\| \leq \sup\left(1, \frac{|\pi|}{\chi}\right) .$$

We proceed as in (7.3): applying successively a unimodular shearing transformation $H_4(x) \in Gl_\mu\left(K\left[x, \frac{1}{x}\right]\right)$, constructed as in (17.3), and a unimodular gauge transformation $H_3(x) \in Gl_\mu(K(x))$, constructed as in (3.11) to remove apparent singularities, we obtain a matrix $A_{[H_4H_3H]}(x)$ satisfying hypothesis 1), 2) and 3) plus the inequality:

$$\left\|\frac{A_{[H_4H_3H]}(x) - \mathbb{I}_\mu}{(q-1)x}\right\| \leq \sup\left(1, \frac{|\pi|}{\chi}\right)$$

Set $Q(x) = H_4(x)H_3(x)H(x)$ and $B(x) = A_{[Q]}(x)$. One observes that

- since both $A(x)$ and $B(x)$ satisfy hypothesis 1) and 2), necessarily $Q(x)$ and $Q(x)^{-1}$ have no poles in $D(0, 1^-) \smallsetminus \{0\}$;

- $B_{[UQ^{-1}]}(x) = A(0)$.

Once again there exists a shearing transformation $P(x) \in Gl_\mu\left(K\left[x, \frac{1}{x}\right]\right)$ such that $B(0) = B_{[PUQ^{-1}]}(x)$. Moreover PUQ^{-1} has its coefficients in $K((x))$. Set $V = PUQ^{-1}$. Then the q-difference system $Y(qx) = B(x)Y(x)$ satisfies conditions 1) through 3), with $B_{[V]}(x) = B(0)$, plus the inequality (11.2.1). Since $r(V) = r(U)$, we conclude that it is enough to prove the proposition assuming (11.2.1).

Step 2. Proof of (11.1) assuming (11.2.1).

Let us set

$$U(x) = U_0 + U_1x + U_2x^2 + \dots$$

and

$$A(x) = A_0 + A_1x + A_2x^2 + \dots .$$

Of course one can suppose $U_0 = \mathbb{I}_\mu$. By assumption we have $U(qx)A(x) = A_0U(x)$. Hence a direct calculation shows that

$$q^mU_mA_0 - A_0U_m = A_1U_{m-1} + A_2U_{m-2} + \dots + A_m .$$

Consider the K-linear map

$$\begin{array}{rccc} \Phi_{q^m,A_0} : M_{\mu\times\mu}(K) & \longrightarrow & M_{\mu\times\mu}(K) \\ M & \longmapsto & q^mMA_0 - A_0M \end{array} . \tag{11.2.2}$$

Observe that the eigenvalues of Φ_{q^m,A_0} are precisely of the form $q^m\alpha-\beta$, where α,β are any two eigenvalues of A_0. Therefore the operator Φ_{q^m,A_0} is invertible by hypothesis 3), which means that

$$U_m=\Phi_{q^m,A_0}^{-1}\left(A_1U_{m-1}+A_2U_{m-2}+\cdots+A_m\right)\ .$$

We need to calculate the norm of Φ_{q^m,A_0}^{-1} as a K-linear operator. Since $\Phi_{q^m,A_0}(M)=q^mM(A_0-\mathbb{I}_\mu)-M(q^m\mathbb{I}_\mu-A_0)$ and

$$\|A(x)-\mathbb{I}_\mu\|\leq|q-1|\sup\left(\left\|\frac{A(x)-\mathbb{I}_\mu}{(q-1)x}\right\|,1\right)\leq|q-1|\sup\left(\frac{|\pi|}{\chi},1\right)\ ,$$

we conclude that

$$\left\|\Phi_{q^m,A_0}^{-1}\right\|=\frac{\|\mathrm{adj}\Phi_{q^m,A_0}\|}{\|\det\Phi_{q^m,A_0}\|}\leq\frac{|q-1|^{\mu^2-1}\sup\left(\frac{|\pi|}{\chi},1\right)^{\mu^2-1}}{\prod_{\alpha,\beta\text{ e.v. of }A_0}|q^m\alpha-\beta|}\ .$$

Recursively we obtain the estimate

$$\begin{aligned}
|U_m| &\leq \left\|\Phi_{q^m,A_0}^{-1}\right\|\left(\sup_{i\geq1}|A_i|\right)\left(\sup_{i=1,\dots,m-1}|U_i|\right)\\
&\leq \prod_{i=1}^{m}\left\|\Phi_{q^m,A_0}^{-1}\right\|\|A(x)-\mathbb{I}_\mu\|^m\\
&\leq \frac{\sup\left(\frac{|\pi|}{\chi},1\right)^{\mu^2m}}{|\det A_0|^{\mu m}}\left(\prod_{\alpha,\beta\text{ e.v. of }A_0}\prod_{i=1}^{m}\left|\frac{1-q^m\alpha\beta^{-1}}{1-q}\right|\right)^{-1}\ .
\end{aligned}$$

We deduce from (20.2) that

$$r(U(x))=\liminf_{m\to\infty}|U_m|^{-\frac{1}{m}}\geq\inf\left(|\pi|,\chi\right)^{\mu^2}|\det A_0|^{\mu}\prod_{\alpha,\beta}\mathrm{type}_q(\alpha\beta^{-1})\ .$$

□

For further reference, we point out that the Step 1 above is actually a proof of the following statement:

Lemma 11.3. *Let $Y(qx)=A(x)Y(x)$ be a q-difference equation, with $A(x)\in Gl_\mu(E_0')$, having only apparent singularities in $D(0,1^-)\smallsetminus\{0\}$ and a regular singularity at 0. We suppose that there exists $U(x)\in Gl_\mu(K((x)))$ satisfying $A_{[U]}(x)\in Gl_\mu(K)$. Then there exists a gauge transformation matrix $H(x)\in Gl_\mu(E_0')$ such that:*

1) $A_{[H]}(x)\in Gl_\mu(E_0)$;

2) any two eigenvalue α,β of $A_{[H]}(0)$ are either equal or $\alpha\beta^{-1}\notin q^{\mathbb{Z}}$;

3) $r(U(x))=r(H(x)U(x))$.

12. A sharper estimate.

In this section we are going to the deduce from (11.1) a sharper estimate for q-difference system in $\mathcal{H}_q^\ell$ (cf. (7.3)), relying on the existence of the weak Frobenius structure.

