On q-summation and confluence

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Abstract

This paper is divided in two parts. In the first part we consider a convergent q-analog of the divergent Euler series, with $q \in (0,1)$, and we show how the Borel sum of a generic Gevrey formal solution to a differential equation can be uniformly approximated on a convenient sector by a meromorphic solution of a corresponding q-difference equation. In the second part, we work under the assumption $q \in (1,+\infty)$. In this case, at least four different q-Borel sums of a divergent power series solution of an irregular singular analytic q-difference equations are spread in the literature: under convenient assumptions we clarify the relations among them.

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Introduction

Let $\mathbb{C}[[x]]$ be the ring of formal power series with complex coefficients and $\mathbb{C}\{x\}$ the ring of germs of analytic functions at zero. A divergent formal power series $\hat{f} = \sum_{n\geq 0} f_n x^{n+1} \in x\mathbb{C}[[x]] \smallsetminus x\mathbb{C}\{x\}$ is said to be a generic Gevrey series if it is solution of a differential equation of the form

$$(1) \qquad a_n(x) \left(x^2\partial\right)^n y(x) + a_{n-1}(x) \left(x^2\partial\right)^{n-1} y(x) + \cdots + a_0(x)y(x) = g(x)\,,$$

where $\partial = \frac{d}{dx}$ and $a_0(x), \dots, a_n(x), g(x) \in \mathbb{C}\{x\}$, with $a_0(0)a_n(0) \neq 0$. This implies that its formal Borel transform $\mathcal{B}(\hat{f}) = \sum_{n\geq 0} \frac{f_n}{n!}\xi^n \in \mathbb{C}\{\xi\}$ is a germ of an analytic non entire function. The most important example is the Euler series

$$\hat{E}(x) = \sum_{n\geq 0} (-1)^n n! x^{n+1}\,,$$

which is solution of the differential equation $x^2\partial y + y = x$. A generic Gevrey series has the following properties: $\mathcal{B}(\hat{f})$ can be analytically continued along almost all direction $d \in (-\pi, \pi]$ and the Laplace integral along the half line $e^{id}\mathbb{R}^+$:

$$\mathcal{S}^d(\hat{f}) = \int_0^{e^{id}\infty} \mathcal{B}(\hat{f})(\xi) e^{-\xi/x} d\xi\,,$$

called sum of $\hat{f}$ in the direction d, represents a convergent solution of (1), analytic on a convenient sector and asymptotic to $\hat{f}$ at zero: this is the first result of the well-known theory of summation of divergent series (*cf.* [RM90, Mal95, LR90, LR95]).

In the last fifteen years analogous summation theories for q-difference equations have been developed (*cf.* [Zha99, MZ00, Zha02, RZ02, DSK05]). This last sentence already shows one issue in the topic: there are many q-summation theories in the literature and the relations among them are not clear.

Let us consider a q-deformation of the Euler series, namely:

$$\hat{E}_q(x) = \sum_{n\geq 0} (-1)^n [n]_q^! x^{n+1}\,,$$

where $[n]_q = 1 + q + \cdots + q^{n-1}$ and $[n]_q^! = [n]_q[n-1]_q \cdots [1]_q$. This series converges coefficientwise to $\hat{E}(x)$ when $q \to 1$ and is solution of the q-difference equation

$$x^2 d_q y + y = x\,,\ \text{with } q \in \mathbb{C}^* \text{ and } d_q y(x) = \frac{y(qx) - y(x)}{(q-1)x},$$

which is a discretization, in an obvious sense, of the so-called Euler differential equation $x^2 \partial y + y = x$. A first dichotomy immediately appears: when $|q| < 1$ the series $\hat{E}_q$ is a germ of analytic function, converging for $|x| < |1-q|$, while for $|q| > 1$ the series $\hat{E}_q$ diverges. This is itself quite a curious fact, that we have investigated in the present paper.

As far as the divergent case $|q| > 1$ is concerned, another dichotomy immediately shows up: authors have been using two formal Borel transforms, namely

$$\mathcal{B}_q(\hat{f}) = \sum_{n \geq 0} \frac{f_n}{[n]_q^!} \xi^n \text{ and } B_q(\hat{f}) = \sum_{n \geq 0} \frac{f_n}{q^{n(n-1)/2}} \xi^n\,.$$

Notice that we have $\mathcal{B}_q(\hat{E}_q) = \frac{1}{1+\xi}$ and $B_q(\hat{E}_q) = \hat{E}_p(\xi)$, with $p = q^{-1}$. Each one of these formal Borel transforms will be seen to naturally determine two summation procedures, so that we end up with at least four summation procedures: understanding the relations among them is a natural question. Notice that from an arithmetic point of view, $\mathcal{B}_q$ and B_q are deeply different (*cf.* [And00]).

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The present paper is divided in two parts: in the first one we consider the case $q \in (0,1) \subset \mathbb{R}$, while in the second one we study different summation procedures under the assumption $q \in (1, \infty)$. Let us make a few comments on these assumptions:

• We assume that the parameter q is real: this simplifies the exposition, although it is not always completely necessary.

• Authors writing on q-difference equations say sometimes that choosing q smaller or greater than one is only a matter of convention: as we explain below, this is not true in the present situation, and the two cases need to be investigated separately.

Let $q \in (0,1) \subset \mathbb{R}$. In this case $\hat{E}_q$ is the Taylor expansion at 0 of a meromorphic function $\mathcal{E}_q$ on $\mathbb{C}$, whose poles are a discrete subset of the negative real axis $\mathbb{R}^-$. In §1 we prove the uniform convergence of $\mathcal{E}_q$ on the compacts of $\mathbb{C} \smallsetminus \mathbb{R}^-$ to the analytic continuation $\mathcal{E}$ of the Borel sum of $\hat{E}$ in the direction $\mathbb{R}^+$. The proof of this result is based on the development of $\mathcal{E}_q$ at ∞, which is a q-deformation of the classical expansion of $\mathcal{E}$ at ∞[1]:

$$\mathcal{E}(x) = (\log x - \gamma) e^{\frac{1}{x}} + \sum_{n \geq 1} \frac{\sum_{1 \leq k \leq n} \frac{1}{k}}{n!} \left(\frac{1}{x}\right)^n\,,$$

where $\gamma = \lim_{n \to \infty} \left(\sum_{k=1}^n \frac{1}{k} - \ln(n)\right)$ is the Euler constant. In the same spirit, using Sauloy's canonical solutions at ∞ of a fuchsian q-difference operators [Sau00] and his result on their confluence when $q \to 1$, we can prove the main theorem of the first part. Namely, let $y(q,x) = \sum_{n \geq 0} y_n(q) x^{n+1} \in x\mathbb{C}[[x]]$ be a family of formal power series, with $q \in (\eta, 1]$, for some $\eta \in (0,1)$. We suppose that the $y_n(q)$'s are continuous functions of q and that the family $\phi(q,\xi) = \mathcal{B}_q y(q,x) \in \mathbb{C}\{\xi\}$ is solution of a family of equations over $\mathbb{P}^1_{\mathbb{C}}$, fuchsian and non resonant at ∞[2]. Then (*cf.* Theorem 2.6 below):

[1] *cf.* [EMOT81, page 261], where we have set $a = c = 1$ and made the variable change $x \mapsto 1/x$.

[2] For a precise definition of a *family of equations over* $\mathbb{P}^1_{\mathbb{C}}$*, fuchsian and non resonant at* ∞, *cf.* Assumption 2.7 below.

Theorem 1. *Let $d \in [0, 2\pi)$ be such that $\phi(1, x)$ is holomorphic on a domain containing the half line $e^{id}\mathbb{R}^+$. Then for any $x \in V := \{|\arg x - d| < \frac{\pi}{2}\}$ we have*

$$\lim_{q \to 1^-} y(q, x) = \mathcal{S}^d(y(1, x)) = \int_0^{e^{id}\infty} \phi(1, \xi) e^{-\xi/x} d\xi \,,$$

the convergence being uniform on the compacts of V.

This result immediately implies two corollaries (*cf.* §2.3 below). First of all, let $y(x) = \sum_{n \geq 0} y_n x^{n+1} \in x\mathbb{C}[[x]]$ be a series such that $\phi(\xi) = \sum_{n \geq 0} \frac{y_n}{n!} \xi^n$ is solution of a fuchsian differential equation $\sum_{i=0}^{\mu} A_i(\xi)(x\partial)^i \phi = 0$ on $\mathbb{P}^1_{\mathbb{C}}$, non resonant at ∞. One can construct a family of power series $\mathbf{y}_q(x)$, with $q \in (0, 1)$, such that $\mathcal{B}_q \mathbf{y}_q(\xi)$ is solution of $\sum_{i=0}^{\mu} A_i(\xi)(xd_q)^i \phi = 0$ and $\mathbf{y}_q(x)$ converges coefficientwise to $y(x)$ when $q \to 1^-$. Then:

Corollary 2. *The family $\mathbf{y}_q(x)$ converges uniformly to the Borel sum $\mathcal{S}^d(y(\xi))$ of $y(x)$, when $q \to 1^-$, on the compacts of a convenient sector $V = \{|\arg x - d| < \pi/2\}$.*

A second corollary is about the sum of confluent hypergeometric series. Let us consider $a, b \in \mathbb{C}$, with $a - b \notin \mathbb{Z}$, and the basic hypergeometric function:

$$\Phi(a, b; q, x) = \sum_{n \geq 0} \frac{(q^a; q)_n (q^b; q)_n}{(q; q)_n} \left(\frac{x}{1 - q} \right)^n ,$$

where $(a; q)_0 = 1$ and $(a; q)_n = (1 - a)(1 - qa) \cdots (1 - q^{n-1}a)$ for any integer $n \geq 1$.

Corollary 3. *The analytic function $\Phi(a, b; q, x)$ converges uniformly to the Borel sum of the hypergeometric confluent series*

$${}_2F_0(a, b; -; x) = \sum_{n \geq 0} \frac{(a)_n (b)_n}{n!} x^n ,$$

with $(a)_0 = 1$ and $(a)_n = a(a + 1) \cdots (a + n)$ for any integer $n \geq 1$, on the compacts of a convenient sector centered at 0, when $q \to 1^-$.

Finally, notice that the result on the confluence of $\mathcal{E}_q$ can be deduced from theorem 1.

The second part of the paper deals with the summation of divergent q-series when $q \in (1, \infty) \subset \mathbb{R}$. Following the scheme of the first part, we start our investigation studying four summations of the series $\hat{E}_q(x)$. We consider the q-exponential and the classical Theta function (here $p = q^{-1}$):

$$e_q(x) = \sum_{n \geq 0} \frac{x^n}{[n]_q^!} \,, \qquad \theta_p(x) = \sum_{n \in \mathbb{Z}} p^{n(n-1)/2} x^n \,.$$

We replace, in the classical Laplace integral, the exponential function by e_q or θ_p. By using an usual integral or a discrete q-analogue[3], denoted $\int_{\lambda p^{\mathbb{Z}}} f d_p\xi$, we get four different *q-Borel sums of* $\hat{E}_q(x)$:

$$\mathcal{E}_q^d(x) = \frac{q - 1}{\ln q} \int_0^{e^{id}\infty} \frac{1}{(1 + \xi) e_q(q\frac{\xi}{x})} d\xi, \qquad E_q^d(x) = \frac{q}{\ln q} \int_0^{e^{id}\infty} \frac{\hat{E}_p(\xi)}{\theta_p(q\frac{\xi}{x})} d\xi;$$

$$\mathcal{E}_q^{[\lambda]}(x) = \frac{q}{1 - p} \int_{\lambda p^{\mathbb{Z}}} \frac{1}{(1 + \frac{\xi}{1-p}) e_q(q\frac{\xi}{(1-p)x})} d_p\xi, \quad E_q^{[\lambda]}(x) = \frac{q}{1 - p} \int_{\lambda p^{\mathbb{Z}}} \frac{\hat{E}_p(\xi)}{\theta_q(q\frac{\xi}{x})} d_p\xi,$$

[3] It's the so-called Jackson's integral, which is an infinite sum that approximates the associated usual integral. The precise definition is in Appendix A below.

where $d \in (-\pi, \pi)$ and $\lambda \notin -p^{\mathbb{Z}}$. We prove that $E_q^d(x) = \mathcal{E}_q^d(x)$ on the sector $\arg(x) \in (-2\pi, 2\pi)$ of the Riemann surface of the logarithm and that $\mathcal{E}_q^{[\lambda]}(x) = E_q^{[\lambda]}(x)$ for any $x \in \mathbb{C} \smallsetminus (p-1)\lambda q^{\mathbb{Z}}$. Moreover we can explicitly determine the functions $\mathcal{E}_q^d(x) - \mathcal{E}_q^{[\lambda]}(x)$ and $\mathcal{E}_q^{[\lambda]}(x) - \mathcal{E}_q^{[\mu]}(x)$ for $x \in \mathbb{C} \smallsetminus \mathbb{R}^-$, in terms of the Theta function. Finally, we establish the following relation between $\mathcal{E}_q^d(x)$ and $\mathcal{E}_q^{[\lambda]}(x)$ (*cf.* Corollary 3.10 below):

$$\mathcal{E}_q^d(x) = \frac{1}{\ln q} \int_1^q \mathcal{E}_q^{[\lambda]}(x) \frac{d\lambda}{\lambda}\,.$$

In an analogous way, for a formal power series $\hat{f} \in \mathbb{C}[[x]]$ such that $\mathcal{B}_q \hat{f}$ is an analytic function with a q-exponential growth of order one at ∞ we can define its sums $\mathcal{S}_q^d \hat{f}$, $\mathcal{S}_q^{[\lambda]} \hat{f}$, $S_q^d \hat{f}$ and $S_q^{[\lambda]} \hat{f}$. Using some explicit results for the Tschakaloff series and a q-convolution product adapted to the situation we can prove the following (*cf.* Theorem 4.14):

Theorem 4. *Let $\hat{f} \in \mathbb{C}[[x]]$ be a generic q-Gevrey series. Then for any $\lambda \in \mathbb{C}^* \smallsetminus \cup_{i=1}^n \mu_i q^{\mathbb{Z}}$, for convenient $\mu_1, \ldots, \mu_n \in \mathbb{C}^*$, and almost all direction $d \in (-\pi, \pi)$, we have $\mathcal{S}_q^d \hat{f} = S_q^d \hat{f}$ and $\mathcal{S}_q^{[\lambda]} \hat{f} = S_q^{[\lambda]} \hat{f}$. Moreover:*

$$\mathcal{S}_q^d \hat{f} = \frac{1}{\ln q} \int_{e^{id}}^{qe^{id}} \mathcal{S}_q^{[\lambda]} \hat{f} \frac{d\lambda}{\lambda}\,.$$

* * *

The theory of irregular singular q-difference equations is nowadays relatively well understood. This paper deals with two of the questions that are still without answer, namely:

1. Thanks to the work of J. Sauloy [Sau00] we know how to "uniformly approximate" the global monodromy of a fuchsian differential equation on $\mathbb{P}^1_{\mathbb{C}}$, in terms of the Birkhoff matrices of a family of q-difference equations deforming the given differential one. Of course an analogous result is expected be true for the Stokes phenomenon: actually the *confluence* of the Stokes matrices is studied for some functional equations linked to classical special functions (*cf.* for instance [Zha02]). The main theorem of the first part of this article goes in the direction of a discrete deformation of the Stokes phenomenon: differently from previous authors, we consider a discrete convergent deformation of the divergent differential datum. This approach is not really explored and the present result surely does not exhaust its possible applications.

2. In the second part of the paper we study the relations between the different kind of q-Borel sums considered in the literature. We prove the relations among them for a generic Gevrey series. This is a first step towards the proof of a general result for a divergent solution of a q-difference equations, having a Newton polygon with more than one slope.

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Part I. Convergent q-Borel and q-Laplace transform and confluence: the case $q < 1$

We suppose that $q \in (0,1) \subset \mathbb{R}$ and we set $p = q^{-1}$.

The first part of the paper is organized as follows. First of all we study the properties of the q-deformation $\mathcal{E}_q(x) = \sum_{n\geq 0}(-1)^n [n]_q^! x^n$ of the Euler series: namely we give two integral representations for $\mathcal{E}_q(x)$, and use them for proving that $\mathcal{E}_q(x)$ converges uniformly to the Borel sum of $\hat{E}(x) = \sum_{n\geq 0}(-1)^n n! x^n$ in the direction $\mathbb{R}^+$, uniformly on the compacts of a convenient sector. Then we give an analogous result for general q-series, deforming coefficientwise a Gevrey series of order 1. In appendix A we recall some general facts on the Jackson integral, while in appendix B we prove a degenerate q-Watson formula for Heine's series that we need in §1 for the proof of Proposition 1.5.

