AN ULTRAMETRIC VERSION OF THE MAILLET-MALGRANGE THEOREM FOR NONLINEAR q-DIFFERENCE EQUATIONS

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ABSTRACT. We prove an ultrametric q-difference version of the Maillet-Malgrange theorem, on the Gevrey nature of formal solutions of nonlinear analytic q-difference equations. Since $\deg_q$ and ord_q define two valuations on $\mathbb{C}(q)$, we obtain, in particular, a result on the growth of the degree in q and the order at q of formal solutions of nonlinear q-difference equations, when q is a parameter. We illustrate the main theorem by considering two examples: a q-deformation of "Painlevé II", for the nonlinear situation, and a q-difference equation satisfied by the colored Jones polynomials of the figure 8 knots, in the linear case.

We also consider a q-analog of the Maillet-Malgrange theorem, both in the complex and in the ultrametric setting, under the assumption that $|q| = 1$ and a classical diophantine condition.

INTRODUCTION

In 1903, E. Maillet [Mai03] proved that a formal power series solution of an algebraic differential equation is Gevrey. B. Malgrange [Mal89] generalized and made more precise Maillet's statement in the case of an analytic nonlinear differential equation. Finally C. Zhang [Zha98] proved a q-difference-differential version of the Maillet-Malgrange theorem. In the meantime a Gevrey theory for linear q-difference-differential equations has been largely developed; *cf.* for instance [Ram78], [Béz92b], [NM93], [FJ95].

In this paper we prove an analogue of the Maillet-Malgrange theorem for ultrametric nonlinear analytic q-difference equations, under the assumption $|q| \neq 1$. It generalizes to nonlinear q-difference equations a theorem of Bézivin and Boutabaa; *cf.* [BB92]. The proof follows [Mal89].

The same technique allows to prove a Maillet-Malgrange theorem for q-difference equations when $|q| = 1$, both in the complex and in the ultrametric setting, under a classical diophantine hypothesis: this result generalizes the main result of [Béz92a] and answers a question asked therein. Notice that the problem of nonlinear differential equation in the ultrametric setting is treated in [SSa],[SS81],[SSb], where a p-adic avatar of diophantine conditions on the exponents is also assumed.

One of the reasons that makes the ultrametric statement interesting is the possible application to the case when q is a parameter (*cf.* §2 below). For instance, when q is a parameter, Corollary 5 (*cf.* below) becomes:

Theorem 1. *Suppose that we are given a nontrivial algebraic nonlinear q-difference equation*

$$F(q, x, y(x), \ldots, y(q^n x)) = 0,$$

i.e. $F(q, x, w_0, \ldots, w_n) \in \mathbb{C}[q, x, w_0, \ldots, w_n]$ nonidentically zero, with a formal solution $y(x) = \sum_{h \geq 0} y_h x^h \in \mathbb{C}(q)[[x]]$. Then there exist nonnegative numbers s, s' such that

$$\limsup_{h \to \infty} \frac{1}{h} \left(\deg_q y_h - s \frac{h(h-1)}{2} \right) < +\infty$$

and

$$\limsup_{h \to \infty} \frac{1}{h} \left(\mathrm{ord}_q y_h - s' \frac{h(h-1)}{2} \right) > -\infty.$$

We could give a more precise statement in which $1/s$ and $-1/s'$ (with the convention $1/0 = +\infty$) are slopes of the Newton polygon of the linearized q-difference operator of $F(q, x, y(x), \ldots, y(q^n x)) = 0$ along $y(x)$ (*cf.* Theorem 6).

In classical literature on special functions, q is frequently a parameter. Basic hypergeometric equations are the most classical example in the linear case, while the q-analogue of Painlevé equations are nonlinear examples, that has been largely studied in the last years.[1] This ultrametric "*q-adic*" approach to the study of a family of functional equations depending on a parameter is peculiar to q-difference equations.

Acknowledgement. I would like to thank Changgui Zhang, whose questions are at the origin of this paper, and Jean-Paul Bézivin for his attentive reading of the manuscript and his numerous interesting comments.