Proposition 12.1. *Let $Y(qx) = A(x)Y(x)$ be a q-difference system in $\mathcal{H}_q^\ell$ in its normal form and let $\chi(A,q) > |\pi|^{\frac{1}{p^{\ell-1}}}$. The matrix $U(x) \in Gl_\mu(K[\![x]\!])$ such that $A_{[U]} = A(0)$ satisfies the inequality*

$$r(U(x)) \geq \inf\left(|\pi|^{\frac{1}{p^\ell}}, \chi(A,q)\right)^{\mu^2} |\det A(0)|^{\frac{\mu}{p^\ell}} \prod_{\alpha,\beta \text{ e.v. of } A(0)} \mathrm{type}_q\left(\alpha\beta^{-1}\right) .$$

Proof. By theorem 7.4 stating the existence of the weak Frobenius structure, there exists a matrix $H(x) \in Gl_\mu(E_0') \cap M_{\mu\times\mu}(E_0)$ such that:

1) $H(0) \in Gl_\mu(K)$ and $|H(0)| = \left|H(0)^{-1}\right| = 1$;

2) $A_{[H]}(x) = F(x^{p^\ell})$;

3) the eigenvalues of $A(0)$ and $F(0)$ coincide;

4) the q^{p^ℓ}-system $V(q^{p^\ell}X) = F(X)V(X)$, with $X = x^{p^\ell}$, has only ordinary q-orbits in $D(0,1^-) \smallsetminus \{0\}$;

5) $\chi(F, q^{p^\ell}) = \chi(A,q)^{p^\ell}$.

By (17.2) there exists $V(X) \in Gl_\mu(K[\![X]\!])$ such that $F_{[V]}(X) = F(0)$. On the other hand we have also $F_{[UH^{-1}]}(x) = A(0)$. It follows from (17.4) that there exists a shearing transformation $Q(x) \in Gl_\mu\left(K\left[x, \frac{1}{x}\right]\right)$ such that $Q(x)U(x) = V(x)H(x)$. Hence $r(U(x)) = r(V(x)) = r(V(X))^{\frac{1}{p^\ell}}$.

By (11.1) we have

$$r(V(X)) \geq \inf\left(|\pi|, \chi(A,q)^{p^\ell}\right)^{\mu^2} |\det F(0)|^{\mu} \prod_{\alpha,\beta \text{ e.v. of } F(0)} \mathrm{type}_{q^{p^\ell}}\left(\alpha\beta^{-1}\right)$$

and hence

$$r(U(x)) \geq \inf\left(|\pi|^{\frac{1}{p^\ell}}, \chi(A,q)\right)^{\mu^2} |\det A(0)|^{\frac{\mu}{p^\ell}} \prod_{\alpha,\beta \text{ e.v. of } A(0)} \mathrm{type}_{q^{p^\ell}}\left(\alpha\beta^{-1}\right)^{\frac{1}{p^\ell}} .$$

We finish the proof by observing that

$$\mathrm{type}_q(\alpha) = \liminf_{n\to\infty} |1 - q^n\alpha|^{1/n} \leq \liminf_{n\to\infty} |1 - q^{p^\ell n}\alpha|^{1/p^\ell n} = \left(\mathrm{type}_{q^{p^\ell}}(\alpha)\right)^{\frac{1}{p^\ell}} .$$

□

13. More general statements.

Let $Y(qx)=A(x)Y(x)$ be a q-difference system such that $A(x)\in Gl_\mu(E'_0)$, having only apparent singularities in $D(0,1^-)\smallsetminus\{0\}$. We suppose that there exists $U(x)\in Gl_\mu(K((x)))$ satisfying $A_{[U]}(x)\in Gl_\mu(K)$. Then we have:

Corollary 13.1.

1) If $\chi(A,q)\leq|\pi|$. Then

$$r(U(x))\geq \chi(A,q)^{\mu^2}|\det A(0)|^\mu \prod_{\alpha,\beta\text{ e.v. of }A(0)} \mathrm{type}_q\left(\alpha\beta^{-1}\right)\ .$$

2) Suppose $|\pi|<\chi(A,q)<1$ and let ℓ be a positive integer such that $|\pi|^{\frac{1}{p^\ell}}\geq \chi(A,q)>|\pi|^{\frac{1}{p^{\ell-1}}}$. Then

$$r(U(x))\geq \chi(A,q)^{\mu^2}|\det A(0)|^{\frac{\mu}{p^\ell}} \prod_{\alpha,\beta\text{ e.v. of }A(0)} \mathrm{type}_q\left(\alpha\beta^{-1}\right)$$

3) If $\chi(A,q)=1$. Then

$$r(U(x))\geq \prod_{\alpha,\beta\text{ e.v. of }A(0)} \mathrm{type}_q\left(\alpha\beta^{-1}\right)$$

Proof. Statement 1) follows immediately from (11.3) and (11.1), while statement 2) follows from (11.3) and (12.1). Let us prove 3). By (11.3) and (12.1), for any $\ell\in\mathbb{Z}_{\geq 0}$, we have $\chi(A,q)>|\pi|^{\frac{1}{p^{\ell-1}}}$, hence

$$r(U(x))\geq |\pi(1-q)|^{\frac{1}{p^\ell}}\,|\det A(0)|^{\frac{\mu}{p^\ell}} \prod_{\alpha,\beta\text{ e.v. of }A(0)} \mathrm{type}_q\left(\alpha\beta^{-1}\right)\ .$$

We finish by letting $\ell\to+\infty$. □

Appendix

A. Twisted Taylor expansion of p-adic analytic functions

14. Analytic functions over a q-invariant open disk.

In this section we prove proposition 1.4. Let us recall its statement:

Proposition 14.1. *Let $D = D(\xi, \rho^-)$ be a q-invariant open disk. The map*

$$\begin{array}{cccc} T_{q,\xi}: & \mathcal{A}_D & \longrightarrow & K\{x-\xi\}_{q,\rho} \\ & f(x) & \longmapsto & \sum_{n\geq 0} \frac{d_q^n f}{[n]_q^!}(\xi)(x-\xi)_{q,n} \end{array}$$

is a q-difference algebras isomorphism. Moreover, for all $f \in \mathcal{A}_D$, the series $T_{q,\xi}(f)(x)$ converges uniformly to $f(x)$ over any closed disk $D(\xi, \eta^+)$, with $0 < \eta < \rho$.