1 Convergent q-Euler series

The series

$$\mathcal{E}_q(x) = \sum_{n\geq 0}(-1)^n [n]_q^! x^{n+1} ,$$

where $[0]_q = 1$, $[n]_q = \frac{q^n-1}{q-1}$ and $[n]_q^! = [n]_q[n-1]_q \cdots [1]_q$, represents a germ of analytic function at 0. If one considers the q-derivation:

$$d_q y = \frac{y(qx) - y(x)}{(q-1)x}$$

and observe that $d_q x^n = [n]_q x^{n-1}$ for any $n \in \mathbb{Z}$, $n \geq 1$, then one easily sees that $\mathcal{E}_q(x)$ verifies the functional equation :

$$x^2 d_q y + y = x ,$$

that can be rewritten in the form:

$$y(x) = \frac{x}{x+1-q} y(qx) - \frac{(q-1)x}{x+1-q} .$$

By substitution of x by $q^n x$, we deduce that

$$y(q^n x) = \frac{q^n x}{q^n x + 1 - q} y(q^{n+1}x) - \frac{(q-1)q^n x}{q^n x + 1 - q} ,$$

which implies that $\mathcal{E}_q(x)$ can be continued to an analytic function on $\mathbb{C} \smallsetminus \{(q-1)q^n : n \in \mathbb{Z},\, n \leq 0\}$. The discrete spiral of poles $\{(q-1)q^n : n \in \mathbb{Z},\, n \leq 0\}$ of $\mathcal{E}_q$ turns out to be a spiral of simple poles, as the following lemma shows:

Lemma 1.1. *The analytic function $\mathcal{E}_q$ admits the following expansion*

$$\mathcal{E}_q(x) = (1-q) \sum_{n\geq 0} \frac{(q^{n+1};q)_\infty}{1 + \frac{1-q}{q^n x}} , \tag{1.1.1}$$

where $(a;q)_\infty = \prod_{i=0}^{\infty}(1-q^i a)$.

In particular for any $k \in \mathbb{Z}$, $k \leq 0$, the function $\mathcal{E}_q(x)$ has a simple pole at $(q-1)q^k$. The residue of the differential form $\mathcal{E}_q(x)dx$ at $(q-1)q^k$ is equal to

$$Res_{x=(q-1)q^k} \mathcal{E}_q(x) dx = -(1-q)^2 q^k (q^{1-k};q)_\infty .$$

We recall some standard notations for basic hypergeometric functions

$$\begin{cases} {}_2\phi_1\left(a,b;c;q,x\right) = \sum_{n\geq 0} \frac{(a;q)_n(b;q)_n}{(c;q)_n(q;q)_n} x^n\,, \\ \text{where } (a;q)_0 = 1 \text{ and } (a;q)_n = \prod_{k=1}^n (1-aq^{k-1})) \text{ for } 1\leq n\leq\infty, \end{cases}$$

and the Heine's basic transformation (*cf.* [GR90, §1.4]):

$$(1.1.2) \qquad {}_2\phi_1\left(a,b;c;q,x\right) = \frac{(a;q)_\infty(bx;q)_\infty}{(c;q;)_\infty(x;q)_\infty} {}_2\phi_1\left(c/a,x;bx;q,a\right) \quad (|q|<1,\ |a|<1)\,.$$

Proof. The lemma above is a straightforward application of (1.1.2), in fact:

$$(1.1.3) \qquad \mathcal{E}_q(x) = x\ {}_2\phi_1\left(q,q;0;q,-\frac{x}{1-q}\right)\,.$$

The calculation of the residues of $\mathcal{E}_q(x)$ follows at once. □

1.1 Integral representation

Using the Jackson's integral (*cf.* Appendix A for the definition) we obtain the following integral representation for $\mathcal{E}_q$:

Proposition 1.2. *For any* $x\in\mathbb{C}\smallsetminus\{(q-1)q^n : n\in\mathbb{Z},\, n\leq 0\}$*, we have:*

$$(1.2.1) \qquad \mathcal{E}_q(x) = \int_0^{\frac{x}{1-q}} \frac{(q(1-q)\frac{t}{x};q)_\infty}{t+1} d_qt = \int_{q^{\mathbb{Z}}\frac{x}{1-q}} \frac{(q(1-q)\frac{t}{x};q)_\infty}{t+1} d_qt.$$

Proof. Let us remark that $(q^{-k};q)_\infty = 0$ for any $k\in\mathbb{Z}$, $k\geq 0$. Then it follows from Remark A.5, that Formula (1.1.1) is equivalent to (1.2.1) □

Remark 1.3. A straightforward verification shows that the infinite product $(q(1-q)x;q)_\infty$ represents a germ of analytic function at 0 and that it verifies the equation

$$y(px) = (1+(p-1)(-qx))\,y(x)\,,$$

or equivalently

$$d_py(x) = -qy(x)\,.$$

In the present article, we will denote by $e_p(x)$ and $e_q(x)$ the functions obtained by replacing $n!$ by $[n]_p^!$ and $[n]_q^!$, respectively, in the Tarlor expansion at zero of the exponential function e^x. Since Euler, one knows that $(q(1-q)x;q)_\infty$ coincides with the analytic function at 0:

$$e_p(-qx) := \sum_{n\geq 0} \frac{(-qx)^n}{[n]_p^!}\,,$$

so that Equation (1.2.1) takes the more familiar shape:

$$\mathcal{E}_q(x) = \int_0^{\frac{x}{1-q}} \frac{e_p(-qt/x)}{t+1} d_qt\,,$$

that so closely reminds the Euler integral:

$$\mathcal{E}(x) = \int_0^{+\infty} \frac{e^{-\frac{t}{x}}}{t+1} dt\,.$$

The analytic function $\mathcal{E}(x)$ can be continued to $\mathbb{C}\smallsetminus\mathbb{R}^-$, it is asymptotic at zero to the Euler series $\sum_{n\geq 0}(-1)^n n! x^{n+1}$ and is solution of the differential equation $x^2y'+y=x$. In the following subsection we are going to study the behavior of $\mathcal{E}_q(x)$ with respect to $\mathcal{E}(x)$ when $q\to 1^-$.

1.2 Confluence

Let us denote by $\mathcal{E}(x)$ the analytic continuation to $\mathbb{C} \setminus (-\infty, 0]$ of the Borel sum of $\hat{E}(x)$ in the direction $\mathbb{R}^+$:

$$\mathcal{E}(x) = \int_0^{+\infty} \frac{e^{-\frac{t}{x}}}{t+1} dt, \quad \Re x > 0 \,,$$

and by $\log x$ the analytic continuation to $\mathbb{C} \setminus (-\infty, 0]$ of $\log x$.

Theorem 1.4. *If $q \to 1^-$, the analytic continuation of $\mathcal{E}_q(x)$ converges to $\mathcal{E}(x)$ for any $x \in \mathbb{C} \smallsetminus (-\infty, 0]$ and the convergence is uniform on the compacts of $\mathbb{C} \smallsetminus (-\infty, 0]$.*

The proof of the theorem above relies on the following result (*cf.* §B.2 below for the proof):

Proposition 1.5. *The following identity holds, for any $x \in \mathbb{C}^* \setminus q^{-\mathbb{N}}$:*
(1.5.1)

$$\sum_{n \geq 0} (q;q)_n x^{n+1} = -\left(-qx\frac{\theta'(-qx)}{\theta(-qx)} + 1 + A(q)\right)\left(\frac{q}{x};q\right)_\infty + \sum_{n \geq 1} \frac{a_n}{(q;q)_n} q^{n(n+1)/2} \left(-\frac{1}{x}\right)^n,$$

where

$$\theta(x) = \theta(q,x) = \sum_{n \in \mathbb{Z}} q^{n(n-1)/2} x^n \,,$$

$$A(q) = \sum_{n \geq 0} \frac{q^{n+1}}{q^{n+1}-1}$$

and

$$a_{n+1} = \sum_{k=0}^{n} \frac{1}{q^{k+1}-1}, \quad n \geq 0 \,.$$

Our strategy for the proof of Theorem 1.4 is based on the fact that (1.5.1) is a "deformation" of the following classical formula:

$$\mathcal{E}(x) = (\log x - \gamma) e^{\frac{1}{x}} + \sum_{n \geq 1} \frac{\sum_{1 \leq k \leq n} \frac{1}{k}}{n!} \left(\frac{1}{x}\right)^n, \tag{1.5.2}$$

where γ is the Euler constant:

$$\gamma = \lim_{n \to \infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \ln(n)\right).$$

In fact, taking the logarithmic derivative of the functional equation $\theta(x) = x\theta(qx)$, one proves that the meromorphic function $(q-1)z\frac{\theta'(-z)}{\theta(-z)}$ verifies the equation $y(qx) - y(x) = q - 1$ or equivalently $d_q y(x) = \frac{1}{x}$, therefore it "deforms" the logarithm. On the other hand we have:

$$(q-1)A(q) = \sum_{n \geq 0} \frac{q^{n+1}}{[n+1]_q} \,,$$

whose link to the Euler constant is intuitive. The proof of Theorem 1.4 is a formalization of these ideas.

Proof of Theorem 1.4

If we perform the variable change $x \to \frac{x}{q-1}$ in (1.5.2) and remember that

$$e_p\left(q/x\right) = \left(\frac{-q(1-q)}{x};q\right)_\infty ,$$

then we obtain the expression

$$(1.5.3)\quad \mathcal{E}_q(x) = (1-q)\left[\ell_q\left(\frac{qx}{q-1}\right) + 1 + A(q)\right] e_p\left(\frac{q}{x}\right) + \sum_{n\geq 1} \frac{(q-1)a_n}{[n]_q^!} q^{n(n-1)/2}\left(\frac{q}{x}\right)^n ,$$

in which we have set

$$\ell_q(x) := -x\frac{\theta'(-x)}{\theta(-x)} .$$

We are going to analyze term (1.5.3) by term.

First of all the constant $A(q)$ can be expressed in terms of the logarithmic derivative $\Psi_q(x) = \frac{\Gamma_q'(x)}{\Gamma_q(x)}$ (see (B.1.5), where $\Omega(q) = A(q)$):

$$(1.5.4)\qquad A(q) = -\frac{1}{\ln q}\left(\Psi_q(1) + \ln(1-q)\right) .$$

The following result says how the q-logarithm ℓ_q tends to the usual logarithm.

Lemma 1.6. *Let $\epsilon \in (0,\pi)$ and consider the sector $V_\epsilon = \{x \in \mathbb{C}^* : |\arg x| \leq \pi - \epsilon\}$. Then the following uniform estimate holds for any $(q,x) \in (0,1)\times V_\epsilon$:*

$$(1.6.1)\qquad |\ln q\ \ell_q(-\sqrt{q}\,x) + \log x| \leq \frac{4\pi\, e^{\frac{2\pi}{\ln q}\epsilon}}{(1-e^{\frac{4\pi^2}{\ln q}})(1-e^{\frac{2\pi}{\ln q}\epsilon})} .$$

Proof. The lemma is a consequence of the following classical functional relation for $\theta(x)$ (*cf.* [WW88, §21.51, p. 475], where $\vartheta_3(z|\tau) = \theta(\sqrt{q}\,e^{2\pi i z})$ with $q = e^{2\pi i\tau}$):

$$(1.6.2)\qquad \theta(\sqrt{q}\,x) = \sqrt{\frac{2\pi}{\ln(1/q)}}\, e^{-\frac{\log^2 x}{2\ln q}}\, \theta^*(\sqrt{q^*}\,x^*),$$

where we write

$$x^* = e^{-\frac{2\pi i}{\ln q}\log x}, \quad q^* = e^{\frac{4\pi^2}{\ln q}}$$

and denote by θ^* the Theta function obtained by replacing q by q^*. Indeed, if we take the logarithmic derivative w.r.t. the variable x in (1.6.2) and observe that $\ln q\, x dx^* = -2\pi i x^* dx$, then we obtain the following expression:

$$(1.6.3)\qquad \ln q\ \ell_q(-\sqrt{q}\,x) + \log x = -2\pi i \ell_{q^*}(-\sqrt{q^*}\,x^*),$$

so that we only need to examine $\ell_{q^*}(-\sqrt{q^*}\,x^*)$. The key point of the proof is the fact that $q^* \to 0^+$ when $q \to 1^-$.

For $\epsilon \in (0,\pi)$ we set:

$$r_\epsilon = e^{\frac{2\pi}{\ln q}\epsilon} \in (\sqrt{q^*}\,,1), \quad V_\epsilon^* = \{x \in \mathbb{C} : \frac{\sqrt{q^*}}{r_\epsilon} \leq |x| \leq \frac{r_\epsilon}{\sqrt{q^*}}\}.$$

It's obvious that for any $x \in V_\epsilon$, we have $x^* \in V_\epsilon^*$, so that

$$(1.6.4)\qquad q^* < \frac{q^*}{r_\epsilon} \leq |\sqrt{q^*}\,x^*| \leq r_\epsilon < 1.$$

On the other hand, the following identity, consequence of the Jacobi triple product,

$$X \frac{\theta^{*\prime}(X)}{\theta^*(X)} = \sum_{n\geq 0} \left(\frac{q^{*n} X}{1+q^{*n}X} - \frac{q^{*n+1}}{X+q^{*n+1}} \right) ,$$

combined with the inequality:

$$|1+Xq^{*n}| \geq 1-|X|, \quad |X+q^{*n+1}| \geq |X|-q^* , \text{ for } q^* < |X| < 1,$$

implies that:

$$\text{(1.6.5)} \qquad \sup_{\frac{q^*}{r_\epsilon}\leq |X| \leq r_\epsilon} |\ell_{q^*}(-X)| \leq \frac{r_\epsilon}{1-q^*}\,\frac{1}{1-r_\epsilon} + \frac{q^*}{1-q^*}\,\frac{1}{\frac{q^*}{r_\epsilon}-q^*} \leq \frac{2r_\epsilon}{(1-r_\epsilon)(1-q^\epsilon)} .$$

We get (1.6.1) and hence Lemma 1.6 by combing (1.6.3) and (1.6.5). □

End of the proof of Theorem 1.4. By replacing x by $\sqrt{q}x/(1-q)$ in Lemma 1.6, we have:

$$\ell_q\left(\frac{qx}{q-1}\right) = -\frac{1}{\ln q}\left[\log x + \ln\left(\frac{\sqrt{q}}{1-q}\right) + O\left(e^{2\pi\epsilon/\ln q}\right)\right] .$$

Therefore, we deduce from (1.5.4) that:

$$\text{(1.6.6)} \qquad \ell_q\left(\frac{qx}{q-1}\right) + 1 + A(q) = -\frac{1}{\ln q}\left[\log x + \Psi_q(1) + \frac{\ln q}{2} + O(e^{2\pi\epsilon/\ln q})\right] ,$$

where Ψ_q denotes the logarithmic derivative of Γ_q. As $q \to 1^-$, the function $\Gamma_q(x)$ converges uniformly to $\Gamma(x)$ on any compact of $\mathbb{C}\setminus(-\mathbb{N})$ (*cf.* [Zha01]), so $\Psi_q(x)$ converges to the logarithmic derivative $\Psi(x)$ of the Γ function. From the classical relation $\Psi(1) = -\gamma$, one deduces that $\Psi_q(1) = -\gamma + o(1)$. In other words, (1.6.6) implies the following estimate:

$$\text{(1.6.7)} \qquad (q-1)\left[\ell_q\left(\frac{qx}{q-1}\right) + 1 + A(q)\right] = -\log x + \gamma + o(1)$$

where $o(1)$ denotes a quantity converging to 0 as $q \to 1^-$, uniformly on any compact of $\mathbb{C}\setminus(-\infty, 0]$.

Notice that the exponential function $e^{\frac{1}{x}}$ is the uniform limit on any domain $\{|x| > R > 0\}$ of the p-exponential $e_p(q/x)$, since

$$e_p\left(\frac{1}{x}\right) = \sum_{n\geq 0} \frac{q^{n(n-1)/2}}{[n]_q^!} \left(\frac{q}{x}\right)^n .$$

In the same time, again the dominated convergence Theorem implies that, as $q \to 1^-$,

$$\sum_{n\geq 1} \frac{(q-1)a_n q^{n(n-1)/2}}{[n]_q^!} \left(\frac{q}{x}\right)^n \to \sum_{n\geq 1} \frac{\sum_{k=1}^n \frac{1}{k}}{n!} \left(\frac{1}{x}\right)^n ,$$

uniformly for $|x| > R > 0$. We conclude combining (1.5.3) with (1.6.7). □

2 Confluence of the *convergent* q-analogue of Borel-Laplace summation

Let q be a real number in the open interval $(0, 1)$. We want to generalize, under convenient reasonable assumptions, the results of the previous section.