1. Ultrametric q-analog of the Maillet-Malgrange theorem for $|q| \neq 1$

Let Ω be a complete ultrametric valued field, equipped with the ultrametric norm $|\ |$, and let $q \in \Omega$ *be an element of norm strictly greater that* 1.[2]

1.1. **Digression on the linear case.** We denote by $\Omega\{x\}$ the ring of germs of analytic functions at 0 with coefficients in Ω, *i.e.* the convergent elements of $\Omega[[x]]$. To the linear q-difference equation

$$\mathcal{L}y(x) = \sum_{i=0}^{n} a_i(x) y(q^i x) = 0,$$

with $a_i(x) \in \Omega\{x\}$, we can attach the Newton polygon

$$(1) \qquad N_q(\mathcal{L}) = \text{ convex envelop} \left(\bigcup_{i=0}^{n} \{(i, h) : h \geq \mathrm{ord}_{x=0} a_i(x)\} \right).$$

Of course the polygon $N_q(\mathcal{L})$ has a finite number of finite sides, with *rational* slopes, plus two infinite vertical sides. We adopt the convention that the right vertical side has slope $+\infty$ and the left one has slope $-\infty$.

Bézivin and Boutabaa have proved the following result:

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[1] For some examples of formal solutions of Painlevé equations *cf.* for instance [RGTT01].

[2] We could have chosen the opposite convention $|q| < 1$, which leads to analogous statements.

Theorem 2. ([BB92]) *Let $g(x) \in \Omega\{x\}$ and $y(x) = \sum_{h \geq 0} y_h x^h \in \Omega[[x]]$ be such that $\mathcal{L}y(x) = g(x)$. Then either $y(x) \in \Omega\{x\}$ or there exists a positive slope $r \in]0, +\infty[$ of $N_q(\mathcal{L})$ such that*

$$\sum_{h \geq 0} \frac{y_h}{q^{\frac{h(h-1)}{2r}}} x^h \,,$$

is a convergent nonentire series.

1.2. **Statement of the main result.** Consider an analytic function at 0 of $n+2$ variable, *i.e.* a power series

$$F(x, w_0, w_1, \ldots, w_n) = \sum_{k, k_0, \ldots, k_n > 0} A_{k, k_0, \ldots, k_n} x^k w_0^{k_0} \cdots w_n^{k_n} \in \Omega[[x, w_0, w_1, \ldots, w_n]] \,,$$

such that

$$\limsup_{k + \sum_{i=0}^{n} k_i \to \infty} |A_{k, k_0, \ldots, k_n}|^{\frac{1}{k + \sum_{i=0}^{n} k_i}} < +\infty \,.$$

Remark that we have assumed, with no loss of generality, that $F(0, \ldots, 0) = 0$. We are interested in studying formal solutions of the *nonlinear analytic q-difference equation*

(2) $$F(x, \varphi(x), \varphi(qx), \ldots, \varphi(q^n x)) = 0 \,.$$

To simplify notation for any $\varphi \in \Omega[[x]]$ we set $\Phi = (\varphi(x), \varphi(qx), \ldots, \varphi(q^n x))$, and we denote by σ_q the usual q-difference operator acting on $\Omega[[x]]$:

$$\begin{array}{rcl} \sigma_q : & \Omega[[x]] & \longrightarrow & \Omega[[x]] \,, \\ & \varphi(x) & \longmapsto & \varphi(qx) \,. \end{array}$$

For any formal power series $\varphi(x) \in \Omega[[x]]$, such that $\varphi(0) = 0$, let $\mathcal{F}_\varphi$ be the *linearized q-difference operator of F along φ*:

$$\mathcal{F}_\varphi = \sum_{i=0}^{n} \frac{\partial F}{\partial w_i}(x, \Phi) \sigma_q^i \,.$$

The operator $\mathcal{F}_\varphi$ being linear, we can define its *Newton polygon $N_q(\mathcal{F}_\varphi)$* in the usual way (*cf.* equation 1). We want to prove that, for a solution $\varphi(x)$ of (2), the positive slopes of $N_q(\mathcal{F}_\varphi)$ are linked to the q-Gevrey order of $\varphi(x)$:

Definition 3. A formal power series $\varphi(x) = \sum_{h \geq 0} \varphi_h x^h \in \Omega[[x]]$ is a *q-Gevrey series (of order $s \in \mathbb{R}$)* if the series

$$\sum_{h \geq 0} \frac{\varphi_h}{q^{s\frac{h(h-1)}{2}}} x^h$$

is convergent.