The proof is divided in three steps:

STEP 1. The map $T_{q,\xi}$ is a well-defined ring homorphism.

Proof. The only non-trivial point is that $T_{q,\xi}$ is well-defined. Let $f = \sum_{n\geq 0} f_n(x-\xi)^n \in \mathcal{A}_D$. By (2.1.1), for any integer $k \geq 0$ and any $\eta \in \mathbb{R}$ such that $|(1-q)\xi| \leq \eta < \rho$, we have

$$\left|\frac{d_q^k(f)}{[k]_q^!}(\xi)\right| \leq \left\|\frac{d_q^k}{[k]_q^!} f(x)\right\|_\xi (\eta) \leq \frac{\|f\|_\xi(\eta)}{\eta^k}\ .$$

Hence $\liminf_{n\to\infty} \left|\frac{d_q^k(f)}{[k]_q^!}(\xi)\right|^{-1/n} \geq \eta$, for all η such that $|(1-q)\xi| \leq \eta < \rho$. Letting $\eta \to \rho$ one proves that $T_{q,\xi}(f) \in K\{x-\xi\}_{q,\rho}$. □

STEP 2. Let $y(x) = \sum_{n\geq 0} a_n(x-\xi)_{q,n} \in K\{x-\xi\}_{q,\rho}$. Then $y(x)$ converges uniformly to an analytic function over any closed disk $D(\xi, \eta^+)$, with $0 < \eta < \rho$.

Proof. For any pair of integers i, k, such that $0 \leq i \leq k$, we consider the symmetric polynomial

$$S_k^i(x_1, \dots, x_k) = \begin{cases} 1 & \text{if } i = 0; \\ \displaystyle\sum_{1\leq j_1 < \dots < j_i \leq k} x_{i_1} \cdots x_{j_i} & \text{otherwise.} \end{cases}$$

We notice that

$$\begin{aligned} (x-\xi)_{q,n} &= (x-\xi)(x-q\xi)\cdots(x-q^{n-1}\xi) \\ &= (x-\xi)\left[(x-\xi)+\xi(1-q)\right]\cdots\left[(x-\xi)+\xi(1-q^{n-1})\right] \\ &= \sum_{k=0}^{n} \xi^{n-k} S_n^{n-k}(0, 1-q, \dots, 1-q^{n-1})(x-\xi)^k\ . \end{aligned} \tag{14.1.1}$$

Since $|\xi^{n-k}S_n^{n-k}(1-q^0,1-q,\ldots,1-q^{n-1})| \leq |\xi(1-q)|^{n-k-1} \leq R^{n-k-1} < \rho^{n-k-1}$, the following series converges in K:

$$\alpha_k = \sum_{n=k}^{\infty} a_n \xi^{n-k} S_n^{n-k}(1-q^0, 1-q, \ldots, 1-q^{n-1}) \in K \ .$$

Then the series $\widetilde{y}(x) = \sum_{k\geq 0} \alpha_k (x-\xi)^k \in K\llbracket x-\xi \rrbracket$ converges over D, in fact for any $x_0 \in D$ we have

$$\sup\left(|\xi(1-q)|, |x_0-\xi|\right) \leq R < \rho \ ,$$

and therefore

$$\begin{array}{rcl} 0 & \leq & \limsup\limits_{k\to\infty} |\alpha_k(x_0-\xi)^k| \\ & \leq & \limsup\limits_{k\to\infty} \left| \sum\limits_{n=k}^{\infty} a_n \xi^{n-k} S_n^{n-k}(1-q^0, 1-q, \ldots, 1-q^{n-1})(x_0-\xi)^k \right| \\ & \leq & \limsup\limits_{k\to\infty} \left(\sup\limits_{n\geq k} |a_n|\, |\xi(1-q)|^{n-k} |x_0-\xi|^k \right) \\ & \leq & \limsup\limits_{k\to\infty} \left(\sup\limits_{n\geq k} |a_n|\, R^n \right) = 0 \end{array} \ .$$

This proves that $\widetilde{y}(x) \in \mathcal{A}_D$.

We want to prove that $y(x) = \sum_{n\geq 0} a_n (x-\xi)_{q,n} \in K\{x-\xi\}_{q,\rho}$ converges uniformly to $\widetilde{y}(x)$ over any closed disk $D(\xi,\eta^+)$, with $0<\eta<\rho$. We fix a positive real number $\eta<\rho$ and we set $C_N = \sup_{n>N} |a_n|\eta^n$. We observe that

$$\lim_{N\to\infty} C_N = 0 \ ,$$

since $\eta < \rho$. We have

$$\begin{array}{l} \left\| \sum\limits_{k=0}^{N} \alpha_k (x-\xi)^k - \sum\limits_{n=0}^{N} a_k (x-\xi)_{q,k} \right\|_\xi (\eta) \\ \quad = \quad \left\| \sum\limits_{k=0}^{N} \alpha_k (x-\xi)^k - \sum\limits_{n=0}^{N} a_n \sum\limits_{k=0}^{n} \xi^{n-k} S_n^{n-k}(0, 1-q, \ldots, 1-q^{n-1})(x-\xi)^k \right\|_\xi (\eta) \\ \quad = \quad \left\| \sum\limits_{k=0}^{N} \sum\limits_{n=k}^{\infty} a_n \xi^{n-k} S_n^{n-k}(0, 1-q, \ldots, 1-q^{n-1})(x-\xi)^k \right. \\ \qquad\qquad \left. - \sum\limits_{k=0}^{N} \sum\limits_{n=k}^{N} a_n \xi^{n-k} S_n^{n-k}(0, 1-q, \ldots, 1-q^{n-1})(x-\xi)^k \right\|_\xi (\eta) \\ \quad = \quad \left\| \sum\limits_{k=0}^{N} \left(\sum\limits_{n>N} a_n \xi^{n-k} S_n^{n-k}(1-q^0, 1-q, \ldots, 1-q^{n-1}) \right) (x-\xi)^k \right\|_\xi (\eta) \\ \quad \leq \quad \sup\limits_{k=0,\ldots,N} \left(\eta^k \sup\limits_{n>N} |a_n| \eta^{n-k} \right) \\ \quad \leq \quad \sup\limits_{k=0,\ldots,N} \left(\sup\limits_{n>N} |a_n| \eta^n \right) = C_N \ . \end{array}$$

We conclude that the two series converge to the same sum over $D(\xi,\eta^+)$, for all $0<\eta<\rho$, hence over D. $\square$

STEP 3. End of the proof.