2.1 Definition of the convergent q-Borel and q-Laplace transform

Definition 2.1. *Let $\mathbb{C}\{x\}$ be the ring of the germs of analytic functions in the neighborhood of $x = 0$.*

1. *We call* (convergent) q-Borel transform *the map $\mathcal{B}_q$ given by:*
$$\mathcal{B}_q \quad : \quad x\mathbb{C}\left\{x\right\} \to \mathbb{C}\{\xi\},\ \sum_{n\geq 0} a_n x^{n+1} \mapsto \sum_{n\geq 0} \frac{a_n}{[n]_q^!}\xi^n.$$

2. *The* (convergent) q-Laplace transform $\mathcal{L}_q$ *is defined by*
$$\mathcal{L}_q = \mathcal{B}_q^{-1} \quad : \quad \mathbb{C}\{\xi\} \to x\mathbb{C}\left\{x\right\},\ \sum_{n\geq 0} a_n \xi^n \mapsto \sum_{n\geq 0} a_n [n]_q^! x^{n+1}.$$

Remark 2.2. Notice that the q-Euler series $\mathcal{E}_q(x)$, considered in the previous section, converges for $|x| < 1-q$. Therefore a function $f(x)$ is analytic on an open disc $\{|x| < R\}$, for some $R \in (0,\infty)$, if and only if its q-Borel transform $\mathcal{B}_q f(\xi)$ is analytic for $|\xi| < R/(1-q)$.

Calling $\mathcal{B}_q$ and $\mathcal{L}_q$ q-Borel and q-Laplace transform is somehow an abuse of language: they don't transform convergent series in divergent series and *vice versa*. Nevertheless they have interesting properties and we will show that they play a role in the understanding of the confluence in the irregular case. In fact, when $q \to 1$, they tend coefficientwise to the usual Borel and Laplace transforms, that we will denote $\mathcal{B}_1$ and $\mathcal{L}_1$ respectively.

Let us denote by $e_q(x)$ the generating function associated to the sequence $(\frac{1}{[n]_q^!})_{n\geq 0}$, so that it follows: $e_q(x) = \frac{1}{((1-q)x;q)_\infty}$. An important property of $\mathcal{B}_q$ and $\mathcal{L}_q$ is that they can be expressed both as continuous and discrete integrals:

Proposition 2.3. *Let $f \in x\mathbb{C}\{x\}$ and $\phi \in \mathbb{C}\{\xi\}$ such that $\mathcal{B}_q f = \phi$. Then:*
$$\phi(\xi) = \frac{1}{2\pi i}\int_{|x|=R} \frac{f(x)}{((1-q)\frac{\xi}{x};q)_\infty}\frac{dx}{x^2} = \frac{1}{2\pi i}\int_{|x|=R} f(x) e_q\left(\xi/x\right)\frac{dx}{x^2},$$
$$f(x) = \frac{-1}{2\pi i}\int_{|\xi|=\rho} \phi(\xi)\mathcal{E}_q\left(-\frac{x}{\xi}\right) d\xi,$$
where the radii R and ρ are assumed to be chosen sufficiently small, independently of the variables ξ and x, respectively.

Proof. The first equality is a consequence of the identity
$$\frac{1}{(x;q)_\infty} = \sum_{n\geq 0}\frac{1}{(q;q)_n}x^n$$
and of the residue theorem. Taking into account (1.1.1), the second equality is an application of the residue theorem. □

Corollary 2.4. *Let f and ϕ be as in Proposition 2.3. Then:*
$$\phi(\xi) = (q;q)_\infty \int_0^\xi f((1-q)x)\frac{(\frac{qx}{\xi};q)_\infty}{\theta'(-\frac{x}{\xi})} d_q x,$$

$$f(x) = \int_0^{\frac{x}{1-q}} \left(\frac{(1-q)q\xi}{x};q\right)_\infty \phi(\xi) d_q\xi = \int_0^{\frac{x}{1-q}} e_q\left(qx/\xi\right)^{-1}\phi(\xi) d_q\xi\,. \tag{2.4.1}$$

Remark 2.5. Notice that Formula (2.4.1) generalizes (1.2.1) and can be obtained directly by direct calculations. We give an alternative proof below.

Proof. Taking the derivative with respect to x of the functional equation

$$\theta(q^n x) = x^{-n} q^{-n(n-1)/2} \theta(x)$$

and setting $x = -1$, we obtain

$$\theta'(-q^n) = (-1)^n q^{-n(n+1)/2} \theta'(-1) = (-1)^n q^{-n(n+1)/2} (q;q)_\infty^3 .$$

The residues formula and Equation (1.1.1) imply that

$$\phi(\xi) = \frac{(1-q)\xi}{(q;q)_\infty} \sum_{n\geq 0} (-1)^n \frac{f((1-q)\xi q^n)}{(q;q)_n} q^{n(n+3)/2}$$

and that

$$f(x) = x \sum_{n\geq 0} q^n (q^{n+1};q)_\infty \phi\left(\frac{q^n x}{1-q}\right) .$$

This ends the proof. □

2.2 Main result

The formulas above suggest the convergence of the q-Laplace transform $\mathcal{L}_q\phi$ to the classical Laplace transform $\mathcal{L}^d\phi$ (in the direction $d \in (0, 2\pi)$):

$$\mathcal{L}^d\phi(x) = \int_0^{\infty e^{id}} \phi(\xi) e^{-\frac{\xi}{x}} d\xi \,, \tag{2.5.1}$$

where ϕ is supposed to be holomorphic in a neighborhood of $\xi = 0$ and to be analytically continued in an open sector $\{|\arg\xi - d| < \epsilon\}$ with at most an exponential growth at infinity.

Theorem 2.6. *Let $y(q,x) = \sum_{n\geq 0} y_n(q) x^{n+1} \in x\mathbb{C}[[x]]$ be a family of formal power series, with $q \in (\eta, 1]$, for some $\eta \in (0,1)$. We suppose that the $y_n(q)$'s are continuous functions of q and that the family $\phi(q,\xi) = \mathcal{B}_q y(q,x) \in \mathbb{C}\{\xi\}$ is solution of a family of equations over $\mathbb{P}^1_{\mathbb{C}}$, fuchsian and non resonant at ∞, in the sense of Assumption 2.7 below.*

Let $d \in [0, 2\pi)$ be such that $\phi(1,x)$ is holomorphic on a domain containing the half line $[0, e^{id}\infty)$. Then for any $x \in V := \{|\arg x - d| < \frac{\pi}{2}\}$ we have

$$\lim_{q\to 1^-} y(q,x) = \mathcal{L}^d\phi(1,\xi) \,,$$

the convergence being uniform on any compact of V.

Notice that $y(q,x) = \mathcal{L}_q\phi(q,\xi)$, so that the result above is actually a result about the confluence of q-summation. Moreover $\phi(q,\xi)$ is meromorphic over $\mathbb{C}^*$ and its poles are contained in a finite set of lines passing through the origin. Also for $\phi(1,\xi)$ there are only a finite numbers of direction d that are forbidden: the anti-Stokes directions.

Assumption 2.7. We suppose that:

1. The series $\phi(1,\xi)$ is solution of a differential equation $\mathcal{N}_1\phi(1,\xi) = \sum_{i=0}^{\mu} A_i(1,\xi)\delta^i\phi(1,\xi) = 0$, where $\delta = \xi\frac{d}{d\xi}$, $A_i(1,\xi) \in \mathbb{C}[\xi]$, and the operator $\mathcal{N}_1$ is fuchsian at 0 and ∞. Moreover we suppose that the exponents of $\mathcal{N}_1\phi(1,\xi) = 0$ at ∞ are non resonant.

2. The series $\phi(q,\xi)=\mathcal{B}_q y(q,x)$, $q\in(0,1)$, are solutions of a linear q-difference operator $\mathcal{N}_q\phi(q,\xi)=\sum_{i=0}^{\mu}A_i(q,\xi)\delta_q^i\phi(q,\xi)=0$, where $\delta_q=\xi d_q$, $A_i(q,\xi)\in\mathbb{C}[\xi]$, and $\mathcal{N}_q$ is fuchsian at 0 and ∞[4].

3. The Newton-Ramis polygons of $\mathcal{N}_q$ coincide for any $q\in(\eta,1]$, and the coefficients $A_i(q,\xi)$ tends uniformly to $A_i(1,\xi)$ when $q\to 1$, on any compact of $\mathbb{P}^1_{\mathbb{C}}$. This implies in particular that for q sufficiently closed to 1, the exponents of $\mathcal{N}_q$ at ∞ are non resonant.

4. For any q sufficiently closed to 1 there exists a constant gauge transformation $C(q)\in Gl_\mu(\mathbb{C})$ such that the constant term at ∞ of the matrix

$$C(q)^{-1}\left(\begin{array}{c|ccc} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \\ \hline -\frac{A_0(q,x)}{A_\mu(q,x)} & -\frac{A_1(q,x)}{A_\mu(q,x)} & \dots & -\frac{A_{\mu-1}(q,x)}{A_\mu(q,x)} \end{array}\right)C(q)$$

is in the Jordan normal form. We suppose that for $q\in(\eta,1]$ the entries of the matrix $C(q)$ are continuous functions of q and that the form of the Jordan blocks is independent of q.

2.3 Applications

Notice that the assumptions of Theorem 2.6 are verified in the following two natural situations.

2.8 ("Constant coefficient deformation" of a differential equation). For a linear differential equation $\sum_{i=0}^{\mu}A_i(x)\delta^i y=0$, a possible trivial deformation is given by $\sum_{i=0}^{\mu}A_i(x)\delta_q^i y=0$. One verifies that if $\sum_{i=0}^{\mu}A_i(\xi)\delta^i y=0$ satisfies the first point of Assumption 2.7, then $\sum_{i=0}^{\mu}A_i(\xi)\delta_q^i y=0$ verifies automatically the next three assumptions, provided that $1-q$ is small enough. Therefore we have:

Corollary 2.9. *Let $y(x)=\sum_{n\geq 0}y_n x^{n+1}\in x\mathbb{C}[[x]]$ be a Gevrey series of order one such that $\phi(\xi)=\mathcal{B}_1 y(x)$ is solution of a fuchsian differential equation $\sum_{i=0}^{\mu}A_i(x)\delta^i\phi=0$ on $\mathbb{P}^1_{\mathbb{C}}$, non resonant at ∞. Consider a family of power series $\mathbf{y}_q(x)$, with $q\in(0,1)$, such that $\mathcal{B}_q(\mathbf{y}_q)(\xi)$ is solution of $\sum_{i=0}^{\mu}A_i(\xi)\delta_q^i\phi=0$ and $\mathbf{y}_q(x)$ converges coefficientwise to $y(x)$ when $q\to 1^-$.*

Then the family $\mathbf{y}_q(x)$ converges uniformly to the Borel sum of $y(x)$, when $q\to 1^-$, on the compacts of a convenient sector $V=\{|\arg x-d|<\pi/2\}$.

2.10 (Confluent hypergeometric case). Take $\phi(q,\xi)$ to be the basic hypergeometric series:

$$\phi(q,\xi)=\begin{cases} {}_2\Phi_1(q^a,q^b;q;q,x)=\displaystyle\sum_{n\geq 0}\frac{(q^a;q)_n(q^b;q)_n}{(q;q)_n(q;q)_n}x^n\,, & \text{for } q\in(0,1);\\[2ex] {}_2F_1(a,b;1;x)=\displaystyle\sum_{n\geq 0}\frac{(a)_n(b)_n}{n!n!}x^n\,, & \text{if } q=1;\end{cases}$$

where $a,b\in\mathbb{C}$, with $a-b\notin\mathbb{Z}$. Then Theorem 2.6 says that:

[4]This means that the only non vertical slope of the Newton-Ramis polygon, *i.e.* of the convex envelope of the set

$$\{(i,j):\ A_i(q,x)\neq 0,\ \mathrm{ord}_{x=0}A_i(q,x)\leq j\leq \deg_x A_i(q,x)\}\,,$$

are horizontal.

Corollary 2.11. *The basic hypergeometric analytic function*

$$\sum_{n\geq 0} \frac{(q^a;q)_n(q^b;q)_n}{(q;q)_n} \left(\frac{x}{1-q}\right)^n$$

converges uniformly to the Borel sum of the hypergeometric confluent series

$${}_2F_0(a,b;-;x) = \sum_{n\geq 0} \frac{(a)_n(b)_n}{n!} x^n$$

on the compacts of a convenient sector centered at 0*, when* $q \to 1^-$.

Of course the results above can be generalized. In fact, for any $\ell \geq 2$, and any *generic* choice of the parameters $a_1,\dots,a_\ell,b_1,\dots,b_{\ell-2} \in \mathbb{C}$, the analytic basic hypergeometric function

$$\sum_{n\geq 0} \frac{(q^{a_1};q)_n \cdots (q^{a_\ell};q)_n}{(q^{b_1};q)_n \cdots (q^{b_{\ell-2}};q)_n (q;q)_n} \left(\frac{x}{1-q}\right)^n$$

converges uniformly to the Borel sum of the hypergeometric confluent series

$${}_\ell F_{\ell-2}(a_1,\dots,a_\ell;b_1,\dots,b_{\ell-2};x) = \sum_{n\geq 0} \frac{(a_1)_n \cdots (a_\ell)_n}{(b)_1 \cdots (b)_{\ell-2} n!} x^n$$

on the compacts of a convenient sector centered at 0, when $q \to 1^-$.

2.4 Proof of Theorem 2.6

We know that our germs $\phi(q,\xi)$, $q \in (\eta, 1]$, of analytic functions at 0 admit an analytic continuation along d. Moreover, for $q<1$, the functions $\phi(q,\xi)$ are actually meromorphic over $\mathbb{C}$, which means that they are linear combination of a basis of solutions of $\mathcal{N}_q y = 0$ at ∞. The main point of the proof is the careful choice of such a basis, which will allow us to prove that $\phi(q,x)$ converges uniformly to $\phi(1,\xi)$ on an *infinite* sector containing the direction d. Of course this ends the proof since Equations (2.4.1) and (2.5.1) imply that for any $x \in V$ we have:

$$\begin{aligned}
\lim_{q\to 1} y(q,x) &= \lim_{q\to 1} \int_0^{\frac{x}{1-q}} \left(\frac{(1-q)q\xi}{x};q\right)_\infty \phi(q,\xi) d_q\xi \\
&= \int_0^{\infty e^{i\arg(x)}} \lim_{q\to 1} \left(e_q\left(\frac{q\xi}{x}\right)^{-1} \phi(q,\xi)\right) d\xi \qquad (2.11.1)\\
&= \int_0^{\infty e^{i\arg(x)}} \phi(1,\xi) e^{-\frac{\xi}{x}} d\xi\,.
\end{aligned}$$

The theorem results of the combination of two lemmas. First of all let us prove the uniform convergence around zero:

Lemma 2.12. *The family* $\phi(q,\xi)$ *converges uniformly to* $\phi(1,x)$*, when* $q \to 1$*, on a closed disk centered at* 0.