We can state our main result:

Theorem 4. *Let $\varphi(x) \in x\Omega[[x]]$ be a formal solution of the equation (2) and let $r \in]0, +\infty]$ be the smallest positive slope of the Newton polygon of $\mathcal{F}_\varphi$. If $\frac{\partial F}{\partial w_n}(x, \Phi) \neq 0$, then $\varphi(x)$ is a q-Gevrey series of order $1/r$.*[3]

As a consequence we obtain:

Corollary 5. *Let $\varphi(x) \in x\Omega[[x]]$ be a formal solution of equation (2). If $F(x, w_0, w_1, \ldots, w_n)$ is not identically zero, then $\varphi(x)$ is a q-Gevrey series (of some nonspecified order).*

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[3]We have implicitly set $1/+\infty = 0$.

1.3. **When q is a parameter...** Suppose that $F(q,x,w_0,\dots,w_n) \in \mathbb{C}[q,x,w_0,\dots,w_n]$, where q is a parameter, and that we have a formal solution $\varphi(x) = \sum_{h\geq 0} y_h x^h \in \mathbb{C}(q)[[x]]$.[4] Up to equivalence, there are exactly two ultrametric norm over $\mathbb{C}(q)$ such that q has norm different than 1. For any $f(q) \in \mathbb{C}[q]$ they are defined by

(1) $|f(q)|_{q^{-1}} = d^{-\deg_q f(x)}$;
(2) $|f(q)|_q = d^{\operatorname{ord}_q f(q)}$;

where $d \in]0,1[$ is a fixed real number. Of course, $|\ |_q$ and $|\ |_{q^{-1}}$ extends to $\mathbb{C}(q)$ by multiplicativity. Notice that $|q|_q = d < 1$ and $|q|_{q^{-1}} = d^{-1} > 1$.

Taking Ω to be the completion of $\mathbb{C}(q)$ with respect to $|\ |_q$ (resp. $|\ |_{q^{-1}}$), we immediately see that Theorem 1 is a particular case of Corollary 5 and that Theorem 4 becomes:

Theorem 6. *Let*
$$\frac{\partial F}{\partial w_n}(q,x,y(x),\dots,y(q^n x)) \neq 0\,.$$

If $r \in]0,+\infty]$ (resp. $r' \in [-\infty,0[$) is the smallest positive slope (resp. the largest negative slope) of $\mathcal{F}_\varphi$, then
$$\limsup_{h\to\infty} \frac{1}{h}\left(\deg_q y_h - s\frac{h(h-1)}{2}\right) < +\infty\,, \ \text{with } s = 1/r,$$

and
$$\limsup_{h\to\infty} \frac{1}{h}\left(\operatorname{ord}_q y_h - s'\frac{h(h-1)}{2}\right) > -\infty\,, \ \text{with } s' = -1/r'.$$

2. Examples

2.1. **Colored Jones polynomial of figure 8 knot.** We consider the q-difference equation satisfied by the generating function of the sequence of invariants of the figure 8 knot called the colored Jones polynomials (*cf.* [Gar04, §3]):
$$J(q,n) = \sum_{k=0}^{n} q^{nk}(q^{-n-1};q^{-1})_k(q^{-n+1};q)_k \in \mathbb{Z}[q,q^{-1}]\,, \ \forall n \in \mathbb{N}\,.$$