Proof. By the previous step, one can define a map $S : K\{x-\xi\}_{q,\rho} \longrightarrow \mathcal{A}_D$. Since ξ is a limit point of $q^{\mathbb{N}}\xi$, for any $f(x), g(x) \in \mathcal{A}_D$ we have

$$\begin{aligned}\frac{d_q^n f}{[n]_q^!}(\xi) = \frac{d_q^n g}{[n]_q^!}(\xi)\ ,\ \forall n \in \mathbb{Z}_{n\geq 0} \quad &\Leftrightarrow \quad f(q^n\xi) = g(q^n\xi) \text{ for all integers } n \geq 0 \\ &\Leftrightarrow \quad f = g\ .\end{aligned}$$

Then one deduces that $T_{q,\xi}^{-1} = S$ from the fact that

$$\frac{d_q^n}{[n]_q^!}\left(S \circ T_{q,\xi}(f)\right)(\xi) = \frac{d_q^n f}{[n]_q^!}(\xi) \text{ and } \frac{d_q^n}{[n]_q^!}\left(T_{q,\xi} \circ S(g)\right)(\xi) = \frac{d_q^n g}{[n]_q^!}(\xi)$$

for any $f \in \mathcal{A}_D$, any $g \in K\{x-\xi\}_{q,\rho}$ and any nonnegative integer n. $\square$

This completes the proof of (14.1).

15. Analytic functions over non connected analytic domain.

Proposition 14.1 can be generalized to the case of analytic functions over convenient q-invariant non-connected analytic domains:

Definition 15.1. *We call* q-disk of center $\xi \in \mathbb{A}_K^1$ and radius $\eta \in \mathbb{R}_{>0}$ *the set*

$$q^{\mathbb{Z}} D(\xi,\eta^-) := \bigcup_{n\in\mathbb{Z}} D(q^n\xi,\eta^-)\ .$$

Remark 15.2. Of course, if n_0 is the smallest positive integer such that $|(1-q^{n_0})\xi| < \eta$, the q-disk D is a disjoint union of the open disks $D(q^n\xi,\eta^-)$, with $0 \leq n < n_0$, each one of them being q^{n_0}-invariant.

The algebra $\mathcal{A}_D$ of analytic functions over a q-disk D is the direct product of the algebras of analytic functions over each connected component. Since the analytic domain D is q-invariant $\mathcal{A}_D$ has a structure of q-difference algebra. Hence one can define a q-expansion map

$$\begin{array}{rcl} T_{q,\xi} \quad : \mathcal{A}_D & \longrightarrow & K\,[\![x-\xi]\!]_q = \left\{\sum_{n\geq 0} a_n (x-\xi)_{q,n}\ :\ a_n \in K\right\} \\ f(x) & \longmapsto & \displaystyle\sum_{n\geq 0} \frac{d_q^n(f)}{[n]_q^!}(\xi)(x-\xi)_{n,q} \end{array},$$

that we still call $T_{q,\xi}$.

Proposition 15.3. *Let $f(x) = \sum_{n\geq 0} a_n(x-\xi)_{q,n}$ be a formal series such that the a_n's are elements of K and $\liminf_{n\to\infty} |a_n|^{-1/n} = \rho$. Then $f(x)$ is a q-expansion of an analytic function over a q-disk D if and only if*

$$\frac{\rho}{|(q-1)\xi\pi_q|} > 1 \ .$$

Let n_0 be the number of connected components of D and $\widetilde{\rho}$ their radius. Then n_0 is the smallest positive integer such that

$$\frac{\rho}{|(1-q)\xi\pi_q|} > |\pi_{q^{n_0}}|^{-1/n_0} \tag{15.3.1}$$

and ρ, $\widetilde{\rho}$ and n_0 are linked by the relation:

$$\frac{\widetilde{\rho}}{|(q^{n_0}-1)\xi\pi_{q^{n_0}}|} = \left(\frac{\rho}{|(q-1)\xi\pi_q|}\right)^{n_0} \ . \tag{15.3.2}$$

Remark 15.4. We recall that $\pi_q \in K$ satisfies $|\pi_q| = \lim_{n\to\infty} |[n]_q^!|^{1/n}$ (cf. (3.4)).

It follows immediately by (3.3) and (15.3) that:

Corollary 15.5. *The system*

$$Y(qx) = A(x)Y(x) \ , \ \textit{with } A(x) \in Gl_\mu(\mathcal{M}_{D(\xi,\rho^-)}),$$

has an analytic fundamental solution over a q-disk centered at ξ if and only if the matrix $A(x)$ does not have any poles in $q^{\mathbb{N}}\xi$, $\det A(x)$ does not have any zeros in $q^{\mathbb{N}}\xi$ and

$$\limsup_{n\to\infty} \left|\frac{G_n(\xi)}{[n]_q^!}\right|^{1/n} < |(q-1)\xi\pi_q|^{-1} \ .$$

16. Proof of proposition 15.3.

The main result of this section is the proof proposition 15.3 (cf. (16.5) below), that we will deduce from the more general result (16.2).

Let $\xi \neq 0$ and $f(x) = \sum_{n\geq 0} a_n(x-\xi)_{q,n} \in K\,[\![x-\xi]\!]_q$. For any non-negative integer k it makes sense to define formally

$$\frac{d_q^k(f)}{[k]_q^!}(\xi) = a_k$$

and

$$f(q^k\xi) = \sum_{n\geq 0} \alpha_n \xi^n (q^k-1)_{q,n} \ .$$

An element $f(x) \in K\,[\![x-\xi]\!]_q$ is uniquely determined by $(f(q^k\xi))_{k\geq 0}$ or by $(a_k)_{k\geq 0}$, knowing that these two sequences are linked by relations that can be deduced by (1.2). Therefore $K\,[\![x-\xi]\!]_q$ is not a local ring and we have:

Lemma 16.1. *If $\xi \neq 0$ the natural morphism of q-difference algebras, defined for any $n_0 \geq 1$,*

$$
\begin{array}{rccc}
 & K [\![x-\xi]\!]_q & \longrightarrow & \prod_{i=0}^{n_0-1} K \left[\!\left[x-q^i\xi\right]\!\right]_{q^{n_0}} \\
(16.1.1) & & & \\
 & f(x) & \longmapsto & \left(\sum_{n\geq 0} \frac{d_{q^{n_0}}^n(f)(q^i\xi)}{[n]^!_{q^{n_0}}}(x-q^i\xi)_{q^{n_0},n}\right)_{i=0,\dots,n_0-1}
\end{array}
$$

is an isomorphism. In particular $K [\![x-\xi]\!]_q$ is not a domain.