Proof. Let us write $\phi(q,\xi) = \sum_{n\geq 0} \phi_n(q) x^n$ for any $q \in (\eta, 1]$. Then there exists $N>0$ such that for any $n>N$ the coefficients $\phi_n(q)$ verify a well defined recursive relation whose coefficients do not degenerate[5]. A direct estimates of the recursive relation allows to conclude

[5]Notice that we have not assumed that 0 is a regular singular point with non resonant exponents at 0.

that $|\phi_n(q)| \leq C^n$ for a convenient real constant C, any $n > N$ and any $q \in (\eta, 1]$. This estimate, together with the fact that $\phi_n(q)$ is a continuous function of q, implies the uniform convergence on a convenient closed disk centered at 0 (*cf.* for instance the estimates in [Sau02, Lemma 1.2.6]). □

The last assumption in 2.7 implies that $\phi(q,x)$ is a linear combination, whose coefficients are entries of the matrix $C(q)$, of the canonical solutions at ∞, constructed in [Sau00, §1], using a q-analog of the Frobenius method. As noticed in [Sau00, §3], the uniform part of the canonical solution at ∞ converges uniformly on any compact of $\mathbb{P}^1_{\mathbb{C}} \smallsetminus \{0\}$ where it is analytic, to the uniform part of the solutions of $\mathcal{N}_1 y = 0$ constructed with the classical Frobenius methods, once that the gauge transformation $C(1)$ has been applied to the companion matrix. Since the entries of $C(q)$ converges to the entries of $C(1)$ by assumption, to obtain the uniform convergence in a neighborhood of ∞ it is enough to control the convergence of the so called *log-car* matrix[6]. Let $\zeta = 1/\xi$. The uniform convergence of $(1-q)\zeta\theta'(q,\zeta)/\theta(q,\zeta)$ over the infinite sector $\{|\arg(\zeta)| < \pi - \varepsilon\}$ to $\log \zeta$ is already proved in Lemma 1.6. We need an analogous result for $\theta(q,\zeta)/\theta(q,cx)$ which is solution of the q-difference equation $y(q\zeta) = cy(\zeta)$, $c \in \mathbb{C}^*$. We give a proof of the needed estimate, although it is a classical result:

Lemma 2.13. *Let $c(q) \in \mathbb{C}^*$ be a function of $q \in (0,1)$ such that $\lim_{q\to 1} \frac{\log c}{\ln q} = \gamma$, $\epsilon \in (0,\pi)$ and consider the sector $V_\epsilon := \{\zeta \in \mathbb{C}^* : |\arg \zeta| \leq \pi - \epsilon\}$. As $q \to 1^-$, the following uniform estimate holds uniformly over V_ϵ:*

$$\frac{\theta(q,\zeta)}{\theta(q,c\zeta)} = \zeta^\gamma (1 + o(1-q)) .$$

Proof. Let us consider again the modular equation (1.6.2) :

$$\theta(q,\sqrt{q}\,) = \sqrt{\frac{2\pi}{\ln(1/q)}}\, e^{-\frac{\log^2 x}{2\ln q}}\, \theta(q^*, \sqrt{q^*}\, x^*),$$

where

$$x^* = e^{-\frac{2\pi i}{\ln q}\ \log x}, \quad q^* = e^{\frac{4\pi^2}{\ln q}} .$$

We observe that $(\zeta/\sqrt{q})^* = -\zeta^*$ and $(c\zeta/\sqrt{q})^* = -c^*\,\zeta^*$. Therefore we obtain:

$$\frac{\theta(q,\zeta)}{\theta(q,c\zeta)} = e^{\frac{\log c}{\ln q}\ (\log(\frac{\zeta}{\sqrt{q}}) + \frac{\log c}{2})}\ \frac{\theta(q^*, -\sqrt{q^*}\,\zeta^*)}{\theta(q^*, -\sqrt{q^*}\,c^*\,\zeta^*)} .$$

As in the proof of Lemma 1.6, we observe that for any $\zeta \in V_\epsilon$, $\zeta^* \in V_\epsilon^*$; see (1.6.4) for more details. Moreover, when X and Y denote two complex numbers such that $|X|, |Y| \in (q^*, 1)$, we have the estimate:

$$\left|\frac{\theta(q^*,X)}{\theta(q^*,Y)}\right| \leq \frac{(-|X|;q^*)_\infty\,(-\frac{q^*}{|X|};q^*)_\infty}{(|Y|;q^*)_\infty\,(\frac{q^*}{|Y|};q^*)_\infty} = \frac{\theta(q^*,|X|)}{\theta(q^*,-|Y|)} .$$

An elementary calculation using (1.6.4) allows to conclude, since $q^* \to 0^+$ and $c^* \to e^{-2\pi i\gamma}$. □

[6]The terminology comes from the juxtaposition of the terms "logarithm" and "character", meaning the solution matrix of a constant coefficient differential (resp. q-difference) system is obtained by a combinatoric procedure from the logarithm (resp. q-logarithm) and a family of characters (resp. q-characters). A solution in a regular singular point, whose exponents are non resonant, is given by the product of an analytic matrix, called uniform part, by the "log-car" matrix.

We are choosing here as a q-logarithm the logarithmic derivative of the Jacobi θ function and as q-characters convenient quotient of the θ functions. For more details in the q-difference setting *cf.* [Sau00].

Resuming, the function $\phi(q,\xi)$ is a linear combination, with coefficients that are continuous functions of q, of a canonical basis of solutions at ∞: we have proved that both the canonical solutions and the coefficients of the linear combination admit a uniform limit in a bounded sector containing d, centered at ∞ and of arbitrary radius $R>0$. Combined with Lemma 2.12, this means that $\phi(q,x)$ converges uniformly to $\phi(1,x)$ in a neighborhood of the direction d, which allows to conclude the proof.

A Jackson's integral

Definition A.1. *We set*

$$F(x)=\int_0^x f(t)d_qt=(1-q)x\sum_{n\geq 0}f(q^nx)q^n\,,$$

whenever the right hand side converges.

Remark A.2.

1. Notice that if $F(x)$ is well-defined then $d_qF(x)=f(x)$.

2. If $f(x)$ is continuous on the closed disk $D(0,r^+)$, then $F(x)$ is well defined for any $x\in D(0,r^+)$. In fact there exists $M>0$ such that $|f(q^nx)q^n|\leq M|q|^n$, which guarantees the convergence of the infinite sum.

Proposition A.3.

1. If $f(x)$ is an analytic function over the disk $D(0,r^-)$, then $F(x)$ is also analytic over $D(0,r^-)$.

2. If $F(x)$ is analytic over the disk $D(0,r^-)$ and if $G(x)$ is another analytic function over $D(0,r^-)$ such that $d_qG(x)=f(x)$, then

$$\int_0^x f(t)d_qt=G(x)-G(0)\,.$$

Proof. 1. It follows from the fact that $F(x)$ is a uniformly convergent series of analytic function over $D(0,r-\varepsilon^+)$, for any $r>\varepsilon>0$.

2. It follows immediately from the remark that the subfield of constants of the ring of analytic function over $D(0,r^-)$ with respect to the operator $f(x)\mapsto f(qx)$ is $\mathbb{C}$. In fact this implies that $F(x)-G(x)\in\mathbb{C}$.

□

Definition A.4. *Let us fix a q-orbit $q^{\mathbb{Z}}\alpha\subset\mathbb{C}$ and suppose that for any $x\in q^{\mathbb{Z}}\alpha$ the integral $F(x)=\int_0^x f(t)d_qt$ is well-defined. Then we set*

$$\int_{q^Z\alpha}f(t)d_qt=\lim_{\substack{|x|\to\infty\\ x\in q^{\mathbb{Z}}\alpha}}\int_0^x f(t)d_qt\,.$$

Remark A.5. A straightforward calculation shows that

$$\int_{q^Z\alpha}f(t)d_qt=(1-q)\alpha\sum_{n\in\mathbb{Z}}f(q^n\alpha)q^n$$

and in particular that

$$\int_{q^{\mathbb{Z}}\alpha}f(t)\frac{d_qt}{t}=(1-q)\sum_{n=-\infty}^{+\infty}f(q^n\alpha)\,,$$

whenever the right side converges.

B Expansion of $\mathcal{E}_q(x)$ at ∞

The purpose of this section is the proof of Proposition 1.5. We recall the notation

$$a_{n+1} = \sum_{k=0}^{n} \frac{1}{q^{k+1}-1}, \quad n \geq 0, \tag{B.0.1}$$

$$A(q) = \sum_{n\geq 0} \frac{q^{n+1}}{q^{n+1}-1},$$

and

$$\theta(x) = \sum_{n\in\mathbb{Z}} q^{n(n-1)/2} x^n,$$

and the statement of the proposition:

Proposition B.1. *For any $x \in \mathbb{C}^* \setminus q^{-\mathbb{N}}$:*

(B.1.1)

$$\sum_{n\geq 0} (q;q)_n x^{n+1} = -\left(-qx\frac{\theta'(-qx)}{\theta(-qx)} + 1 + A(q)\right)\left(\frac{q}{x};q\right)_\infty + \sum_{n\geq 1} \frac{a_n}{(q;q)_n} q^{n(n+1)/2}\left(-\frac{1}{x}\right)^n.$$

The proof of the proposition above is based on a Watson's formula for basic hypergeometric functions, which is the analogue of a Barnes' formula for Gauss hypergeometric function. Barnes (*cf.* [WW88, §14.51] and [Bar08]) proved that if $|\arg(-x)| < \pi$, $c \notin \mathbb{Z}_{\leq 0}$ and $a - b \notin \mathbb{Z}$, then the analytic continuation of ${}_2F_1(a,b;c;x)$ for $|x| > 1$ is given by:

$$\begin{aligned} {}_2F_1(a,b;c;x) &= \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-x)^{-a}{}_2F_1(a, a-c+1, a-b+1; x^{-1}) \\ &\quad + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-x)^{-b}{}_2F_1(b, b-c+1, b-a+1; x^{-1}). \end{aligned}$$

G.N. Watson (*cf.* [GR90, §4.3]) proved a formula of the same kind for Heine series, namely if:

$$x \notin q^{-\mathbb{N}} \cup \left(\frac{cq}{ab}q^{\mathbb{N}}\right), \quad c \notin q^{-\mathbb{N}}, \quad \frac{a}{b} \notin q^{\mathbb{Z}}, \quad abcx \neq 0. \tag{B.1.2}$$

then

$$\begin{aligned} {}_2\phi_1(a,b;c;q,x) &= \frac{(b, c/a;q)_\infty}{(c, b/a;q)_\infty}\, \frac{\theta(-ax)}{\theta(-x)}\ {}_2\phi_1\left(a, aq/c; aq/b; q, \frac{cq}{abx}\right) \\ &\quad + \frac{(a, c/b;q)_\infty}{(c, a/b;q)_\infty}\, \frac{\theta(-bx)}{\theta(-x)}\ {}_2\phi_1\left(b, bq/c; bq/a; q, \frac{cq}{abx}\right), \end{aligned} \tag{B.1.3}$$

where $(\alpha_1, ..., \alpha_k; q)_n = \prod_{i=1}^{k} (\alpha_i; q)_n$. We are going to consider a degeneration of Watson's formula letting $b \to a$ and $c \to 0$. In this way we obtain an expression for ${}_2\phi_1(a, a; 0; q, x)$ that we can apply to

$$\mathcal{E}_q(x) = x\,{}_2\phi_1\left(q, q; 0; q, -\frac{x}{1-q}\right).$$

B.1 Degenerate cases of the Watson's formula

Let us first consider the case $b \to aq^m$, where m denotes a non-negative integer. For this purpose, we introduce the following notation

$$\Omega_{m+1}(x) = \sum_{k=0}^{m} \frac{q^k x}{q^k x - 1}$$

and

$$\Omega_0(x) = 0, \quad \Omega(x) := \Omega_\infty(x) = \lim_{m\to\infty} \Omega_m(x).$$

Notice that $\Omega_m(x)$ may be identified to a logarithmic derivative as follows:

$$\Omega_m(x) = \frac{x\frac{d}{dx}(x;q)_m}{(x;q)_m} = x\frac{d}{dx}\log(x;q)_m, \quad m \in \mathbb{N}\cup\{\infty\}.$$

Since $(x;q)_{m+n} = (x;q)_m\,(xq^m;q)_n$ for any m, $n \in \mathbb{N}$, it follows that

$$\Omega_{m+n}(x) = \Omega_m(x) + \Omega_n(q^m x), \quad \Omega(x) = \Omega_m(x) + \Omega(q^m x). \tag{B.1.4}$$

Let

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}\,, \text{ defined for any } x \notin (-\mathbb{N}),$$

be the Jackson's Gamma function (*cf.* [GR90, §10.1]). It's useful to remark that, if $\Psi_q(x) = \dfrac{d}{dx}\log\Gamma_q(x)$, then

$$\Psi_q(x) = -\ln q\,\Omega(q^x) - \ln(1-q), \quad \Psi_q(x+m) = \Psi_q(x) + \ln q\,\Omega_m(q^x)\,, \tag{B.1.5}$$

for any non-negative integer m.

We recall that we have set

$$\ell(x) := \ell_q(x) = -x\frac{\theta'(-x)}{\theta(-x)}\,,$$

where $\theta'(-x)$ denotes the derivative of θ w.r.t. at the variable x. From the Jacobi's triple formula $\theta(x) = (q, -x, -q/x; q)_\infty$, one deduces the following relation:

$$\ell(x) = -\Omega(x) + \Omega\left(\frac{q}{x}\right). \tag{B.1.6}$$

Since $\Omega_1(x) + \Omega_1(\frac{1}{x}) = 1$, putting $m = 1$ in (B.1.4) allows to obtain the following relation:

$$\ell(qx) - \ell(x) = 1,$$

which means that, for any given non-zero complex number λ, the function $x \mapsto \ell(\lambda x)$ is a q-logarithm. From (B.1.5), one gets the following link between ℓ_q and Ψ_q:

$$\ell(q^x) = \frac{1}{\ln q}\,\left(\Psi_q(x) - \Psi_q(1-x)\right)\,. \tag{B.1.7}$$

Proposition B.2. *Let m be a non-negative integer and let a, c be non-zero complex numbers. Suppose that $a \notin q^{-\mathbb{N}}$, $c \notin q^{-\mathbb{N}}$ and $c/a \notin q^{-\mathbb{N}}$. Then, the following formula holds:*

$$\begin{aligned} {}_2\phi_1(a, aq^m; c; q, x) \;=\; & \frac{(aq^m, c/a;q)_\infty}{(c, q^m;q)_\infty}\,\frac{\theta(-ax)}{\theta(-x)}\,P_m(a,c,x) + \frac{(a, cq^{-m}/a;q)_\infty}{(c,q;q)_\infty\,(q^{-m};q)_m}\,\frac{\theta(-aq^m x)}{\theta(-x)} \\ & \times\left\{[C_m(a,c) + \ell(aq^m x)]\,\Phi_m(a,c,x) + \sum_{n=1}^{\infty} C_{m,n}(a,c)\,\phi_{m,n}(a,c,x)\right\}\,, \end{aligned} \tag{B.2.1}$$

where

$$P_m(a,c,x) = \frac{(q;q)_{m-1}}{(a;q)_m}\sum_{n=0}^{m-1}\frac{(a, aq/c;q)_n}{(q, q^{1-m};q)_n}\left(\frac{cq^{1-m}}{a^2 x}\right)^n,$$

$$C_m(a,c) = \Omega(q) + \Omega(q^{1+m}) - \Omega(cq^{-m}/a) - \Omega(aq^m) + 1,$$

$$\Phi_m(a,c,x) = {}_2\phi_1(aq^m, aq^{1+m}/c; q^{1+m}; q, \frac{cq^{1-m}}{a^2x}) = \sum_{n\geq 0} \phi_{m,n}(a,c,x),$$

$$\phi_{m,n}(a,c,x) = \frac{(aq^m, aq^{1+m}/c;q)_n}{(q,q^{1+m};q)_n} \left(\frac{cq^{1-m}}{a^2x}\right)^n$$

and

$$C_{m,n}(a,c) = \Omega_n(aq^m) + \Omega_n(aq^{1+m}/c) - \Omega_n(q^{1+m}) - \Omega_n(q).$$

When $m = 0$, $P_m(a,c,x) = 0$.

Remark B.3. Equations (B.2.1) is a q-analog of [EMOT81, p. 109, (7)].