The series $\mathcal{J}(x) = \sum_{n\geq 0} J(q,n)x^n \in \mathbb{C}(q)[[x]]$ satisfies the linear q-difference equation

$$\left[q\sigma_q(q^2+\sigma_q)(q^5-\sigma_q^2)(1-\sigma_q^2)\right]y(x)-$$

$$x\Big[\sigma_q^{-1}(1+\sigma_q)\Big(q^4+\sigma_q\left(q^3-2q^4\right)+\sigma_q^2\left(-q^3+q^4-q^5\right)$$

$$+\sigma_q^3\left(-2q^4+q^5\right)+\sigma_q^4 q^4\Big)(q^5-q^2\sigma_q^2)(1-\sigma_q)\Big]y(x)+$$

$$x^2\Big[q^5(1-\sigma_q)(1+\sigma_q)(1-q^3\sigma_q^2)\left(q^8+\sigma_q(q^9-2q^8)-\sigma_q^2(-q^7+q^8-q^9)+q^7\sigma_q^3+q^8\sigma_q^4\right)\Big]y(x)-$$

$$x^3\left[q^{10}\sigma_q(1-\sigma_q)(1+q^2\sigma_q)(1-q^5\sigma_q^2)\right]y(x) = 0\,.$$

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[4]The results that follows are actually true when we replace $\mathbb{C}$ by any field.

The finite slopes of the Newton polygon are: $-1/2$, 0, $1/2$. It is clear looking at the leading term of $J(q,n)$ that $\mathcal{J}(x)$ cannot be a converging series for the norms $|\ |_q$ and $|\ |_{q^{-1}}$. Therefore it follows from Bézivin and Boutabaa theorem that

$$\limsup_{n \to 0} \frac{1}{n}\left(\deg_q J(q,n) - 2\frac{n(n-1)}{2}\right) < +\infty$$

and

$$\limsup_{n \to 0} \frac{1}{n}\left(\operatorname{ord}_q J(q,n) + 2\frac{n(n-1)}{2}\right) > -\infty\,.$$

Notice that modulo the AJ conjecture (*cf.* [Gar04, §1.4]), those slopes are the same as the ones defined in [CCG$^+$94].

2.2. A q-deformation of the second Painlevé equation.

Let us consider the nonlinear q-difference equation associated to the analytic funtion at $(0,1,1,1)$:[5]

$$F(x, w_{-1}, w_0, w_1) = (w_0 + x)(w_0 w_1 - 1)(w_0 w_{-1} - 1) - qx^2 w_0\,,$$

namely

(3) $$(y(x) + x)(y(x)y(qx) - 1)(y(x)y(q^{-1}x) - 1) - qx^2 y(x) = 0\,.$$

It is a q-deformation of P_{II}. Let $\varphi(x) \in \mathbb{C}(q)[[x]]$, with $\varphi(0) = 1$, be a formal solution of equation (3). Then

$$\begin{aligned}
\mathcal{F}_\varphi &= \sum_{i=-1}^{1} \frac{\partial F}{\partial w_i}(x, \varphi(q^{-1}x), \varphi(x), \varphi(qx))\sigma_q^i \\[2mm]
&= \Big[\big(\varphi(x) + x\big)\big(\varphi(x)\varphi(qx) - 1\big)\varphi(x)\Big]\sigma_q^{-1} \\[2mm]
&+ \Big[\big(\varphi(x)\varphi(qx) - 1\big)\big(\varphi(x)\varphi(q^{-1}x) - 1\big) + \big(\varphi(x) + x\big)\varphi(qx)\big(\varphi(x)\varphi(q^{-1}x) - 1\big) \\[2mm]
&\quad + \big(\varphi(x) + x\big)\big(\varphi(x)\varphi(qx) - 1\big)\varphi(q^{-1}x) - qx^2\Big]\sigma_q^0 \\[2mm]
&+ \Big[(\varphi(x) + x)\varphi(x)(\varphi(x)\varphi(q^{-1}x) - 1)\Big]\sigma_q
\end{aligned}$$