Proof. It is enough to observe that an element $f \in \prod_{i=0}^{n_0-1} K \left[\!\left[x-q^i\xi\right]\!\right]_{q^{n_0}}$ is uniquely determined by

$$
\left(\left(f(q^{nn_0+i}\xi)\right)_{n\geq 0}\right)_{i=0,\dots,n_0-1} .
$$

□

The isomorphism (16.1.1) induces an isomorphism between "converging q-series":

Proposition 16.2. *We set $\widetilde{q} = q^{n_0}$. Let*

$$
g = \sum_{n\geq 0} g_n (x-\xi)_{q,n} \in K [\![x-\xi]\!]_q
$$

and let

$$
f = \left(\sum_{n\geq 0} f_{i,n}(x-\xi)_{\tilde{q},n}\right)_{0\leq i<n_0} \in \prod_{i=0}^{n_0-1} K \left[\!\left[x-q^i\xi\right]\!\right]_{\tilde{q}}
$$

be its image via (16.1.1). We set

$$
\rho = \liminf_{n\to\infty} |g_n|^{-1} \text{ and } \widetilde{\rho} = \inf_{0\leq i<n_0} \left(\liminf_{n\to\infty} |f_{i,n}|^{-1}\right) .
$$

Then

$$
\frac{\widetilde{\rho}}{|(\tilde{q}-1)\xi\pi_{\tilde{q}}|} = \left(\frac{\rho}{|(q-1)\xi\pi_q|}\right)^{n_0} .
$$

In order to prove proposition 16.2, we need a technical lemma:

Lemma 16.3. *1) Let $g \in K\llbracket x-\xi \rrbracket_q$; then*
(16.3.1)

$$\begin{aligned}(\widetilde{q}-1)^k \xi^k d_{\widetilde{q}}^k(g)(\xi) &= (-1)^k \sum_{h=0}^{n_0 k} \sum_{\frac{h}{n_0}\le j\le k} (-1)^j \binom{k}{j}_{\tilde{q}^{-1}} \\ &\cdot \widetilde{q}^{\frac{-j(j-1)}{2}} \binom{n_0 j}{h}_q \cdot q^{\frac{h(h-1)}{2}} (q-1)^h \xi^h d_q^h(g)(\xi)\ .\end{aligned}$$

2) Let $f=(f_i)\in\prod_{i=0}^{n_0-1} K\left[\left[x-q^i\xi\right]\right]_{\tilde{q}}$. Then
(16.3.2)

$$\begin{aligned}(q-1)^k \xi^k d_q^k(f)(\xi) &= (-1)^k \sum_{h=0}^{\left[\frac{n_0}{k}\right]} \sum_{\substack{0\le i<n_0 \\ hn_0+i\le k}} \binom{k}{hn_0+i}_{q^{-1}} q^{\frac{-(hn_0+i)(hn_0+i-1)}{2}} \\ &\cdot \sum_{j=0}^{h} \binom{h}{j}_{\tilde{q}} \widetilde{q}^{\frac{j(j-1)}{2}} q^{ij} (\widetilde{q}-1)^j \xi^j d_{\widetilde{q}}^j(f_i)(q^i\xi)\ .\end{aligned}$$

Proof. The two formulas can be obtained by applying twice the formulas relating d_q and σ_q (cf. (1.2)), taking into account that $d_{\tilde{q}}(f(q^i\xi)) = q^i d_{\tilde{q}}(f)(q^i\xi)$. □

16.4. Proof of proposition 16.2.

The proof is based on the following property of the limit of a sequence (a_n) of real non-negative numbers (cf. [AB01, II, 1.8 and 1.9]):

(16.4.1) $$\sup\left(1, \limsup_{n\to\infty} a_n^{1/n}\right) = \limsup_{n\to\infty}\left(\sup_{s\le n} a_s\right)^{1/n}\ .$$

STEP 1. Proof of (16.2) when one of the (equivalent) conditions
- $\rho < |(q-1)\xi\pi_q|$,
- $\widetilde{\rho} < |(\widetilde{q}-1)\xi\pi_{\tilde{q}}|$
is satisfied.
Suppose first that $\rho < |(q-1)\xi\pi_q|$. Since $\rho^{-1}|\pi_q| > 1$, equality (16.4.1) implies that

$$\begin{aligned}\rho^{-1}|\pi_q| &= |\pi_q| \limsup_{n\to\infty} \left|\frac{d_q^n(g)}{[n]_q^!}(\xi)\right|^{1/n} \\ &\le \limsup_{n\to\infty} \left|d_q^n(g)(\xi)\right|^{1/n} \\ &= \limsup_{n\to\infty}\left(\sup_{s\le n}\left|d_q^s(g)(\xi)\right|\right)^{1/n}\ .\end{aligned}$$

Since $|(q-1)\xi\pi_q|\rho^{-1}>1$ we deduce from (16.3.2) that

$$\begin{aligned}
|(q-1)\xi\pi_q|\rho^{-1} &\leq \limsup_{n\to\infty}\left|(q-1)^n\xi^n d_q^n(g)(\xi)\right|^{1/n}\\
&\leq \limsup_{n\to\infty}\left(\sup_{s\leq\left[\frac{n}{n_0}\right],l=0,\dots,n_0-1}\left|(\widetilde{q}-1)^s\xi^s d_{\widetilde{q}}^s(g)(q^l\xi)\right|\right)^{1/n}\\
&= \sup\left(1,|(\widetilde{q}-1)\xi\pi_{\widetilde{q}}|^{1/n_0}\widetilde{\rho}^{-1/n_0}\right)\\
&= (|(\widetilde{q}-1)\xi\pi_{\widetilde{q}}|\widetilde{\rho})^{-1/n_0}
\end{aligned}$$

and hence that $\rho\geq|(q-1)\xi\pi_q|\left(\frac{\tilde{\rho}}{|(\tilde{q}-1)\xi\pi_{\tilde{q}}|}\right)^{1/n_0}$. Moreover the last inequality implies that $\widetilde{\rho}<|(\widetilde{q}-1)\xi\pi_{\widetilde{q}}|$.