Proof. Letting $b = aq^m\epsilon$ in (B.1.3) gives raise to the following formula:

$$\begin{aligned} {}_2\phi_1(\ a\ , aq^m\epsilon; c; q, x) &= \frac{(aq^m\epsilon, c/a;q)_\infty}{(c, q^m\epsilon;q)_\infty} \frac{\theta(-ax)}{\theta(-x)}\, {}_2\phi_1\left(a, aq/c; q^{1-m}/\epsilon; q, \frac{cq^{1-m}}{a^2\epsilon x}\right) \\ &+ \frac{(a, cq^{-m}/(a\epsilon);q)_\infty}{(c, q^{-m}/\epsilon;q)_\infty} \frac{\theta(-aq^m\epsilon x)}{\theta(-x)}\, {}_2\phi_1\left(aq^m\epsilon, aq^{1+m}\epsilon/c; q^{1+m}\epsilon; q, \frac{cq^{1-m}}{a^2\epsilon x}\right), \end{aligned} \tag{B.3.1}$$

Suppose that $m \geq 1$. Since

$$(X;q)_{n+m} = (X;q)_m\,(Xq^m;q)_n, \quad (X;q)_m = (-X)^m\,(q^{1-m}/X;q)_m\,q^{m(m-1)/2}$$

and

$$\theta(q^mX) = X^{-m}\,q^{-m(m-1)/2}\,\theta(X),$$

we obtain

$$\begin{aligned} {}_2\phi_1\left(a, aq/c; q^{1-m}/\epsilon; q, \frac{cq^{1-m}}{a^2\epsilon x}\right) &= \sum_{n=0}^{m-1} \frac{(a, aq/c;q)_n}{(q, q^{1-m}/\epsilon;q)_n} \left(\frac{cq^{1-m}}{a^2\epsilon x}\right)^n \\ &+ \quad \frac{(a, cq^{-m}/a;q)_m}{(q,\epsilon;q)_m} \left(\frac{q}{a\epsilon x}\right)^m \sum_{n=0}^{\infty} \frac{(aq^m, aq^{1+m}/c;q)_n}{(q^{1+m}, q/\epsilon;q)_n} \left(\frac{cq^{1-m}}{a^2\epsilon x}\right)^n \end{aligned}$$

and

$$\frac{(a, cq^{-m}/(a\epsilon);q)_\infty}{(c, q^{-m}/\epsilon;q)_\infty} \frac{\theta(-aq^m\epsilon x)}{\theta(-x)} = -\epsilon\, \frac{(a, cq^{-m}/(a\epsilon);q)_\infty}{(c, q/\epsilon;q)_\infty\,(\epsilon;q)_{m+1}} \left(\frac{q}{ax}\right)^m \frac{\theta(-a\epsilon x)}{\theta(-x)}.$$

Hence, we can re-write (B.3.1) as follows:

$${}_2\phi_1(a, aq^m\epsilon; c; q, x) = A(\epsilon) + \frac{B_1(\epsilon)C_1(\epsilon) - B_2(\epsilon)C_2(\epsilon)}{\epsilon - 1}, \tag{B.3.2}$$

where

$$A(\epsilon) = \frac{(aq^m\epsilon, c/a;q)_\infty}{(c, q^m\epsilon;q)_\infty} \frac{\theta(-ax)}{\theta(-x)} \sum_{n=0}^{m-1} \frac{(a, aq/c;q)_n}{(q, q^{1-m}/\epsilon;q)_n} \left(\frac{cq^{1-m}}{a^2\epsilon x}\right)^n,$$

$$B_1(\epsilon) = \epsilon\, \frac{(a, cq^{-m}/(a\epsilon);q)_\infty}{(c, q/\epsilon;q)_\infty\,(\epsilon q;q)_m} \left(\frac{q}{ax}\right)^m \frac{\theta(-a\epsilon x)}{\theta(-x)},$$

$$C_1(\epsilon) = {}_2\phi_1\left(aq^m\epsilon, aq^{1+m}\epsilon/c; q^{1+m}\epsilon; q, \frac{cq^{1-m}}{a^2\epsilon x}\right),$$

$$B_2(\epsilon) = \frac{(a, cq^{-m}/a;q)_\infty}{(c, q\epsilon;q)_\infty\,(q;q)_m} \frac{(aq^m\epsilon;q)_\infty}{(aq^m;q)_\infty} \left(\frac{q}{a\epsilon x}\right)^m \frac{\theta(-ax)}{\theta(-x)},$$

$$C_2(\epsilon) = \sum_{n=0}^{\infty} \frac{(aq^m, aq^{1+m}/c; q)_n}{(q^{1+m}, q/\epsilon; q)_n} \left(\frac{cq^{1-m}}{a^2 \epsilon x}\right)^n .$$

Since $B_1(1) = B_2(1)$ and $C_1(1) = C_2(1)$, letting $\epsilon \to 1$ in (B.3.2) allows us to get the following relation:

$${}_2\phi_1(a, aq^m; c; q, x) = A(1) + [B_1'(1) - B_2'(1)]\, C + B\, [C_1'(1) - C_2'(1)] \ ,$$

with $C = C_1(1)$, $B = B_1(1)$. By direct computation, it yields:

$$B_1'(1) = \left[1 - \Omega(cq^{-m}/a) + \Omega(q) - \Omega_m(q) + \ell(ax)\right] B, \quad B_2'(1) = [\Omega(aq^m) - \Omega(q) - m]\, B,$$

$$C_1'(1) = \sum_{n=1}^{\infty} \frac{(aq^m, aq^{1+m}/c; q)_n}{(q^{1+m}, q; q)_n} \left[\Omega_n(aq^m) + \Omega_n(aq^{1+m}/c) - \Omega_n(q^{1+m}) - n\right] \left(\frac{cq^{1-m}}{a^2 x}\right)^n$$

and

$$C_2'(1) = \sum_{n=1}^{\infty} \frac{(aq^m, aq^{1+m}/c; q)_n}{(q^{1+m}, q; q)_n} \left[\Omega_n(q) - n\right] \left(\frac{cq^{1-m}}{a^2 x}\right)^n .$$

Notice also that

$$B = \frac{(a, cq^{-m}/a; q)_\infty}{(c, q; q)_\infty\, (q; q)_m} \left(\frac{q}{ax}\right)^m \frac{\theta(-ax)}{\theta(-x)} = \frac{(a, cq^{-m}/a; q)_\infty}{(c, q; q)_\infty\, (q^{-m}; q)_m} \frac{\theta(-aq^m x)}{\theta(-x)} ,$$

which ends the proof when $m \geq 1$.

If $m = 0$, the term $A(\epsilon)$ disappears, so $P_0(a, c, x) = 0$. □

Consider now two cases for which the hypothesis of Proposition B.2 are not all fulfilled: $c/a \in q^{-\mathbb{N}}$ or $c = 0$. Let $k \in \mathbb{N}$ and $c/a = q^{-k}\epsilon$, with $\epsilon \to 1$. Then the limit

$$\lim_{\epsilon \to 1} (q^{-k}\epsilon; q)_\infty\, \Omega(q^{-m-k}\epsilon) = -(q^{-k}; q)_k\, (q; q)_\infty,$$

implies the following:

Corollary B.4. *Let k, $m \in \mathbb{N}$ and $a \in \mathbb{C}^*$. The following equality holds:*
(B.4.1)

$${}_2\phi_1(a, aq^m; aq^{-k}; q, x) = \frac{(q^{-k-m}; q)_k}{(aq^{-k}; q)_k} \frac{\theta(-ax)}{\theta(-x)}\, {}_2\phi_1\left(aq^m, q^{1+m+k}; q^{1+m}; q, \frac{q^{1-k-m}}{ax}\right) .$$

Taking now $c \to 0$ in (B.2.1) gives the following formula:

$$\begin{aligned} {}_2\phi_1(a, aq^m; 0; q, x) \ = \ & \frac{(aq^m; q)_\infty}{(q^m; q)_\infty} \frac{\theta(-ax)}{\theta(-x)} P_m(a, x) + \frac{(a; q)_\infty}{(q; q)_\infty\, (q^{-m}; q)_m} \frac{\theta(-aq^m x)}{\theta(-x)} \\ & \times \left\{ [C_m(a) + \ell(aq^m x)]\, \Phi_m(a, x) + \sum_{n=1}^{\infty} C_{m,n}(a)\, \phi_{m,n}(a, x) \right\} , \end{aligned} \tag{B.4.2}$$

where

$$P_m(a, x) = \frac{(q; q)_{m-1}}{(a; q)_m} \sum_{n=0}^{m-1} \frac{(a; q)_n\, q^{n(n-1)/2}}{(q, q^{1-m}; q)_n} \left(-\frac{q^{2-m}}{ax}\right)^n ,$$

$$C_m(a) = \Omega(q) + \Omega(q^{1+m}) - \Omega(aq^m) + 1,$$

$$\Phi_m(a, x) = {}_1\phi_1(aq^m; q^{1+m}; q, \frac{q^2}{ax}) = \sum_{n \geq 0} \phi_{m,n}(a, x),$$

$$\phi_{m,n}(a,x) = \frac{(aq^m;q)_n\, q^{n(n-1)/2}}{(q,q^{1+m};q)_n} \left(-\frac{q^2}{ax}\right)^n$$

and

$$C_{m,n}(a) = \Omega_n(aq^m) + n - \Omega_n(q^{1+m}) - \Omega_n(q).$$

Once again, when $m = 0$ and $P_m(a,x) = 0$, we have:
(B.4.3)

$$ {}_2\phi_1(a,a;0;q,x) = \frac{(a;q)_\infty}{(q;q)_\infty}\, \frac{\theta(-ax)}{\theta(-x)} \left\{ [C_0(a) + \ell(ax)]\, \Phi_0(a,x) + \sum_{n=1}^{\infty} C_{0,n}(a)\, \phi_{0,n}(a,x) \right\} ,$$

where

$$C_0(a) = 2\Omega(q) - \Omega(a) + 1, \quad \Phi_0(a,x) = {}_1\phi_1(a;q;q,\frac{q^2}{ax}) = \sum_{n\geq 0} \phi_{0,n}(a,x),$$

$$\phi_{0,n}(a,x) = \frac{(a;q)_n\, q^{n(n-1)/2}}{(q,q;q)_n} \left(-\frac{q^2}{ax}\right)^n , \quad C_{0,n}(a) = \Omega_n(a) + n - 2\Omega_n(q).$$

B.2 Proof of Proposition 1.5

The equality ${}_1\phi_1(q;q;q,X) = (X;q)_\infty$, plus (B.4.3), where we have set $a = q$, implies the following formula:

$$\sum_{n\geq 0} (q;q)_n x^n = -\frac{1}{x} \left\{ [\Omega(q) + 1 + \ell(qx)]\ (q/x;q)_\infty + \sum_{n\geq 1} [n - \Omega_n(q)]\, \frac{q^{n(n-1)/2}}{(q;q)_n} \left(-\frac{q}{x}\right)^n \right\} .$$

Thus, one may obtain Proposition 1.5 by taking into account the following equalities:

$$\ell(qx) = -qx\, \frac{\theta'(-qx)}{\theta(-qx)}, \quad a_n = \Omega_n(q) - n, \quad A_q = \Omega(q).$$

Part II. Summation of divergent q-series and confluence: the case $q > 1$

Important. From now on, we fix $q \in (1,+\infty) \subset \mathbb{R}$, so that $p = q^{-1} \in (0,1)$.

In this second part we consider four types of q-summation (*cf.* Definition 4.12 below): our purpose is studying the relations among them. First of all, we investigate the different sums of the q-Euler series $\sum_{n\geq 0} (-1)^n [n]_q^! x^{n+1}$ and their properties (*cf.* §3 below). Then we prove a general result for generic q-Gevrey series (*cf.* Theorem 4.14), based on the study of the Tschakaloff series

$$T_q(x) = \sum_{n\geq 0} q^{n(n-1)/2} x^{n+1} ,$$

and of a convenient q-convolution product.

Notation. We set:

$$\theta_p(x) := \theta(p,x) = \sum_{n\in\mathbb{Z}} p^{n(n-1)/2} x^n = (p;p)_\infty (-x;p)_\infty (-p/x;p)_\infty \tag{2.4.4}$$

and

$$e_q(x) := (-(1-p)x;p)_\infty = \sum_{n\geq 0} \frac{x^n}{[n]_q^!} . \tag{2.4.5}$$

Remark that $e_q(x) = e_p(-x)^{-1}$ and that $\theta_p(x) = x\theta_p(px)$.

3 The divergent q-Euler series

Since $q > 1$, the q-Euler series

$$\hat{\mathcal{E}}_q(x) = \sum_{n\geq 0}(-1)^n [n]_q^! x^{n+1}$$

is obviously divergent for any $x \in \mathbb{C}^*$, as the Euler series $\sum_{n\geq 0}(-1)^n n! x^{n+1}$. The corresponding q-difference equation is

$$x^2 d_q y + y = x\,.$$

3.1 Definition of different sums of the q-Euler series

Let us consider the *q-Borel transforms of* $\hat{\mathcal{E}}_q(x)$ (for the general definition,*cf.* §4.1):

$$\psi(\xi) := \frac{1}{1+\xi} \text{ and } \phi(\xi) := \mathcal{E}_p(\xi)\,.$$

In the following, we will identify $\mathcal{E}_p(\xi)$ to its analytic continuation on $\mathbb{C} \setminus ((p-1)q^{\mathbb{N}})$. For any $d \in (-\pi, \pi)$ and $\lambda \notin -p^{\mathbb{Z}}$, we set:

$$\mathcal{E}_q^d(x) = \frac{q-1}{\ln q}\int_0^{e^{id}\infty} \frac{\psi(\xi)}{e_q(q\frac{\xi}{x})}d\xi, \quad \arg x \in (d-\pi, d+\pi); \tag{3.0.6}$$

$$\mathcal{E}_q^{[\lambda]}(x) = \frac{q}{1-p}\int_{\lambda p^{\mathbb{Z}}} \frac{\psi\left(\frac{\xi}{1-p}\right)}{e_q\left(q\frac{\xi}{(1-p)x}\right)}\, d_p\xi, \quad x \notin (p-1)\lambda q^{\mathbb{Z}}; \tag{3.0.7}$$

$$E_q^d(x) = \frac{q}{\ln q}\int_0^{e^{id}\infty} \frac{\phi(\xi)}{\theta_p(q\frac{\xi}{x})}d\xi, \quad \arg x \in (d-\pi, d+\pi); \tag{3.0.8}$$

$$E_q^{[\lambda]}(x) = \frac{q}{1-p}\int_{\lambda p^{\mathbb{Z}}} \frac{\phi(\xi)}{\theta_p(q\frac{\xi}{x})}\, d_p\xi, \quad x \notin (p-1)\lambda q^{\mathbb{Z}}; \tag{3.0.9}$$

Proposition 3.1. *(1) The functions $\mathcal{E}_q^d$ and E_q^d can be analytically continued on the sector $\{|\arg x| < 2\pi\}$ of the Riemann surface of the logarithm.*

(2) The functions $\mathcal{E}_q^{[\lambda]}$ and $E_q^{[\lambda]}$ are analytic on $\mathbb{C}^* \setminus (-(1-p)\lambda q^{\mathbb{Z}})$, the point $-(1-p)\lambda q^n$ being a simple pole for any integer $n \in \mathbb{Z}$.

Proof. The functions $\mathcal{E}_q^d$ and E_q^d are *a priori* defined for $|\arg x - d| < \pi$ and d varies in $(-\pi, \pi)$. The second assertion is straightforward. □

We will denote by $\mathcal{E}_q$ and E_q the analytic continuation of $\mathcal{E}_q^d$ and E_q^d, respectively, on the open sector $V(-2\pi, 2\pi) := \{x \in \tilde{\mathbb{C}}^* : |\arg x| < 2\pi\}$ on the Riemann surface of the logarithm. We recall the following result:

Proposition 3.2 ([RZ02, Thm. 2.1] and [Zha02, Thm. 1.3.2])**.** *The function $\mathcal{E}_q(x)$ (resp. $\mathcal{E}_q^{[\lambda]}$) admits $\hat{\mathcal{E}}_q(x)$ as q-Gevrey asymptotic expansion at $x = 0$ in the sector $\{\arg x < 3\pi/2\}$. In particular they are solution of $x^2 d_q y + y = x$.*

The following theorem is about the comparison between the four summations of $\hat{\mathcal{E}}_q$ we have just introduced:

Theorem 3.3. *$\mathcal{E}_q(x) = E_q(x)$ and $\mathcal{E}_q^{[\lambda]}(x) = E_q^{[\lambda]}(x)$.*

First, we need to prove the following two lemmas.

Lemma 3.4. *For any $x \in \mathbb{C}$ such that $\arg x \in (-2\pi, 0)$ we have:*

$$\mathcal{E}_q(xe^{2\pi i}) - \mathcal{E}_q(x) = E_q(xe^{2\pi i}) - E_q(x) = -2\pi i \frac{q-1}{\ln q} \frac{1}{e_q(-\frac{q}{x})}. \tag{3.4.1}$$

In particular: $\mathcal{E}_q(xe^{2\pi i}) - E_q(xe^{2\pi i}) = \mathcal{E}_q(x) - E_q(x)$.