A formal solution of equation (3) is give by

$$\varphi(x) = \frac{{}_1\Phi_1(0; -q; q, -q^2 x)}{{}_1\Phi_1(0; -q; q, -qx)} = 1 + \frac{q}{1+q}x + \cdots,$$

where ${}_1\Phi_1(0; -q; q, x)$ is a basic hypergeometric series:

$${}_1\Phi_1(0; -q; q, -qx) = \sum_{h \geq 0} \frac{q^{h(h-1)}}{(-q;q)_h(q;q)_h} x^h\,,$$

and

$$(a;q)_h = (1-a)(1-aq)\ldots(1-aq^{h-1})\,.$$

A direct and straightforward calculation shows that the Newton polygon of $\mathcal{F}_\varphi$ is *regular singular*, meaning that it has only one finite horizontal slope of length 2,

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[5]This example is studied in [KMN$^+$05, §3.5] and [RGTT01, Eq.(2.55)], where many other examples can be found.

plus the two vertical sides. Therefore Theorem 4 implies that the solution $\varphi(x) = 1 + \sum_{h \geq 1} \varphi_h x^h$ considered above verifies:

$$\limsup_{h \to \infty} \frac{1}{h} \deg_q \varphi_h < +\infty \text{ and } \limsup_{h \to \infty} \frac{1}{h} \mathrm{ord}_q \varphi_h > -\infty \,.$$

In other words, the solution $\varphi(x) \in \mathbb{C}(q)\,[[x]]$ is convergent for both the norm $|\ |_q$ and the norm $|\ |_{q^{-1}}$.

We could have also remarked that $_1\Phi_1(0; -q; q, x)$ is a solution of the linear equation

$$\sigma_q^{-2}(\sigma_q - 1)(\sigma_q + 1)y(x) + q^2 x y(x) = 0 \,,$$

whose Newton polygon has only a horizontal finite slope. This means that $_1\Phi_1(0; -q; q, x)$ is convergent for both $|\ |_q$ and $|\ |_{q^{-1}}$, and hence that $\varphi(x)$ is also convergent.

3. Proofs

3.1. **Proof of Theorem 4.** The proof follows [Mal89]. It relies on the ultrametric implicit function theorem; *cf.* [A'C69], [Ser06], [SS81].

We set $\varphi(x) = \sum_{h \geq 1} \varphi_h x^h$. For any $k \in \mathbb{N}$, let

1. $\varphi_k(x) = \sum_{h=0}^k \varphi_h x^h$;
2. $\psi(x)$ be a formal power series such that $\varphi(x) = \varphi_k(x) + x^k \psi(x)$;
3. $\Psi(x) = (\psi(x), \psi(qx), \dots, \psi(q^n x))$ and $\Phi_k(x) = (\varphi_k(x), \varphi_k(qx), \dots, \varphi_k(q^n x))$.

Let $W = (w_0, \dots, w_n)$, $Z = (z_0, \dots, z_n)$. By taking the Taylor expansion of $F(x, W + Z)$ at W we obtain:

$$F(x, W + Z) = F(x, W) + \sum_{i=0}^n \frac{\partial F}{\partial w_i}(x, W) z_i + \sum_{i,j=0}^n H_{i,j}(x, W, Z) z_i z_j \,,$$

where $H(x, W, Z)$ is an analytic function of $2n + 3$ variables in a neighborhood of zero. Hence we can write:

$$
\begin{aligned}
0 = F(x, \Phi) \quad = \quad & F(x, \Phi_k(x)) + x^k \sum_{i=0}^n \frac{\partial F}{\partial w_i}(x, \Phi_k) q^{ik} \sigma_q^i \psi \\
& + x^{2k} \sum_{i,j=0}^n H_{i,j}(x, \Phi_k(x), x^k \Psi(x)) q^{(i+j)k} \sigma_q^i \psi \sigma_q^j \psi \,.
\end{aligned}
$$
(4)

To finish the proof we have to distinguish two cases: $r < +\infty$ and $r = +\infty$.