Let us suppose now that $\widetilde{\rho}<|(\widetilde{q}-1)\xi\pi_{\widetilde{q}}|$. Then

$$\widetilde{\rho}^{-1}|\pi_{\widetilde{q}}|\leq\sup_{i=0,\dots,n_0-1}\left(\limsup_{n\to\infty}\left(\sup_{s\leq n}\left|d_{\widetilde{q}}^s(f)(q^i\xi)\right|\right)^{1/n}\right)\ .$$

We deduce by (16.3.1) that

$$\begin{aligned}
|(\widetilde{q}-1)\xi\pi_{\widetilde{q}}|\widetilde{\rho}^{-1} &\leq \limsup_{n\to\infty}\left|d_{\widetilde{q}}^n(f)(\xi)\right|^{1/n}\\
&\leq \limsup_{n\to\infty}\left(\sup_{s\leq nn_0}\left|(q-1)^s\xi^s d_q^s(f)(\xi)\right|\right)^{1/n}\\
&\leq |(q-1)\xi|^{n_0}|p|^{n_0/(p-1)}\rho^{-n_0}
\end{aligned}$$

and hence that $\widetilde{\rho}\geq|(\tilde{q}-1)\xi\pi_{\tilde{q}}|\left(\frac{\rho}{|(q-1)\xi\pi_q|}\right)^{n_0}$. Notice that this inequality implies that $\rho<|(q-1)\xi\pi_q|$

STEP 2. Proof of (16.2) when one of the (equivalent) conditions
- $\rho\geq|(q-1)\xi\pi_q|$,
- $\widetilde{\rho}\geq|(\widetilde{q}-1)\xi\pi_{\widetilde{q}}|$

is satisfied.

We choose $\gamma\in K$ such that

$$|\gamma|>|(q-1)\xi\pi_q|^{-1}\rho\ .$$

By setting $\xi=\gamma\zeta$ and $x=\gamma t$, we identify g with an element of $K\,[\![t-\zeta]\!]_q$ satisfying the hypothesis of Step 1. We deduce from the estimate in Step 1 that

$$\begin{aligned}
|\gamma|^{-1}\rho &= \frac{|(q-1)\zeta\pi_q|}{|(\widetilde{q}-1)\zeta\pi_{\widetilde{q}}|^{1/n_0}}(|\gamma|^{-1}\widetilde{\rho})^{1/n_0}\\
&= |\gamma|^{-1}\frac{|(q-1)\xi\pi_q|}{|(\widetilde{q}-1)\xi\pi_{\widetilde{q}}|^{1/n_0}}\widetilde{\rho}^{1/n_0}\ .
\end{aligned}$$

□

16.5. Proof of proposition 15.3.

We briefly recall the statement of (15.3). Let $f(x) = \sum_{n\geq 0} a_n(x-\xi)_{q,n}$ be a formal series such that $a_n \in K$ and $\liminf_{n\to\infty} |a_n|^{-1/n} = \rho$. We have to prove that $f(x)$ is a q-expansion of an analytic function over a q-disk $q^{\mathbb{Z}}D(\xi,\widetilde{\rho}^-)$ if and only if

$$\frac{\rho}{|(q-1)\xi\pi_q|} > 1 \ . \tag{16.5.1}$$

Let $f(x)$ satisfy (16.5.1). Since $\frac{\rho}{|(q-1)\xi\pi_q|} > 1$ there exists a minimal positive integer n_0 such that

$$\left(\frac{\rho}{|(q-1)\xi\pi_q|}\right)^{n_0} |\pi_{q^{n_0}}| > 1 \ .$$

Then set

$$\widetilde{\rho} = \left(\frac{\rho}{|(q-1)\xi\pi_q|}\right)^{n_0} |\pi_{q^{n_0}}||(q^{n_0}-1)\xi| \ .$$

It follows from (1.4) and (15.3) that $f(x)$ is the q-expansion of an analytic function over q-disk $q^{\mathbb{Z}}D(\xi,\widetilde{\rho}^-)$ satisfying (15.3.1) and (15.3.2).

Conversely, let $f(x)$ be the q-expansion of an analytic function over a q-disk $D = q^{\mathbb{Z}}D(\xi,\widetilde{\rho}^-)$. Let n_0 be the number of connected components of D and $\widetilde{q} = q^{n_0}$. Then $\widetilde{\rho} > |(\widetilde{q}-1)\xi|$. It follows from (16.2) that $\rho > |(q-1)\xi\pi_q|$. □

B. Basic facts about regular singularities of q-difference systems

In this chapter we briefly recall some properties of regular singular q-difference systems, that we have used in III and IV. A complete exposition can be found in [vdPS97, Ch. 12], [Sau00a] and [Sau02b].

17. Regular singular q-difference systems.

Let K be an algebraically closed field of characteristic 0 and $K((x)) = K[[x]]\left[\frac{1}{x}\right]$ the field of Laurent series. For any $q \in K$, it has a natural structure of q-difference algebra, hence one can consider a q-difference system with coefficients in $K((x))$:

$$Y(qx) = A(x)Y(x) \ .$$

Definition 17.1. *The q-difference system $Y(qx) = A(x)Y(x)$ is said to have a* regular singularity at 0 *(or is* regular singular at 0*) if there exists $U(x) \in Gl_\mu(K((x)))$ such that the matrix $A_{[U]}(x) = U(qx)A(x)U(x)^{-1} \in Gl_\mu(K[[x]])$.*

Lemma 17.2. [Sau00a, 1.1.3] *Suppose $A(x) \in Gl_\mu(K[\![x]\!])$. If any two eigenvalues α,β of $A(0)$ are such that either $\alpha=\beta$ or $\alpha\beta^{-1} \notin q^{\mathbb{Z}}$, then one can construct $U(x) \in Gl_\mu(K[\![x]\!])$ such that $A_{[U]}(x) = A(0)$.*

By a convenient gauge transformation one can always assume that the hypothesis of the lemma above are satisfied:

Proposition 17.3. [Sau00a, 1.1.1] *Suppose that $A(x) \in Gl_\mu(K[\![x]\!])$ and for any eigenvalue α of $A(0)$ choose an integer n_α. Then there exists a matrix $H(x)$ constructed by alternatively multiplying constant matrices in $Gl_\mu(K)$ and diagonal matrices of the form*

$$\begin{pmatrix} \mathbb{I}_{\mu_1} & & \\ & \mathbb{I}_{\mu_2} x^{\pm 1} & \\ & & \mathbb{I}_{\mu_3} \end{pmatrix} , \text{ with } \mu_1+\mu_2+\mu_3=\mu , \tag{17.3.1}$$

such that, for any eigenvalue α of $A(0)$, $q^{n_\alpha}\alpha$ is an eigenvalue of $A_{[H]}(0)$.