Proof. A variable change in the integral defining $\mathcal{E}_q^d$ (resp. E_q^d) shows that $\mathcal{E}_q^d(xe^{2\pi i}) = \mathcal{E}_q^{d-2\pi}(x)$ (resp. $E_q^d(xe^{2\pi i}) = E_q^{d-2\pi}(x)$). We are reduced to calculate $\mathcal{E}_q^{d-2\pi}(x) - \mathcal{E}_q^d(x)$ and hence to calculate a residue at $\xi = -1$. In an analogous way, using formula (1.1.1), we obtain

$$\begin{aligned}
E_q^d(xe^{2\pi i}) - E_q^d(x) &= -\frac{q2\pi i}{\ln q} \sum_{n\geq 0} \mathrm{Res}_{\xi=(p-1)q^n} \left(\frac{\mathcal{E}_p(\xi)}{\theta_p\left(q\frac{\xi}{x}\right)} \right) \\
&= -\frac{q(p-1)2\pi i}{\ln q} \sum_{n\geq 0} \frac{q^n(p^{n+1};p)_\infty}{\theta_p\left(q\frac{(p-1)q^n}{x}\right)} \\
&= \frac{q(p-1)2\pi i}{\ln q} \frac{(p;p)_\infty}{\theta_p\left(\frac{1-q}{x}\right)} \sum_{n\geq 0} \frac{p^{n(n-1)/2}}{(p;p)_n} \\
&= -\frac{q(p-1)2\pi i}{\ln q} \frac{\left(p, -\frac{x}{q-1}; p\right)_\infty}{\theta_p\left(\frac{1-q}{x}\right)}.
\end{aligned}$$

The Jacobi triple product formula for θ_p immediately allows to conclude. □

Lemma 3.5. *Let us consider the homogenous q-difference equation*

$$x^2 d_q y = y. \tag{3.5.1}$$

Let y_0 be a meromorphic solution of (3.5.1) on the domain $\Omega = \{0 < |x| < R\}$. Suppose that one of the following hypotheses is verified:

- *the function y_0 is analytic on Ω;*
- *there exists $\mu \in \mathbb{C}^*$ such that $\mu \notin (1-p)p^{\mathbb{N}}$ and such that the function y_0 has only simple poles contained in the set $\mu p^{\mathbb{N}}$;*

then y_0 is identically zero.

Proof. Notice that $1/e_p(q/x)$ is a uniform solution to (3.5.1). Hence, there exists a q-invariant function $K(x)$ such that $y_0(x) = K(x)/e_p(q/x)$. Identifying $K(x)$ to an elliptic function, one ends the proof noticing that $(1-p)p^{\mathbb{Z}}$ is the only spiral of poles of $e_p(q/x)$. □

Proof of Theorem 3.3. Lemma 3.4 implies that $h^d(x) := E_q^d(x) - \mathcal{E}_q^d(x)$ is an analytic solution of (3.5.1) on $\mathbb{C}^*$. We deduce from Lemma 3.5 that $h^d \equiv 0$.

The difference $E_q^{[\lambda]}(x) - \mathcal{E}_q^{[\lambda]}(x)$ has only simple poles on $-\lambda(1-p)q^{\mathbb{Z}}$. Since $\lambda \notin -p^{\mathbb{Z}}$ we conclude applying Lemma 3.5. □

3.2 q-integral and continuous integral

Although both $\mathcal{E}_q(x)$ and $\mathcal{E}_q^{[\lambda]}$ are solutions of the q-difference equation $x^2 d_q y + y = x$, they have a deeply different nature. In fact, while $\mathcal{E}_q(x)$ is meromorphic on the whole Riemann surface $\widetilde{\mathbb{C}}^*$ of the logarithme, the function $\mathcal{E}_q^{[\lambda]}$ is a uniform function: more precisely, it is analytic on $\mathbb{C}^*$ minus a spiral of simple poles.

Let us consider the projection:

$$\begin{array}{rccc} \pi: &]0,+\infty[\times\mathbb{R} \cong \widetilde{\mathbb{C}}^* & \longrightarrow & \mathbb{C}^* \\ & (r,\alpha) & \longmapsto & re^{2i\pi\alpha} \end{array}.$$

Of course, we can identify $\mathcal{E}_q^{[\lambda]}$ to its pull back *via* π on $\widetilde{\mathbb{C}}^*$, *i.e.* to a meromorphic function on $\widetilde{\mathbb{C}}^*$, and study the solution $\mathcal{E}_q(x) - \mathcal{E}_q^{[\lambda]}$ of $x^2 d_q y + y = 0$. We have the following result (we identify all meromorphic function on $\mathbb{C}^*$ to their pull back on $\widetilde{\mathbb{C}}^*$):

Proposition 3.6. *Let $\lambda \in \mathbb{C}^* \setminus (-q^{\mathbb{Z}})$. For any $x \in \tilde{\Omega}_\lambda := \pi^{-1}\left(\mathbb{C}^* \setminus [-\lambda(1-p);q]\right)$, we have:*

$$\mathcal{E}_q(x) - \mathcal{E}_q^{[\lambda]}(x) = -2\pi i \frac{q-1}{\ln q}\left(L_{(-\lambda(1-p))^*,q^*}(x^*) - C_{\lambda,q}\right) e_q(-q/x)^{-1},$$

where we have used the following notation:

$$x^* = e^{-2\pi i \frac{\log x}{\ln q}}, \qquad q^* = e^{4\pi^2/\ln q},$$

$$L_{a,q}(x) = -\frac{x}{a}\frac{\theta_p'(-\frac{x}{a})}{\theta_p(-\frac{x}{a})} = \ell_p\left(\frac{x}{a}\right), \quad C_{\lambda,q} = L_{(-\lambda(1-p))^*,q^*}\left((1-p)^*\right).$$

Since $L_{a,q}(\frac{x}{a'}) = L_{aa',q}(x) = \ell_p(\frac{x}{aa'})$ and $(xx')^* = x^*(x')^*$, the theorem above can be rephrased in the following statement:

Corollary 3.7. *Let $p^* = 1/q^* = e^{-4\pi^2/\ln q}$. Then*

$$\mathcal{E}_q(x) - \mathcal{E}_q^{[\lambda]}(x) = -2\pi i \frac{q-1}{\ln q}\left[\ell_{p^*}\left(\left(-\frac{x}{\lambda(1-p)}\right)^*\right) - \ell_{p^*}\left(\left(-\frac{1}{\lambda}\right)^*\right)\right] e_q(-q/x)^{-1}.$$

The proof of Proposition 3.6 is based on the following two lemmas:

Lemma 3.8. *Let $\lambda \in \mathbb{C}^* \setminus (-p^{\mathbb{Z}})$ and $\tilde{\Omega}_\lambda := \pi^{-1}(\mathbb{C}^* \setminus [-\lambda(1-p);q]) \subset \tilde{\mathbb{C}}^*$. For any $x \in \tilde{\Omega}_\lambda$ we set:*

$$U_\lambda(x) = (\mathcal{E}_q(x) - \mathcal{E}^{[\lambda]}(x))e_q(-q/x).$$

Then

$$U_\lambda(xe^{2\pi i}) - U_\lambda(x) = -2\pi i \frac{q-1}{\ln q}$$

and

$$U_\lambda(qx) = U_\lambda(x).$$

Proof. The proof follows from Lemma 3.4, taking into account the functional equation of $e_q(-q/x)$:

$$e_q(-q/x) = \left(1 - \frac{q-1}{x}\right) e_q(-1/x)$$

□

Lemma 3.9. *Let $a \in \mathbb{C}^*$ and consider the function $L_{a,q}$ defined for $x \in \mathbb{C}^* \setminus [a;q]$ as above,* i.e. *:*

$$L_{a,q}(x) = \ell_p\left(\frac{x}{a}\right) = -\frac{x}{a}\frac{\theta_p'(-\frac{x}{a})}{\theta_p(-\frac{x}{a})}\,.$$

Then, up to an additive constant, $L_{a,q}$ is the only solution of the q-difference equation $y(qx) - y(x) = 1$, which is analytic on $\mathbb{C}^* \setminus aq^{\mathbb{Z}}$ and has only simple poles at $aq^{\mathbb{Z}}$.

Proof. The functional equation for $L_{a,q}$ is obtained deriving the equation $-\frac{x}{a}\theta_p(-p\frac{x}{a}) = \theta_p(-\frac{x}{a})$. The uniqueness (up to a constant) comes from the remark that there are no non constant elliptic function having only a simple pole in a fundamental domain. $\square$

Proof of Proposition 3.6. Let us consider the modular variable change:

$$q \longmapsto q^* = e^{4\pi^2/\ln q}\,, \quad x \longmapsto x^* = e^{-2\pi i \frac{\log x}{\ln q}}\,.$$

In the notation of Lemma 3.8 above let $W(x^*) = U_\lambda(x)$. Then:

$$W(x^*q^*) - W(x^*) = -2\pi i\frac{q-1}{\ln q}, \quad W(x^*e^{-2\pi i}) = W(x^*).$$

Equivalently, $x^* \mapsto W(x^*)$ represents a uniform solution to a first order q^*-difference equation. By Lemma 3.9, there exists a constant $C \in \mathbb{C}$ such that

$$W(x^*) = -2\pi i\frac{q-1}{\ln q}L_{(-\lambda(1-p))^*,q^*}(x^*) + C, \quad x \in \tilde{\Omega}_\lambda\,.$$

We calculate the constant $C = C_{\lambda,q}$ setting $x = 1-p$ and $x^* = e^{-2\pi i\frac{\ln(1-p)}{\ln q}}$. Since $e_q(-q/x)$ has a zero for $x = 1-p$, we obtain the exact expression for C. $\square$

The main result of this section is:

Theorem 3.10. *For any $x \in \mathbb{C}^* \setminus]-\infty, 0[$ we have:*

$$\mathcal{E}_q(x) = \frac{1}{\ln q}\int_1^q \mathcal{E}_q^{[\lambda]}(x)\frac{d\lambda}{\lambda}.$$

The theorem follows from the combination of Corollary 3.7 and the following lemma:

Lemma 3.11. *Let $p^* = 1/q^* = e^{-\frac{4\pi^2}{\ln}q}$. For z close enough to 1, we have:*

$$\int_1^q \ell_{p^*}\left(\left(-\frac{z}{\lambda}\right)^*\right)\frac{d\lambda}{\lambda} = \int_1^q \ell_{p^*}\left(\left(-\frac{1}{\lambda}\right)^*\right)\frac{d\lambda}{\lambda}\,.$$

Proof. Let $\mu = -\lambda^{-1}$. From the identity $x^* = e^{-2\pi i\frac{\log x}{\ln q}}$ we deduce that:

$$\frac{d\lambda}{\lambda} = -\frac{d\mu}{\mu} = -\frac{\ln q}{2\pi i}\frac{d\mu^*}{\mu^*}\,.$$

Therefore for $t = \mu^*$ we obtain:

$$\int_1^q \ell_{p^*}\left(\left(-\frac{z}{\lambda}\right)^*\right)\frac{d\lambda}{\lambda} = \frac{\ln q}{2\pi i}\int_{\mathcal{C}_{(-z)^*}} \ell_{p^*}(t)\frac{dt}{t}\,,$$

where $\mathcal{C}_{(-z)^*}$ is the positive oriented circle, centered at 0 and passing through the point $(-z)^*$. Observing that, for z close enough to 1, the meromorphic function $t \mapsto \ell_{p^*}(t)$ has no poles in the annulus between $\mathcal{C}_{(-z)^*}$ and $\mathcal{C}_{(-1)^*}$, we conclude applying Cauchy Theorem. $\square$

Remark 3.12. In Theorem 3.10, we could have replaced the interval $[1,q]$ with a path of the form $[a, qa]$, for any $a \in \mathbb{C} \setminus (-\infty, 0]$.

3.3 Comparing sums along different spirals

Notice that if $\lambda q^{\mathbb{Z}} = \mu q^{\mathbb{Z}}$, *i.e.* if λ and μ are two complex numbers congruent modulo q, the two discrete sums $\mathcal{E}_q^{[\lambda]}$ and $\mathcal{E}_q^{[\mu]}$ coincide. On the other side, if $\lambda q^{\mathbb{Z}} \neq \mu q^{\mathbb{Z}}$ these sums are trivially distinct, since the sets of their poles are distinct. This simple remark underlines a fundamental difference between the continuous and the discrete summations. In fact, when we make the direction d vary we are actually constructing an analytic continuation of $\mathcal{E}_q^d$ on $\widetilde{\mathbb{C}}^*$, while when we make $\lambda q^{\mathbb{Z}}$ vary, we obtain a whole family of distinct meromorphic solution of $x^2 d_q y + y = x$. This implies that the "discrete Stokes phenomenon" for the discrete summation has a different nature from the classical differential Stokes phenomenon. It is described in the following theorem:

Theorem 3.13. *For $\lambda, \mu \in \mathbb{C}^* \setminus (1-p)q^{\mathbb{Z}}$ we have:*

$$\mathcal{E}_q^{[\lambda]} - \mathcal{E}_q^{[\mu]} = \frac{K(\lambda,\mu,x)}{e_p(\frac{q}{x})}\,,$$

where:

$$K(\lambda,\mu,x) = C\frac{\theta_p(-\frac{\lambda}{\mu})\theta_p(\frac{1-p}{x})\theta_p(\frac{(1-p)\lambda\mu}{x})}{\theta_p(\frac{(1-p)\lambda}{x})\theta_p(\frac{(1-p)\mu}{x})}\,,$$

where C is a constant depending only on q.

Proof. The function $\mathcal{E}_q^{[\lambda]}(x) - \mathcal{E}_q^{[\mu]}(x)$ being solution of the homogeneous equation $x^2 d_q y = -y$, has the form

$$\mathcal{E}_q^{[\lambda]} - \mathcal{E}_q^{[\mu]} = \frac{K(\lambda,\mu,x)}{e_p(\frac{q}{x})}\,,$$

where $K(\lambda,\mu,x)$ is q-invariant function in each variable (x,λ,μ).

We want a more precise description of $K(\lambda,\mu,x)$. Notice that $\mathcal{E}_q^{[\lambda]} - \mathcal{E}_q^{[\mu]}$ has only two spirals of simple poles: $-(1-p)\lambda p^{\mathbb{Z}}$ and $-(1-p)\mu p^{\mathbb{Z}}$. Since any q-invariant uniform function can be written as a quotient of Theta functions, we obtain:

$$K(\lambda,\mu,x) = \frac{C(\lambda,\mu)\theta(\frac{\alpha}{x})\theta(\frac{\beta}{x})}{\theta(\frac{(1-p)\lambda}{x})\theta(\frac{(1-p)\mu}{x})}\,,$$

where $\alpha\beta = (1-p)^2\lambda\mu$. Moreover the factor $e_q(-q/x)$ in

$$K(\lambda,\mu,x) = e_q(-q/x)(\mathcal{E}_q^{[\lambda]} - \mathcal{E}_q^{[\mu]}),$$

implies that $K(\lambda,\mu,x)$ has a spiral of simple zeros at $(1-p)p^{\mathbb{Z}}$, which implies that we can chose either α or β equal to $-(1-p)$. We conclude that $\{\alpha,\beta\} = \{-(1-p), -(1-p)\lambda\mu\}$.

We have to calculate $C(\lambda,\mu)$. The poles of $K(\lambda,\mu,x)$ with respect to the variable λ forms two spirals: $-\frac{x}{1-p}p^{\mathbb{Z}}$ and $-p^{\mathbb{Z}}$, hence:

$$C(\lambda,\mu) = \frac{\theta(-\frac{\lambda}{\mu})}{\theta(\lambda)}C(\mu)\,.$$

A similar argument shows that $C(\mu) = C/\theta(p\mu)$. $\square$

Remark 3.14. One can express the constant C in terms of q-series. For instance, setting $x = 1$ and letting $\lambda \to \mu = 1$ in $K(\lambda,\mu,x)$, we can express C as a value of a derivative.