Case 1. $r < +\infty$. We are going to choose $k \geq \sup(k_1, k_2 + l + 1)$, where k_1, k_2, l are constructed as follows (*cf.* figure below). First of all let $(n', l) \in \mathbb{N}^2$ be the point of $N_q(\mathcal{F}_\varphi)$ which verifies the two properties:

1. l is the smallest real number such that $(j, l) \in N_q(\mathcal{F}_\varphi)$ for some $j \in \mathbb{R}$;
2. n' is the greatest real number such that $(n', l) \in N_q(\mathcal{F}_\varphi)$.

Let us consider the polynomial

$$\mathcal{L}(T) = \sum_{i=0}^{n'} \left[\frac{1}{x^l} \frac{\partial F}{\partial w_i}(x, \Phi) \right]_{x=0} T^i \,.$$

We chose k_1 to be a positive integer such that for any $k \geq k_1$, the polynomial $\mathcal{L}(T)$ does not vanishes at q^k, and $k_2 \geq r(n - n')$. Notice that for any $k \geq k_2 + l$, the

smallest positive slope of $N_q(\mathcal{F}_{\varphi_k})$ is equal to r and the point (n', l) is the "lowest" point of $N_q(\mathcal{F}_{\varphi_k})$ with greater abscissae.
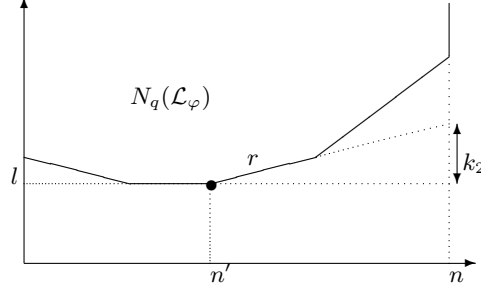
Remark that for any $k \geq \sup(k_1, k_2 + l + 1)$ we have

$$\mathrm{ord}_{x=0} \sum_{i=0}^{n} \frac{\partial F}{\partial w_i}(x, \Phi_k) q^{ik} \sigma_q^i \psi \geq \mathrm{ord}_{x=0} \psi(x) + \inf_{i=0,\ldots,n} \mathrm{ord}_{x=0} \frac{\partial F}{\partial w_i}(x, \Phi_k) \geq l + 1\,.$$

Therefore we can write the linear part of equation (4) in the form

$$\frac{1}{x^l} \sum_{i=0}^{n} \frac{\partial F}{\partial w_i}(x, \Phi_k) q^{ik} \sigma_q^i \psi = \mathcal{L}(q^k \sigma_q) \psi + x \widetilde{\mathcal{L}}(x, \sigma_q) \psi\,,$$

where $\widetilde{\mathcal{L}}(x, \sigma_q)$ is an analytic functional. Moreover we deduce from equation (4) that

$$\mathrm{ord}_{x=0} F(x, \Phi_k) \geq k + l + 1\,,$$

so that there exists an analytic function $M(x, w_0, \ldots, w_n)$ such that equation (4) divided by x^{l+k} becomes

$$(5) \qquad \mathcal{L}(q^k \sigma_q) \psi + x \widetilde{\mathcal{L}}(x, \sigma_q) \psi + x M(x, x^k \Psi) = 0\,.$$

Since $\mathcal{L}(q^k \sigma_q)$ is a linear operator with constant coefficients and $\mathcal{L}(q^h) \neq 0$ for any $h \geq k$, equation (5) admits one unique formal solution $\psi(x) \in x\Omega\,[[x]]$, whose coefficients can be constructed recursively.