In particular one can construct a matrix $H(x)$ such that any two eigenvalues α,β of $A_{[H]}(0)$ satisfy either $\alpha=\beta$ or $\alpha\beta^{-1}\notin q^{\mathbb{Z}}$.

The idea of the proof of the proposition above is that one has first to consider a constant gauge matrix Q such that $Q^{-1}A(x)Q$ has the constant term in the Jordan normal form. Then by using a gauge matrix of the form (17.3.1) one multiplies the eigenvalue of a chosen block by $q^{\pm 1}$ of $A(0)$. By iterating the algorithm one obtain the desired gauge transformation. One calls a gauge matrix constructed as in the previous proposition *shearing transformation*.

Corollary 17.4. *Let $U(x), V(x) \in Gl_\mu(K((x)))$ be two gauge transformation matrix such that both $A_{[U]}$ and $A_{[V]}$ are in $Gl_\mu(K)$. Then there exists a shearing transformation $H(x)$ such that $HU = V$.*

The corollary follows immediately by proposition (17.3), by observing that the Jordan normal forms of $A_{[U]}$ and $A_{[V]}$ coincide, modulo the fact that the eigenvalues of $A_{[U]}$ are the eigenvalues of $A_{[V]}$ multiplied by an integer power of q.

18. From q-difference systems to q-difference equations.

Consider now a q-difference equation

$$\mathcal{L}y = a_\mu(x)y(q^\mu x) - a_{\mu-1}(x)y(q^{\mu-1}x) - \cdots - a_0(x)y(x) = 0 ,$$

with $a_i(x) \in K((x))$, for all $i=0,\dots,\mu-1$, and $a_\mu(x)=1$. The origin is said to be a *regular singularity of $\mathcal{L}y=0$* if and only if the *Newton Polygon*, i.e. the convex hull in $\mathbb{R}^2$ of

$$\{(i,j)\in\mathbb{Z}^2 :\ i=0,\dots,\mu \text{ and } j \geq \mathrm{ord}_x a_i(x)\} ,$$

has only one finite slope equal to 0. Obviously one has:

Lemma 18.1. *If* 0 *is a regular singularity for* $\mathcal{L}y = 0$*, then* $a_i(x)$ *does not have any pole at* 0*, for any* $i = 0, \dots, \mu - 1$*. In particular,* $a_0(0) \neq 0$*.*

It follows that the q-difference system

$$Y(qx) = A_{[H]}(x)Y(x)\ ,\ \text{with } A_{[H]}(x) = \left(\begin{array}{c|c} 0 & \\ \vdots & \mathbb{I}_{\mu-1} \\ 0 & \\ \hline a_0(x) & a_1(x)\dots a_{\mu-1}(x) \end{array}\right) \tag{18.1.1}$$

has a regular singularity at 0.

The converse is also true:

Proposition 18.2. ([Sau00a, Annexe B] and [Sau02b, §2, in particular 2.2.6, (ii)]) *A* q*-difference system over* $K((x))$ *has a regular singularity at* 0 *if and only if the associated* q*-difference equation* via *the cyclic vector lemma has a regular singularity at* 0*.*

18.3. Second order q-difference equations.

Consider a second order regular singular q-difference equation

$$y(q^2x) - P(x)y(qx) - Q(x)y(x) = 0\ . \tag{18.3.1}$$

Then the associated q-difference system is given by

$$Y(qx) = A(x)Y(x)\ ,\ \text{with } A(x) = \begin{pmatrix} 0 & 1 \\ Q(x) & P(x) \end{pmatrix}$$

and the eigenvalues α, β of $A(0)$ are solutions of the second order equation

$$T^2 - P(0)T - Q(0) = 0\ . \tag{18.3.2}$$

By (17.2), if $\alpha \neq \beta$ and $\alpha\beta^{-1} \notin q^{\mathbb{Z}}$, there exists $U(x) \in Gl_\mu(K[\![x]\!])$ such that

$$U(qx)A(x)U(x)^{-1} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}\ .$$

If we call $e_\alpha(x)$ (resp. $e_\beta(x)$) a solution of $y(qx) = \alpha y(x)$ (resp. $y(qx) = \beta y(x)$) in a convenient q-difference extension of $K((x))$, then we have

$$\left[U(qx)^{-1}\begin{pmatrix} e_\alpha(qx) & 0 \\ 0 & e_\beta(qx) \end{pmatrix}\right] = A(x)\left[U(x)^{-1}\begin{pmatrix} e_\alpha(x) & 0 \\ 0 & e_\beta(x) \end{pmatrix}\right]\ .$$

If $(u_\alpha(x), u_\beta(x))$ is the first row of $U(x)^{-1}$, then $e_\alpha(x)u_\alpha(x), e_\beta(x)u_\beta(x)$ is a basis of solutions of (18.3.1), meaning that $e_\alpha(x)u_\alpha(x)$ and $e_\beta(x)u_\beta(x)$ are linearly independent over the field of constants and they span the vector space of solutions of $\mathcal{L}y = 0$. Observe also that $e_\alpha(x), e_\beta(x)$ are linearly independent over $K((x))$ (that can be proved as in [Sau00a, Annexe A, 4)]).

C. The q-type of a number

The purpose of this appendix is to calculate the radius of convergence of the series

$$\Phi(x) = \sum_{n\geq 0} \frac{(1-q)^n x^n}{(1-q\alpha)\cdots(1-q^n\alpha)}$$

for $\alpha \notin q^{-\mathbb{N}}$, which is a key point in the proof theorem 10.3. Actually the radius of such a series turns out to be equal to $|\pi|\mathrm{type}_q(\alpha)$.

As in chapters III and IV, we assume that $|1-q| < |\pi|$, so that $|1-q| = |\log q|$.