3.4 Confluence

Theorem 3.15. *Let $\mathcal{E}(x)$ be the sum of the classical Euler series in the direction $\mathbb{R}^+$. Then $\mathcal{E}_q(x) \to \mathcal{E}(x)$ if $q \to 1^+$ for any $x \in \mathbb{C}$ such that $\arg x \in (-\pi, \pi)$ and the convergence is uniform on the compacts of such a domain.*

Proof. Notice that for any $t \in \mathbb{R}^+$ we have $e_q(t) \to e^t$ and $e_q(t) \le e^t$. The dominated convergence theorem applied to the q-Laplace transform in a direction $d \in (-\pi, \pi)$ allows to conclude. Moreover, the estimate of $e_q(x)$ being uniform with respect to $d = \arg x$, the uniform convergence on the compacts of $\{|arg x| < \pi\}$ follows at once. □

Corollary 3.16. *The same statement holds for $\mathcal{E}_q^{[\lambda]}(x)$ when $q \to 1^+$.*

Proof. The proof results of the combination of Proposition 3.6 relating $\mathcal{E}_q(x)$ to $\mathcal{E}_q^{[\lambda]}(x)$, Lemma 1.6 on the uniform convergence of the q-logarithm, and the theorem above. □

4 Generic q-Gevrey series

We call *generic q-Gevrey series* a power series $\hat{f} \in \mathbb{C}[[x]]$ satisfying a q-difference equation $\Delta \hat{f} \in \mathbb{C}\{x\}$ for some analytic q-difference operator Δ of the form:

(4.0.1) $\Delta := a_0(x)(x\sigma_q)^n + a_1(x)(x\sigma_q)^{n-1} + ... + a_n(x)\,,$ with $a_j \in \mathbb{C}\{x\},\ a_0(0)a_n(0) \neq 0,$

and $\sigma_q(f(x)) = f(qx)$. This means that the associated Newton polygon has only one finite slope equal to one (*cf.* [Ram92] and [Zha99]).

An explicit calculation (*cf.* also [DV02, Lemma 1.1.10]) shows that

Lemma 4.1. *Let $d_q = \frac{\sigma_q - 1}{x(q-1)}$ and consider a q-difference operator $\Delta \in \mathbb{C}\{x\}[\sigma_q]$. Then Δ can be written as* (4.0.1) *if and only if it can be rewritten in the following form:*

(4.1.1) $\quad \Delta := b_0(x)(x^2 d_q)^n + b_1(x)(x^2 d_q)^{n-1} + ... + b_n(x), \quad b_j \in \mathbb{C}\{x\},\ b_0(0)b_n(0) \neq 0.$

Notice that the q-Euler series $\hat{\mathcal{E}}(x)$ considered in previous section is a generic q-Gevrey series.

4.1 Two formal q-Borel transforms

The classical Borel transform associates to each power series $\sum_{n\ge 0} a_n x^{n+1}$ the *more convergent* (or *less divergent*) power series $\sum_{n\ge 0} \frac{a_n}{n!}\xi^n$. For the solutions of a q-difference equations, the Gevrey "scaling factor" $(n!)^s$ is replaced by the q-Gevrey one: $(q^{sn^2/2})$ (*cf.* [Béz92],[Ram92], [Zha99],[Zha00]). Indeed, in the literature there are (at least) two q-analogs of the factorial $n!$, namely $[n]_q^!$ and $q^{n(n-1)/2}$. The reason for this dichotomy is the identity

$$[n]_q^! = \frac{(q;q)_n}{(1-q)^n} = \frac{(p;p)_n}{(1-p)^n} q^{n(n-1)/2} = [n]_p^! q^{n(n-1)/2}\,,$$

which implies that

(4.1.2)
$$[n]_q^! \sim \frac{q^{n(n-1)/2}}{(1-p)^n}\,,\ \text{when } n \to +\infty\,.$$

Let us consider the following two formal Borel transforms, associated to those q-factorials:

$$\mathcal{B}_q \quad : \quad x\mathbb{C}\,[[x]] \to \mathbb{C}\,[[\xi]]\,,\ \sum_{n\ge 0} a_n x^{n+1} \longmapsto \sum_{n\ge 0} \frac{a_n}{[n]_q^!}\xi^n$$

and

$$B_q \ : \quad x\mathbb{C}\,[[x]] \to \mathbb{C}\,[[\xi]]\,,\ \sum_{n\geq 0} a_n x^{n+1} \longmapsto \sum_{n\geq 0} \frac{a_n}{q^{n(n-1)/2}}\xi^n.$$

Following J.P. Ramis [Ram92] we set:

Definition 4.2. *An entire function ϕ is said to have q-*exponential growth of order 1 at ∞ *if there exist two constants $K>0$ and $\mu>0$ such that*

$$|\phi(x)| < K|x|^\mu e_q(|x|), \quad |x| \to \infty.$$

Remark that the function $e_q(|x|)$ can be replaced by $e^{\ln^2|x|/(2\ln q)}$.

Lemma 4.3 ([Ram92, Prop. 2.1])**.** *Let $\mathbb{E}_q$ be the set of all the entire functions having a q-exponential growth of order 1 at ∞, and let $\mathbb{C}\{x\}$ be the set of all power series having a positive convergence radius. Then $\mathbb{E}_q = \mathcal{B}_q(x\mathbb{C}\{x\}) = B_q(x\mathbb{C}\{x\})$.*

The following function space $\mathbb{H}_q$ has been introduced in [Zha02] and [RZ02]; see also [Zha99].

Definition 4.4. *For any $\lambda \in \mathbb{C}^*$, let $[\lambda;q] = \lambda q^{\mathbb{Z}}$.*

1. *A germ of function ϕ analytic at 0 is said to belong to $\mathbb{H}^{[\lambda;q]}$ if there exist a domain $\Omega \subset \mathbb{C}$ and a real number $r>0$ such that:*
   - *$\cup_{m\geq 0}\{x\in\mathbb{C} : |x-\lambda q^m| < rq^m\} \subset \Omega$;*
   - *ϕ can be continued to be an analytic function on Ω with a q-exponential growth of order 1 at infinity.*
2. *A germ of function ϕ analytic at 0 is said to belong to $\mathbb{H}_q$ if there exist a finite set $\Lambda \subset \mathbb{C}^*$ such that $\phi \in \mathbb{H}^{[\lambda;q]}$ for any $\lambda \in \mathbb{C}^* \setminus \Lambda$.*

Proposition 4.5. *Let $\hat f \in x\mathbb{C}[[x]]$, $\lambda \in \mathbb{C}^*$ and let $\psi = \mathcal{B}_q \hat f$, $\phi = B_q \hat f$. Then $\psi \in \mathbb{H}^{[\lambda;q]}$ if and only if $\phi \in \mathbb{H}^{[(1-p)\lambda;q]}$.*

Moreover, the map $\sum_{n\geq 0} a_n x^n \mapsto \sum_{n\geq 0} a_n [n]_p^! x^n$ induces an automorphism of the vector space $\mathbb{H}_q$.

Proof. Use the integral representation for the corresponding Hadamard product. □

Definition 4.6. *Let $\lambda \in \mathbb{C}^*$ and let $\hat f \in x\mathbb{C}[[x]]$.*

1. *We set $\mathbb{C}\{x\}_q^{[\lambda;q]} = \mathcal{B}_q^{-1}\mathbb{H}^{[\lambda/(1-p);q]} = B_q^{-1}\mathbb{H}^{[\lambda;q]}$. We say that $\hat f \in \mathbb{C}\{x\}_q^{[\lambda;q]}$ is q-*Borel summable along $[\lambda;q]$.
2. *Each element of $\mathbb{DS}\hat f := \{[\lambda;q] : \hat f \notin \mathbb{C}\{x\}_q^{[\lambda;q]}\}$ will be called* singular direction of $\hat f$.
3. *If $\mathbb{DS}\hat f$ is finite, $\hat f$ is called q-*Borel summable *and we denote $\mathbb{C}\{x\}_q$ the set of q-Borel summable series.*

Theorem 4.7 ([Zha02, Thm. 1.2.1])**.** *Every generic q-Gevrey series is q-Borel summable.*

4.2 Different kinds of q-exponential summation

The classical Borel-Laplace exponential summation is based on the Euler's integral representation of the Gamma function, namely

$$n! = \int_0^{\infty} e^{-t} t^{n+1} \frac{dt}{t} \,. \tag{4.7.1}$$

In the definition of a q-summation procedure one must be guided by the q-analogs of this last integral, question investigated since Jackson, Wigeret, Watson, etc...

We recall the following q-analogs of the integral representation of the Euler Gamma function.

Proposition 4.8. *For $d \in (-\pi, \pi)$ and $\lambda \notin (-q^{\mathbb{Z}})$ we have:*

$$[n]_q^! = \frac{q-1}{\ln q} \int_0^{\infty e^{id}} \frac{t^n}{e_q(qt)} \, dt = \frac{q-1}{\ln q} \int_0^{\infty e^{id}} t^n e_p(-qt) \, dt \,, \tag{4.8.1}$$

$$[n]_q^! = q \int_{\lambda p^{\mathbb{Z}}} \frac{t^n}{e_q(qt)} \, d_p t = q \int_{\lambda p^{\mathbb{Z}}} t^n e_p(-qt) \, d_p t \,, \tag{4.8.2}$$

$$q^{n(n-1)/2} = \frac{q}{\ln q} \int_0^{\infty e^{id}} \frac{t^n}{\theta_p(qt)} \, dt \,, \tag{4.8.3}$$

$$q^{n(n-1)/2} = \frac{q}{1-p} \int_{\lambda p^{\mathbb{Z}}} \frac{t^n}{\theta_p(qt)} \, d_p t \,. \tag{4.8.4}$$

Proof. For the proof of the identities above *cf.* [AAR99, pages 549-550]. More precisely, letting $c \to n+1$, $b \to 1$ and $a \to 0$ (resp. letting $c \to n+1$, $b, a \to 0$) in

$$\int_0^{\infty} x^{c-1} \frac{(-ax;p)_\infty (-pb/x;p)_\infty}{(-x;p)_\infty (-p/x;p)_\infty} dx = \frac{(ab;p)_\infty (p^c;p)_\infty (p^{1-c};p)_\infty \pi}{(bp^c;p)_\infty (ap^{-c};p)_\infty (p;p)_\infty \sin(\pi c)},$$

one gets the formula

$$\int_0^{\infty} \frac{x^n}{(-x;p)_\infty} dx = (\ln q) \ (p;p)_n p^{-n(n+1)/2}$$

$$\left(\text{resp.} \ \int_0^{\infty} \frac{x^n}{(-x;p)_\infty (-p/x;p)_\infty} dx = (\ln q) \ p^{-n(n+1)/2} \right),$$

which yields (4.8.1) (resp. (4.8.3)). Similarly, the formulae (4.8.2) and (4.8.4) can be viewed as special cases of

$$\int_0^{\infty} x^{c-1} \frac{(-ax;p)_\infty (-bp/x;p)_\infty}{(-x;p)_\infty (-p/x;p)_\infty} d_p x = \frac{(1-p)(p;p)_\infty (-p^c;p)_\infty (-p^{1-c};p)_\infty (ab;p)_\infty}{(-1;p)_\infty (-p;p)_\infty (ap^{-c};p)_\infty (bp^c;p)_\infty} \,.$$

□

Remark 4.9. In particular, (4.8.4) and (4.8.3) have been studied in [RZ02] and [Zha02] as starting points for the corresponding summation procedures. Other kinds of q-summation are considered in [Zha99] and [MZ00].

Let $d \in [-\pi, \pi)$. We will identify d to the half line $[0, \infty e^{id}) := \mathbb{R}^+ e^{id}$.

Definition 4.10. *We set* $\mathbb{H}_q^d = \cap_{\lambda \in (0,\infty e^{id})} \mathbb{H}^{[\lambda;q]}$.

Remark 4.11. The functional space $\mathbb{H}_q^d$ is exactly the space $\mathbb{H}_{q;1}^d$ introduced in [Zha99].

Let $\lambda \in \mathbb{C}^*$ and $d \in [-\pi, \pi)$. According to Proposition 4.8, the following four *q-Laplace transforms* are well defined:

$$\forall \phi \in \mathbb{H}^{[\lambda;q]}, \quad L_q^{[\lambda]}\phi = \frac{q}{1-p} \int_{\lambda p^{\mathbb{Z}}} \frac{\phi(\xi)}{\theta_p(q\frac{\xi}{x})}\, d_p\xi\,;$$

$$\forall \phi \in \mathbb{H}^{[\lambda/(1-p);q]}, \quad \mathcal{L}_q^{[\lambda]}\phi = \frac{q}{1-p} \int_{\lambda p^{\mathbb{Z}}} \frac{\phi(\xi/(1-p))}{e_q(q\frac{\xi}{(1-p)x})}\, d_p\xi;$$

$$\forall \phi \in \mathbb{H}_q^d, \quad L_q^d\phi = \frac{q}{\ln q} \int_0^{e^{id}\infty} \frac{\phi(\xi)}{\theta_p(q\frac{\xi}{x})} d\xi\,, \quad \mathcal{L}_q^d\phi = \frac{q-1}{\ln q} \int_0^{e^{id}\infty} \frac{\phi(\xi)}{e_q(q\frac{\xi}{x})} d\xi\,.$$

Definition 4.12. *1. If* $\hat{f} \in \mathbb{C}\{x\}_q^{[\lambda;q]}$*, we define its sums in the direction* $[\lambda;q]$ *as follows:*

$$\mathcal{S}_q^{[\lambda]}\hat{f} = \mathcal{L}_q^{[\lambda]}(\mathcal{B}_q\hat{f}), \quad S_q^{[\lambda]}\hat{f} = L_q^{[\lambda]}(B_q\hat{f}).$$

2. If $\hat{f} \in \mathbb{C}\{x\}_q^d$*, define its sums in the direction* d *as follows:*

$$\mathcal{S}_q^d\hat{f} = \mathcal{L}_q^d(\mathcal{B}_q\hat{f}), \quad S_q^d\hat{f} = L_q^d(B_q\hat{f}).$$

Remark 4.13.

• The summation procedures $\hat{f} \to S^{[\lambda;q]}\hat{f}$ and $\hat{f} \to S^d\hat{f}$ are introduced in [Zha02] and [Zha00]: they have many good asymptotic properties.

• Suppose that $\hat{f}$ is q-summable and that d is not a singular direction. Then we have the following *formal* equality (meaning that we exchange carelessly the infinite sum and the integral):

$$\mathcal{S}_q^d\hat{f} = \frac{1}{\ln q} \int_{e^{id}}^{qe^{id}} \mathcal{S}_q^{[\lambda]}\hat{f}\frac{d\lambda}{\lambda} \tag{4.13.1}$$

and

$$S_q^d\hat{f} = \frac{1}{\ln q} \int_{e^{id}}^{qe^{id}} S_q^{[\lambda]}\hat{f}\frac{d\lambda}{\lambda}\,. \tag{4.13.2}$$

To prove that this identity is not only formal, but analytic, one would like to apply the dominated convergence theorem: unfortunately the dominated convergence is a little bit delicate for a general $\hat{f}$, since we don't really control the spirals of poles of the discrete q-Borel sums. Anyway, we will prove (4.13.1) and (4.13.2) for a generic q-Gevrey series (*cf.* Theorem 4.14).

At this stage a natural question arises:

Do we have $S^{[\lambda;q]}\hat{f} = \mathcal{S}^{[\lambda;q]}\hat{f}$ and $S^d\hat{f} = \mathcal{S}^d\hat{f}$?

The answer is clear, and trivially positive, if $\hat{f}(x)$ is a germ of analytic function at zero: in this case all the sums of $\hat{f}(x)$ coincide with f.

The rest of the paper is devoted to the proof of the following theorem:

Theorem 4.14. *Let $\hat{f}$ be a generic q-Gevrey series and let $\lambda \in \mathbb{C}^*$, $d \in [-\pi, \pi)$. Assume that $\lambda \notin \mathbb{DS}(\hat{f})$ and $(0, e^{id}\infty) \cap \mathbb{DS}(\hat{f}) = \emptyset$. Then*

1. *$S_q^{[\lambda]}\hat{f} = \mathcal{S}_q^{[\lambda]}\hat{f}$, on a convenient domain Ω.*
2. *$S_q^d\hat{f} = \mathcal{S}_q^d\hat{f}$ on a convenient sector containing the direction d.*

Moreover we have:

$$\mathcal{S}_q^d\hat{f} = \frac{1}{\ln q}\int_{e^{id}}^{qe^{id}} \mathcal{S}_q^{[\lambda]}\hat{f}\frac{d\lambda}{\lambda}\,.$$

Remark 4.15. Theorems 3.3 and 3.10 are a special case of Theorem 4.14.

Before giving a proof of Theorem 4.14 in §4.5, we make a digression about two essential ingredients of the proof: first we prove the theorem in the special case of the Tschakaloff series; then we introduce a functional space that allows to read, in certain sense, any q-Gevrey series as a finite linear combination of some modified Tschakaloff series.

4.3 The Tschakaloff series

Let us consider another q-analogue of the Euler series:

$$T_q(x) = \sum_{n\geq 0} q^{n(n-1)/2}x^{n+1}\,,$$

called the Tschakaloff series or the partial Theta function. It satisfies the q-difference equation

$$xT_q(qx) - qT_q(x) = -qx\,,$$

that can also be rewritten in the form $x^2(q-1)d_qy + (x-q)y = -qx$.