In order to conclude, we have to estimate the Gevrey order of $\psi(x)$. Let us consider the following Banach Ω-vector space:

$$\mathcal{H}_{s,m} = \left\{ \sum_{h \geq 1} \varphi_h x^h \in \Omega\,[[x]] : \sup_{h \geq 1} |\varphi_h||q|^{hm - s\frac{h(h-1)}{2}} < +\infty \right\}$$

equipped with the norm

$$\left\| \sum_{h \geq 1} \varphi_h x^h \right\|_{s,m} = \sup_{h \geq 1} |\varphi_h||q|^{hm - s\frac{h(h-1)}{2}}\,.$$

Since for any positive rational number s and any pair of positive integers k, h we have

$$|q|^{s\frac{k(k-1)}{2}} |q|^{s\frac{k(k-1)}{2}} \leq |q|^{s\frac{(k+h)(k+h-1)}{2}}\,,$$

the analytic functional

$$A(\lambda, \psi) = \mathcal{L}(q^k \sigma_q) \psi + \lambda x \widetilde{\mathcal{L}}(\lambda x, \sigma_q) \psi + \lambda x M(\lambda x, \lambda^k x^k \Psi)$$

is defined over $\Omega \times \mathcal{H}_{s,n'}$:

$$A(\lambda, \psi) : \Omega \times \mathcal{H}_{s,n'} \longrightarrow \mathcal{H}_{s,0},$$

and verifies

$$A(0,0) = 0 \text{ and } \frac{\partial A}{\partial \psi}(0,0) = \mathcal{L}(q^k \sigma_q).$$

Since $\mathcal{L}(q^k \sigma_q)$ is invertible, the implicit function theorem implies that for any λ in a neighborhood of 0 there exists ψ_λ such that $A(\lambda, \psi_\lambda) = 0$. The formal solution ψ of equation (5) being unique, we must have $\psi_\lambda(x) = \psi(\lambda x)$ for any λ closed to 0, which ends the proof.

Case 2. $r = +\infty$. We chose the point (n', l) as in the previous case: since there are no finite positive slopes, we have $n' = n$. We can define the polynomial $\mathcal{L}(T)$ in the same way as before. So we choose $k_1 \in \mathbb{N}$ such that $\mathcal{L}(q^k) \neq 0$ for any $k \geq k_1$ and $k_2 \in \mathbb{N}$ such that

$$\inf_{i=0,\dots,n} \operatorname{ord}_{x=0} \left(\frac{\partial F}{\partial w_i}(x, \Phi_k) \right) > l$$

for any $k \geq k_2$. We deduce that $\operatorname{ord}_{x=0} F(x, \Phi_k) \geq k + l + 1$ and hence we are reduced, by dividing equation (4) by x^{l+k}, to consider the functional

$$\mathcal{L}(q^k \sigma_q) + \lambda x M(\lambda x, \lambda^k x^k \Psi) = 0.$$

The same argument as above also allows us to conclude the proof in this case.

3.2. **Proof of Corollary 5.**

Following [Mal89], we are going to show by induction on n that Theorem 4 implies Corollary 5. Notice that for $n = 0$ we are in the classical case of Puiseux development of a solution of an algebraic equation (*cf.* [Mal89]). So let us suppose $n \geq 1$.

If there exists a positive integer k such that

$$(6) \qquad\qquad \frac{\partial^k F}{\partial w_n^k}(x, \Phi) \neq 0,$$

we conclude by applying Theorem 4 to the q-difference equation

$$\frac{\partial^{\kappa-1} F}{\partial w_n^{\kappa-1}}(x, \Phi) = 0,$$

where κ is the smallest positive integer verifying equation (6).

We now suppose that for any positive integer k we have $\frac{\partial^k F}{\partial w_n^k}(x, \Phi) = 0$. By taking the Taylor expansion of $F(x, w_0, \dots, w_n)$, we can verify that $F(x, \varphi(x), \dots, \varphi(q^{n-1}x), \psi(x)) \equiv 0$ for any $\psi(x) \in x\Omega[[x]]$. In particular, there exists $\lambda \in \Omega$ such that $F(x, w_0, \dots, w_{n-1}, \lambda x)$ is not identically zero and $F(x, \varphi(x), \dots, \varphi(q^{n-1}x), \lambda x) = 0$. So we are reduced to the case "$n - 1$".

4. Complex q-analog of the Maillet-Malgrange theorem for $|q| = 1$

Let Ω be either the ultrametric field defined in §1 or the complex field $\mathbb{C}$. We choose $q \in \Omega$ such that $|q| = 1$ and q is not a root of unity.

