A NEW SPECTRAL THEORY

A complex analytic intrinsic theory with applications

to

The Connes-Moscovici spectrum which matches the zeros of zeta

and to

Linear perturbations of black-holes

Jean-Pierre Ramis

Académie des Sciences (Institut de France)
and
Institut de Mathématiques de Toulouse

Joint works with Anne Duval, Michèle Loday-Richaud, Emmanuel Paul, Françoise Richard-Jung and Jean Thomann

Séminaire Différentiel, Versailles, 25 novembre 2025

PRESENTATION

A new paradigm

I propose a new paradigm for the spectral theory of ODEs with *analytic* coefficients (in the complex domain).

The classical approach is for real operators and is based on *Hilbert spaces and linear operators* (non necessarily bounded). To the data of the operator one adds *external data*: (two points and) two boundary conditions.

A new paradigm

I propose a new paradigm for the spectral theory of ODEs with *analytic* coefficients (in the complex domain).

The classical approach is for real operators and is based on *Hilbert spaces and linear operators* (non necessarily bounded). To the data of the operator one adds *external data*: (two points and) two boundary conditions.

My approach is based on *analytic continuation* in the complex domain, and *k-summability* of divergent series, otherwise speaking on *wild monodromy*.

A new paradigm

I propose a new paradigm for the spectral theory of ODEs with *analytic* coefficients (in the complex domain).

The classical approach is for real operators and is based on *Hilbert spaces and linear operators* (non necessarily bounded). To the data of the operator one adds *external data*: (two points and) two boundary conditions.

My approach is based on *analytic continuation* in the complex domain, and *k-summability* of divergent series, otherwise speaking on *wild monodromy*.

My analytic spectra are *intrinsic*, the data of the two points and of the boundary conditions are *"contained in the equation"*. I recently discovered that this concept was clearly formulated by Erwin Schrödinger, nearly a century ago, in a letter dated of 27 december, 1925, in relation with the hydrogen spectrum.

The analytic spectra are compatible with some "natural transformations" of ODEs as *s-homotopic transformations* and *gauge transformations*.

The analytic spectra are compatible with some "natural transformations" of ODEs as *s-homotopic transformations* and *gauge transformations*.

There are strong relations with special functions theory and the spectra appearing in natural sciences are very often analytic spectra. There are numerous examples in quantum chemistry and in black holes theory. Our formulation already appeared in some examples, 'between the lines' or explicitely.

The analytic spectra are compatible with some "natural transformations" of ODEs as *s-homotopic transformations* and *gauge transformations*.

There are strong relations with special functions theory and the spectra appearing in natural sciences are very often analytic spectra. There are numerous examples in quantum chemistry and in black holes theory. Our formulation already appeared in some examples, 'between the lines' or explicitely.

The basic idea of the new spectral theory is very simple and natural when one has the good tools. Some are relatively recent.

CLASSICAL SPECTRAL THEORY

Classically a *spectrum* (for a one dimensional problem) is given by *a real linear second order differential equation* and some *external data*:

- two distinct points in $\mathbb{R} \cup \{\pm\infty\}$ defining an interval;
- a boundary condition at each point.

Classically a *spectrum* (for a one dimensional problem) is given by *a real linear second order differential equation* and some *external data*:

- two distinct points in $\mathbb{R} \cup \{\pm\infty\}$ defining an interval;
- a boundary condition at each point.

An elementary example is given by standing waves on a string. When *two ends are fixed*, the spectrum is indexed by $\mathbb{N}^*$.
Model: $y'' + \lambda y = 0$, $y(0) = y(1) = 0$, $\lambda = (n\pi)^2$, $n \in \mathbb{N}^*$.

Classically a *spectrum* (for a one dimensional problem) is given by *a real linear second order differential equation* and some *external data*:

- two distinct points in $\mathbb{R} \cup \{\pm\infty\}$ defining an interval;
- a boundary condition at each point.

An elementary example is given by standing waves on a string. When *two ends are fixed*, the spectrum is indexed by $\mathbb{N}^*$.
Model: $y'' + \lambda y = 0$, $y(0) = y(1) = 0$, $\lambda = (n\pi)^2$, $n \in \mathbb{N}^*$.

In general, there are several possible boundary conditions. An example is to ask that the eigenfunctions are *bounded* on the interval defined by the two points. In this case, I will speak of *naive spectrum*.

Sturm-Liouville theory

I consider operators of the form $D = -\frac{d}{dx}\left(p\frac{d}{dx}\right) + q + \lambda$.
I first consider the case of a segment and for simplicity I suppose that it is $[0,1]$. I suppose that p, q are real and $\mathcal{C}^1$ on $[0,1]$.

Sturm-Liouville theory

I consider operators of the form $D = -\frac{d}{dx}\left(p\frac{d}{dx}\right) + q + \lambda$.
I first consider the case of a segment and for simplicity I suppose that it is $[0, 1]$. I suppose that p, q are real and $\mathcal{C}^1$ on $[0, 1]$.

Regular case. We suppose $p \neq 0$ on $[0, 1]$. We put an homogeneous linear condition on the local solutions at each extremity (vanishing of a linear form on the values of the function and of its derivative). Equivalently *we choose a line of solutions at each extremity*.