The Borel transforms of T_q are:

$$\psi(\xi) = \mathcal{B}_q(T_q) = \sum_{n\geq 0}\frac{\xi^n}{[n]_p^!} = e_p(\xi) = ((1-p)\xi; p)^{-1}$$

and

$$\phi(\xi) = B_q(T_q) = \frac{1}{1-\xi}\,.$$

Proposition 4.16. *Let us fix $\lambda \notin [-1; q]$. Then $\mathcal{S}_q^{[\lambda]}T_q = S_q^{[\lambda]}T_q$.*

Proof. The definition of the Jackson integral (*cf.* §A) and the Jacobi triple product formula (*cf.* (2.4.4)), plus the development of the q-exponential $e_q(x)$ as an infinite product (*cf.* (2.4.5)), imply that:

$$\begin{aligned}\mathcal{L}_q^{[\lambda]}\psi &= \frac{q}{1-p}\int_{\lambda p^{\mathbb{Z}}}\frac{e_p(\xi)}{e_q\left(\frac{q\xi}{(1-p)x}\right)}d_p\xi = \frac{q}{1-p}\int_{\lambda p^{\mathbb{Z}}}\left(\xi, -\frac{q\xi}{x}; p\right)_\infty^{-1} d_p\xi \\ &= \lambda(p;p)_\infty\sum_{n\in\mathbb{Z}}\frac{p^n(-p^{1-n}x/\lambda; p)_\infty}{(p^{n+1}\lambda; p)_\infty\theta_p(p^n\lambda/x)}\,.\end{aligned}$$

Since $\theta_p(x) = p^{n(n-1)/2}x^n\theta_p(p^nx)$ for any $n \in \mathbb{Z}$, we obtain:

$$\mathcal{S}_q^{[\lambda]}T_q = \mathcal{L}_q^{[\lambda]}\psi = \frac{\lambda(p;p)_\infty}{\theta_p(\lambda/x)}\sum_{n\in\mathbb{Z}}\frac{\left(-p^{1-n}\frac{x}{\lambda}; p\right)_\infty}{(p^{n+1}\lambda; p)_\infty}p^{n(n+1)/2}\left(\frac{\lambda}{x}\right)^n\,.$$

On the other hand we have:

$$
\begin{aligned}
S_q^{[\lambda]} T_q = L_q^{[\lambda]} \phi &= \frac{q}{1-p} \int_{p^{\mathbb{Z}}\lambda} \frac{d_p \xi}{(1-\xi)\theta_p(q\xi/x)} \\
&= \lambda \sum_{n\in\mathbb{Z}} \frac{p^n}{(1-p^{n+1}\lambda)\theta_p(p^n\lambda/x)} \\
&= \frac{\lambda}{\theta_p(\lambda/x)} \sum_{n\in\mathbb{Z}} \frac{p^{n(n+1)/2}}{1-p^{n+1}\lambda} \left(\frac{\lambda}{x}\right)^n
\end{aligned}
$$

A straightforward calculation of the residues of $(x,p)_\infty^{-1}$ at $x = p^{-k}$, for $k \geq 0$, gives the formula:

$$
\frac{1}{(x;p)_\infty} = \frac{1}{(p;p)_\infty} \sum_{k\geq 0} \frac{a_k}{1-xp^k}\ , \text{ with } a_k = \frac{1}{(p^{-k};p)_k} = \frac{(-1)^k p^{k(k+1)/2}}{(p;p)_k}\ .
$$

Therefore we obtain:

$$
\mathcal{S}_q^{[\lambda]} T_q \quad = \quad \frac{\lambda}{\theta(\frac{\lambda}{x})} \sum_{\ell\in\mathbb{Z}} \frac{p^{\ell(\ell+1)/2}}{1-p^\ell\lambda} \left(\frac{\lambda}{x}\right)^\ell (a_\ell;p)_\infty\ {}_0\phi_1(-;a_\ell;p,a_\ell), \quad a_\ell = -p^{1-\ell}\frac{x}{\lambda}\,.
$$

The Ramanujan formula (*cf.* [Zha03, Thm. 4.4]):

$$
(x;p)_\infty\ {}_0\phi_1(-;x;p,x) = 1\,,\ \text{for any } x \notin q^{-\mathbb{N}},
$$

implies that $\mathcal{L}_q^{[\lambda]}\psi = L_q^{[\lambda]}\phi$. □

4.4 The functional space H in the Borel plane

We recall that the q-Borel transform B_q associates to a power series $\hat{f} = \sum_{n\geq 0} a_n x^{n+1} \in \mathbb{C}[[x]]$ the power series $\phi = \sum_{n\geq 0} a_n q^{-n(n-1)/2}\xi^n \in \mathbb{C}[[\xi]]$. As we have already pointed out, the q-Borel transform of a generic q-Gevrey series admits a positif radius of convergence and can be continued to an analytic function in the whole complex plane minus a finite number of sets of the form $\lambda q^{\mathbb{N}}$ (*cf.* [Zha99] and [Zha02]).

In this section we want to prove that every generic q-Gevrey series can be expressed by means of "modified Tschakaloff series". Our strategy consists in proving that the q-Borel transform of any generic q-Gevrey series admits an elementary decomposition, by studying the q-convolution product of suitable entire functions by a rational functions. This leads to the construction of a functional space which is somehow spanned by the q-Borel transforms of the modified Tschakaloff series.

Definition 4.17. *We call q-convolution product the following bilinear operator:*

$$
\begin{array}{rccc}
*_q: & \mathbb{C}\{\xi\}\times\mathbb{C}\{\xi\} & \longrightarrow & \xi\mathbb{C}\{\xi\} \\
& \xi^n *_q \xi^m & \longmapsto & q^{-(nm+n+m+1)}\xi^{n+m+1}
\end{array}.
$$

A direct calculation shows that

1. If $\phi = \sum_{n\geq 0}\phi_n\xi^n \in \mathbb{C}\{\xi\}$ and $\psi \in \mathbb{C}\{\xi\}$, then ([MZ00, 1.4.3, where $s = 1$)])

$$
\phi *_q \psi(\xi) = \sum_{\geq 0} \phi_0 q^{-n-1}\xi^{n+1}\psi(q^{-n-1}\xi).
$$

2. $B_q(\hat{f}\hat{g}) = B_q(\hat{f}) *_q B_q(\hat{g})$.

Let K be the set of rational functions bounded at zero and let $\mathbb{E}_q$ be the set of all entire functions admitting at most a q-exponential growth of order 1 at the infinity. We know that $K \cap \mathbb{E}_q = \mathbb{C}[\xi]$ and $\mathbb{E}_q = B_q(x\mathbb{C}\{x\})$ (*cf.* [Ram92]). Notice that the formula $B_q(\hat{f}\hat{g}) = B_q(\hat{f}) *_q B_q(\hat{g})$ identifies $(\mathbb{E}_q, *_q)$ to a commutatif sub-ring of $(\mathbb{C}\{\xi\}, *_q)$.

Definition 4.18. *We define the functional space $H := \cup_{n\geq 0} H_n$ in the following way:*

$$H_{-1} = \{1\}, \quad H_0 = K, \quad H_1 = \mathbb{E}_q *_q K := \{\phi *_q r : \phi \in \mathbb{E}_q, r \in K\}$$

and, for any integer $n \geq 1$,

$$H_{2n} = KH_{2n-1} := \{ru : r \in K, u \in H_{2n-1}\}, \quad H_{2n+1} = \mathbb{E}_q *_q H_{2n} := \{\phi *_q u : \phi \in \mathbb{E}_q, u \in H_{2n}\}.$$

Proposition 4.19. *For any $(r, \phi, u) \in K \times \mathbb{E}_q \times H$, we have $(ru, \phi *_q u) \in H \times H$. In other words, the functional space H is a $(K, \mathbb{E}_q)$-bimodule.*

Proof. It follows immediately from the definition of H. Indeed, if $n \leq m$, then $H_n \subset H_m$. So, we can suppose that $(r, \phi, u) \in K \times \mathbb{E}_q \times H_n$ and hence, $(ru, \phi *_q u) \in H_{n+2} \times H_{n+2} \subset H \times H$. □

Theorem 4.20. *For any $u \in H$, there exist ϕ_0, $\phi_1, \ldots, \phi_n \in \mathbb{E}_q$ and r_0, $r_1, \ldots, r_n \in K$ such that*

$$u = \phi_0 + r_0 + \phi_1 *_q r_1 + \ldots + \phi_n *_q r_n.$$

Moreover, we can suppose that r_1, ..., r_n are rational functions of the form $\frac{1}{(\xi-\lambda_i)^{\nu_i}}$, where c_i, $\lambda_i \in \mathbb{C}^*$ and $\nu_i \in \mathbb{N}$.

Proof. Since $H_m \subset H_{m+1}$, for any $u \in H$ there exists $m \in \mathbb{N}$ such that $u \in H_m$. So we can prove the theorem by induction on m. The cases $m = 0$ and $m = 1$ are trivial. Suppose that $u \in H_{m+1}$. Then there exists $(r, v, \phi) \in K \times H_m \times \mathbb{E}_q$ such that on of the following two cases occurs:

(1) $u = rv$,

(2) $u = \phi * v$,

and, by inductional hypothesis, $v = \phi_0 + r_0 + \sum_{j=1}^{m} \phi_j *_q r_j$. The proof in the case (2) is straightforward, since $\phi *_q (\phi_j *_q r_j) = (\phi *_q \phi_j) *_q r_j$. In the case (1), we need the following elementary lemma.

Lemma 4.21. *For any $(a, b, \ell, n) \in \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{N} \times \mathbb{N}$ such that $a \neq b$, the following decomposition holds:*

$$\frac{1}{(x-a)^\ell}\frac{1}{(x-b)^n} = \sum_{k=0}^{\ell-1} \frac{(n+k)!}{(a-b)^{n+k}k!}\frac{(-1)^k}{(x-a)^{\ell-k}} + \sum_{k=0}^{n-1} \frac{(\ell+k)!}{(b-a)^{\ell+k}k!}\frac{(-1)^k}{(x-b)^{n-k}}.$$

Proof. It enough to take the $(\ell-1)$-th derivative with respect to a and the $n-1$-th derivative with respect to b in the formula

$$\frac{1}{x-a}\frac{1}{x-b} = \frac{1}{a-b}\left(\frac{1}{x-a} - \frac{1}{x-b}\right).$$

. □

Let us go back to the proof of the Theorem 4.20. By linearity, it enough to consider a product of the form $r(\phi *_q r')$, with $r = \frac{1}{(\xi-\lambda)^\ell}$, $r' = \frac{1}{(\xi-\mu)^n)}$ and $\phi = \sum_{k\geq 0} \phi_k \xi^k \in \mathbb{E}_q$. Since

$$r' *_q \phi(\xi) = \sum_{k\geq 0} \phi_k q^{-k-1} \frac{\xi^{k+1}}{(q^{-k-1}\xi - \mu)^n},$$

the decomposition follows from the lemma above. □

Corollary 4.22. *For any $u \in H$, there exist $\lambda_1, \ldots, \lambda_n \in \mathbb{C}^*$ such that u is analytic on the domain $\mathbb{C} \setminus (\cup_{i=1}^m \lambda_i q^{\mathbb{N}})$ and the function U defined by*

$$U(\xi) = u(\xi) \prod_{i=1}^{n} \prod_{m\geq 0} \left(1 - \frac{\xi}{\lambda_i q^m}\right)$$

can be continued to an entire function that has at most a q-exponential growth of order $n+1$ at the infinity.

The corollary results from the combination of the theorem above and the following lemma:

Lemma 4.23. *Let $\phi \in \mathbb{E}_q$, $r = \frac{1}{(\xi-\lambda)^n}$, $n \geq 1$ and $\lambda \in \mathbb{C}^*$. Then $\phi *_q r$ admits $\lambda q^{\mathbb{N}}$ as set of poles and there exist $C > 0$, $m > 0$ such that, for any $\epsilon > 0$,*

$$|\xi q^{-n} - \lambda| > \epsilon \quad \Longrightarrow \quad |\phi *_q r(\xi)| < \frac{C}{\epsilon^n} |\xi|^m |e^{\frac{(\log x)^2}{2\ln q}}|.$$

Proof. Let $\phi = \sum_{k\geq 0} \phi_n \xi^k$. Since $\phi \in \mathbb{E}_q$, there exist A, $B > 0$ such that

$$\forall k \in \mathbb{N}, \quad |\phi_k| < AB^n q^{-n(n+1)/2}.$$

On the other hand,

$$\phi *_q r = \sum_{k\geq 0} \phi_k q^{-k-1} \xi^{k+1} r(q^{-k-1}\xi),$$

which implies directly the lemma. □

4.5 Proof of Theorem 4.14

We start by proving the following preparatory result:

Proposition 4.24. *If $\hat{f} \in x\mathbb{C}[[x]]$ is a generic q-Gevrey series, then its q-Borel transform belongs to H.*

Proof. Let Δ be a linear analytic q-difference operator such that $\Delta\hat{f} = g \in x\mathbb{C}\{x\}$. We know that Δ admits an analytic factorization (*cf.* [Zha99, Prop. 5.1.4], [Sau04, Thm. 1.2.1]):

(4.24.1) $\Delta = (x\sigma_q - \lambda_1)h_1(x\sigma_q - \lambda_2)h_2 ... (x\sigma_q - \lambda_n)h_n, \quad \lambda_j \in \mathbb{C},\ h_j \in \mathbb{C}\{x\},\ h_j(0) = 1\,.$

We suppose that we have chosen n minimal and let us prove the statement by induction on n. We consider first of all the case $n = 1$: we suppose that $(x\sigma_q - \lambda_1)h_1\hat{f} = g$, with $B_q(g) \in H$[7]. This implies that $B_q(h_1\hat{f}) \in H$, since $B_q((x\sigma_q - \lambda_1)h_1\hat{f}) = (q\xi - \lambda_1)B_q(h_1\hat{f})$. Therefore

$$B_q(xh_1(qx)\hat{f}(qx)) = B_q(g) - B_q(\lambda_1 h_1 \hat{f}) \in H\,,$$

with $B_q(xh_1(qx)) \in \mathbb{E}_q$ and $xh_1(qx)\hat{f}(qx) \in x^2\mathbb{C}[[x]]$. So $x\hat{f}(qx) = \widetilde{g}h_1(qx)^{-1}$ and $B_q(x\hat{f}(qx)) = B_q(\widetilde{g}/x) *_q B_q(xh_1(qx)^{-1}) \in H$. Finally $B_q(\hat{f}) \in H$.

For $n > 1$, the inductive hypothesis implies that $B_q((x\sigma_q - \lambda_n)h_n\hat{f}) \in H$, and hence that $B_q(\hat{f}) \in H$. □

[7]Notice that we are assuming that $B_q(g) \in H$ and not $B_q(g) \in \mathbb{E}_q$.

Proof of Theorem 4.14. Applying Theorem 4.20 to $B_q(\hat{f})$, we can write $\hat{f}$ as follows:

$$\hat{f} = f_0 + \hat{e}_0 + f_1\hat{e}_1 + ... + f_n\hat{e}_n,$$

where $f_0, \ldots, f_n \in x\mathbb{C}\{x\}$, $B_q(\hat{e}_0) \in K$ and, for $i = 1,..., n$, $B_q(\hat{e}_i) = \frac{1}{(\xi-\lambda_i)^{\nu_i}}$. So it is enough to prove Theorem 4.14 for a modified Tschakaloff series (*cf.* [MZ00, Prop. 1.4.2]), *i.e.* under the assumption $B_q(\hat{f}) = \frac{1}{(\xi-\lambda)^n}$. By replacing ξ by $-\lambda\xi$, we can suppose that $\lambda = -1$. The case $n = 1$ corresponds exactly to the Tschakaloff divergent series $\hat{\mathcal{E}}$, and the result is stated in Proposition 4.16. If $n > 1$, by considering $\hat{f}(x) = \frac{(-1)^{n-1}}{(n-1)!x^{n-1}}\partial^{n-1}\ \partial a^{n-1}\hat{\mathcal{E}}(ax)\|_{a=1}$, we can easily deduce the wanted result by the help of the dominated convergence theorem.

Concerning the second part of the statement of Theorem 4.14, the decomposition above allows once again to reduce to the case of the Tschakaloff series. The dominated convergence theorem applies with no difficulties to this explicit case (*cf.* Remark 4.13). □

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