To the linear q-difference equation

$$\mathcal{L}y(x) = \sum_{i=0}^{n} a_i(x) y(q^i x) = 0,$$

with $a_i(x) = a_{i,j_i} x^{j_i} + a_{i,j_i+1} x^{j_i+1} + \cdots \in \Omega\{x\}$, we can attach a polynomial

$$Q_{\mathcal{L}}(T) = (T-1) \sum_{i=0}^{n} a_{i,j_i} T^i \,.$$

We recall the result:

Theorem 7. *(cf. [Béz92b, Thm. 6.1] and [BB92, Thm. 6.1]) Let $\varphi(x) \in \Omega\,[[x]]$ be a formal solution of $\mathcal{L}y(x) = 0$. We suppose that*

($\mathcal{H}$) *There exist two constants $c_1, c_2 > 0$, such that for any root u of $Q_{\mathcal{L}}(T)$ and any $n >> 0$ the following inequality is satisfied: $|q^n - u| \geq c_1 n^{-c_1}$.*

Then $\varphi(x)$ is convergent.

In the nonlinear case we have the following result that generalizes [Béz92a, §1]:

Theorem 8. *Let $\varphi(x) \in x\Omega\,[[x]]$ be a formal solution of the q-difference equation*

(7) $$F(x, \varphi(x), \varphi(qx), \ldots, \varphi(q^n x)) = 0 \,,$$

analytic at zero. We make the following assumptions:

(1) $\frac{\partial F}{\partial w_n}(x, \Phi) \neq 0$,

(2) *the polynomial $Q_{\mathcal{F}_\varphi}$ associated to the linear operator $\mathcal{F}_\varphi$ verifies the hypothesis ($\mathcal{H}$).*

Then $\varphi(x)$ is convergent.

Remark 9. Notice that the second hypothesis is always verified in the following cases:

- if $\Omega = \mathbb{C}$ and q and the coefficients of Q are algebraic numbers (*cf.* [Béz92a, 2.2]),
- if Ω is an extension of a number field K equipped with a p-adic valuation, and q and the roots of $Q(T)$ are in K (in this case it is a consequence of Baker's theorem; *cf.* for instance [DV02, §8.3])

Proof of Theorem 8. The first part of the proof of Theorem 4 is completely formal. So once again we are reduced to consider equation (5)

$$\mathcal{L}(q^k \sigma_q)\psi + x\widetilde{\mathcal{L}}(x, \sigma_q)\psi + xM(x, x^k \Psi) = 0 \,.$$

The key-point is the choice of $k >> 1$, so that the Newton polygon of the q-difference operator $\mathcal{L}(q^k \sigma_q) + x\widetilde{\mathcal{L}}(x, \sigma_q)$ coincides with the Newton polygon of $\mathcal{F}_\varphi$, up to a vertical shift.

Let $\mathcal{H}(0, r)$ be the Banach algebra of analytic functions converging over the closed disk $D(0, r^+)$ of center 0 and radius $r > 0$, for r small enough, equipped with the norm

$$\left| \sum_{n \geq 0} a_n X^n \right|_{\mathcal{H}(0,r)} = \sup_{n \geq 0} |a_n| r^n \,.$$

It follows from [Béz92b, Thm. 6.1] and [BB92, Thm. 6.1][6] that the operator $\mathcal{L}(q^k \sigma_q) + x\widetilde{\mathcal{L}}(\lambda x, \sigma_q)$ acts on $\Omega \times \mathcal{H}(0, r)$ and hence

$$A(\lambda, \psi) : \Omega \times \mathcal{H}(0, r) \longrightarrow \mathcal{H}(0, r) \,.$$

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[6]Notice that [BB92, Thm. 6.1] is formulated only for q-difference equations with polynomial coefficients, but the same proof as [Béz92b, Thm. 6.1] works in the analytic case.

The implicit function theorem also allows us to conclude this case.   □

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