19. Basic properties of the q-type of a number.

We recall that for any $\alpha \in K$ we have defined the q-type of α to be the radius of convergence of $\sum_{\substack{n\geq 0\\ \alpha\neq q^n}} \frac{1-q}{1-q^n\alpha} x^n$, while the type of α is the radius of convergence of $\sum_{\substack{n\geq 0\\ \alpha\neq n}} \frac{x^n}{n-\alpha}$.

Proposition 19.1. *For any $\alpha \in K$ we have*

$$\mathrm{type}_q(\alpha) = \begin{cases} \mathrm{type}\left(\dfrac{\log\alpha}{\log q}\right) & \text{if } \left|\dfrac{\alpha-1}{q-1}\right| \leq 1\ , \\[2ex] 1 & \text{otherwise.} \end{cases}$$

Proof. We have

$$\frac{\alpha-q^n}{q-1} = \frac{\alpha-1}{q-1} - \frac{q^n-1}{q-1} = \frac{\alpha-1}{q-1} - [n]_q\ .$$

Hence, if $\left|\frac{\alpha-1}{q-1}\right| > 1$, the radius of convergence of $\sum_{n\geq 0} \frac{q-1}{\alpha-q^n} x^n$ is equal to 1.

Suppose $|\alpha-1| \leq |q-1| < |\pi|$. Then $|\alpha - q^n| \leq |q-1| < |\pi|$ for any integer n and

$$\left|\frac{\alpha-q^n}{q-1}\right| = \left|\frac{\log(\alpha q^{-n})}{\log q}\right| = \left|\frac{\log\alpha}{\log q} - n\right|\ .$$

$\square$

20. Radius of convergence of ${}_1\Phi_1(q;\alpha q;q;(1-q)x)$.

The series $\Phi(x)$, whose radius of convergence we want to estimate, is a basic hypergeometric series. In the literature it is denoted by ${}_1\Phi_1(q;\alpha q;q;(1-q)x)$ (cf. [GR90]).

Lemma 20.1. *For any $\alpha \in K \smallsetminus q^{\mathbb{Z}}$ we have:*

$$\Phi(x) = \frac{1-\alpha}{1-q}\left(\sum_{n\geq 0}\frac{x^n}{[n]_q^!}\right)\left(\sum_{n\geq 0} q^{\frac{n(n+1)}{2}}\frac{(-x)^n}{[n]_q^!}\frac{1-q}{1-q^n\alpha}\right) .$$

Proof. One verifies directly that the series $\Phi(x)$ is a solution of the q-difference equation

$$\begin{array}{rcl} \mathcal{L}\Phi(x) & = & \big[\sigma_q - 1\big]\circ\big[\alpha\sigma_q - ((q-1)x+1)\big]\Phi(x) \\ & = & \alpha\Phi(q^2x) - ((q-1)qx + 1 + \alpha)\,\Phi(qx) + (q-1)qx + 1 = 0 \ . \end{array}$$

Since the roots of the equation (cf. (18.3.2))

$$\alpha T^2 - (\alpha+1)T + 1 = 0$$

are exactly α^{-1} and 1, any solution of $\mathcal{L}y(x) = 0$ of the form $1 + \sum_{n\geq 1} a_n x^n \in K\,[\![x]\!]$ must coincide with $\Phi(x)$. Therefore, to finish the proof of the lemma, it is enough to verify that

$$\Psi(x) = \frac{1-\alpha}{1-q}\left(\sum_{n\geq 0}\frac{x^n}{[n]_q^!}\right)\left(\sum_{n\geq 0} q^{\frac{n(n+1)}{2}}\frac{(-x)^n}{[n]_q^!}\frac{1-q}{1-q^n\alpha}\right)$$

is a solution of $\mathcal{L}y(x) = 0$ and that $\Psi(0) = 1$.

Let $e_q(x) = \sum_{n\geq 0}\frac{x^n}{[n]_q^!}$. Then $e_q(x)$ satisfies the q-difference equation

$$e_q(qx) = ((q-1)x+1)\,e_q(x) \ ,$$

hence

$$\begin{array}{rcl} \mathcal{L}\circ e_q(x) & = & \big[\sigma_q - 1\big]\circ e_q(qx)\circ\big[\alpha\sigma_q - 1\big] \\ & = & e_q(x)\,((q-1)x+1)\,\big[\,((q-1)qx+1)\,\sigma_q - 1\big]\circ\big[\alpha\sigma_q - 1\big] \\ & = & (*)\big[\,((q-1)qx+1)\,\sigma_q - 1\big]\circ\big[\alpha\sigma_q - 1\big] \ , \end{array}$$

where we have denoted with $(*)$ a coefficient in $K(x)$, not depending on σ_q.

Consider the series $E_q(x) = \sum_{n\geq 0} q^{\frac{n(n-1)}{2}}\frac{x^n}{[n]_q^!}$, which satisfies

$$(1-(q-1)x)\,E_q(qx) = E_q(x)$$

and the series

$$g_\alpha(x) = \sum_{n\geq 0} q^{\frac{n(n+1)}{2}}\frac{(-x)^n}{[n]_q^!}\frac{1-q}{1-q^n\alpha} \ .$$

Then

$$\begin{aligned}\mathcal{L}\circ e_q(x)g_\alpha(x) &= (*)\big[\left((q-1)qx+1\right)\sigma_q-1\big]\circ\big[\alpha\sigma_q-1\big]g_\alpha(x)\\ &= (*)\big[\left((q-1)qx+1\right)\sigma_q-1\big]E_q(-qx)\\ &= (*)\big[\left((q-1)qx+1\right)E_q(-q^2x)-E_q(-qx)\big]\\ &= 0\ .\end{aligned}$$

It is enough to observe that $e_q(0)g_\alpha(0)=\frac{1-q}{1-\alpha}$ to conclude that the series $\Psi(x)=\frac{1-\alpha}{1-q}e_q(x)g_\alpha(x)$ coincides with $\Phi(x)$. □

Corollary 20.2. *For any $\alpha\in K\smallsetminus q^{\mathbb{Z}}$, the radius of convergence of*

$$\sum_{n\geq 0}\frac{(1-q)^n x^n}{(1-q\alpha)\cdots(1-q^n\alpha)}$$

is $|\pi|\mathrm{type}_q(\alpha)$.

Proof. It follows immediately from the previous results, since the radius of convergence of $\sum_{n\geq 0}\frac{x^n}{[n]_q^!}$ is $|\pi|$. □

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