Sturm-Liouville theory

I consider operators of the form $D = -\frac{d}{dx}\left(p\frac{d}{dx}\right) + q + \lambda$.
I first consider the case of a segment and for simplicity I suppose that it is $[0,1]$. I suppose that p, q are real and $\mathcal{C}^1$ on $[0,1]$.

Regular case. We suppose $p \neq 0$ on $[0,1]$. We put an homogeneous linear condition on the local solutions at each extremity (vanishing of a linear form on the values of the function and of its derivative). Equivalently *we choose a line of solutions at each extremity*.
With such boundary conditions we have the Lagrange identity

$$\int_0^1 ((Du)v - u(Dv)) = 0.$$

We can interpret D as a *self-adjoint operator on a Hilbert space* and we have the spectral theorem. The eigenvalues are real and non degenerated and form an infinite sequence:

$$\lambda_0 < \lambda_1 < \ldots < \lambda_n < \ldots$$

The eigenfunctions form an orthogonal system.

Sturm-Liouville theory

Singular case

Possible singularities at the endpoints of the interval $]a, b[$, now allowed to be infinite or semi-infinite.
If $p \neq 0$ on $]a, b[$ but vanishes at one extremity, or if $a = -\infty$ or if $b = +\infty$, we are in the singular case and the problem is (a lot) more difficult. This case was studied by Hermann Weyl in his dissertation in 1910 (under the direction of David Hilbert). He introduced the alternative LC (limit circle) vs LP (limit point).

Sturm-Liouville theory

Singular case

Possible singularities at the endpoints of the interval $]a, b[$, now allowed to be infinite or semi-infinite.
If $p \neq 0$ on $]a, b[$ but vanishes at one extremity, or if $a = -\infty$ or if $b = +\infty$, we are in the singular case and the problem is (a lot) more difficult. This case was studied by Hermann Weyl in his dissertation in 1910 (under the direction of David Hilbert). He introduced the alternative LC (limit circle) vs LP (limit point).

Later von Neumann introduced an approach based on *unbounded linear operators* in Hilbert spaces.

Sturm-Liouville theory

Singular case

Possible singularities at the endpoints of the interval $]a, b[$, now allowed to be infinite or semi-infinite.

If $p \neq 0$ on $]a, b[$ but vanishes at one extremity, or if $a = -\infty$ or if $b = +\infty$, we are in the singular case and the problem is (a lot) more difficult. This case was studied by Hermann Weyl in his dissertation in 1910 (under the direction of David Hilbert). He introduced the alternative LC (limit circle) vs LP (limit point).

Later von Neumann introduced an approach based on *unbounded linear operators* in Hilbert spaces.

In applications we are most often in the singular case and the study of the spectrum can be difficult.

ANALYTIC SPECTRA

Intrinsic spectra

Intrinsic spectra

I limit myself to linear second order operators D which are analytic (or meromorphic) on an open subset of the Riemann sphere and mainly to rational operators: $D \in \mathbb{C}(x)[d/dx]$. Even if D is real, I consider the solutions in the complex domain.

In a lot of "interesting" applications in mathematics or in natural sciences, we can do the two following observations on the spectral problems:

Intrinsic spectra

I limit myself to linear second order operators D which are analytic (or meromorphic) on an open subset of the Riemann sphere and mainly to rational operators: $D \in \mathbb{C}(x)[d/dx]$. Even if D is real, I consider the solutions in the complex domain.

In a lot of "interesting" applications in mathematics or in natural sciences, we can do the two following observations on the spectral problems:

1. the two extremities are *singular points* of the operator;

Intrinsic spectra

I limit myself to linear second order operators D which are analytic (or meromorphic) on an open subset of the Riemann sphere and mainly to rational operators: $D \in \mathbb{C}(x)[d/dx]$. Even if D is real, I consider the solutions in the complex domain.

In a lot of "interesting" applications in mathematics or in natural sciences, we can do the two following observations on the spectral problems:

1. the two extremities are *singular points* of the operator;
2. the boundary conditions *can be defined using only the operator*.

Intrinsic spectra

I limit myself to linear second order operators D which are analytic (or meromorphic) on an open subset of the Riemann sphere and mainly to rational operators: $D \in \mathbb{C}(x)[d/dx]$. Even if D is real, I consider the solutions in the complex domain.

In a lot of "interesting" applications in mathematics or in natural sciences, we can do the two following observations on the spectral problems:

1. the two extremities are *singular points* of the operator;
2. the boundary conditions *can be defined using only the operator*.

   2. can be difficult to verify !

Intrinsic spectra

I limit myself to linear second order operators D which are analytic (or meromorphic) on an open subset of the Riemann sphere and mainly to rational operators: $D \in \mathbb{C}(x)[d/dx]$. Even if D is real, I consider the solutions in the complex domain.

In a lot of "interesting" applications in mathematics or in natural sciences, we can do the two following observations on the spectral problems:

1. the two extremities are *singular points* of the operator;
2. the boundary conditions *can be defined using only the operator*.

   2. can be difficult to verify !

I recently discovered that, as he explained in a letter dated december 27, 1925, Erwin Schrödinger did such observations for the radial equation of the hydrogen (relativistic case). The extremities are 0 (*regular singular*) and infinity (*irregular*) and:

... *"with remarkable boundary conditions that the equation 'carries within itself'* [in sich trägt] *and that are not externally predetermined. ..."*

Special solutions and analytic spectra

From special solutions to analytic spectra

Around 2020, Martin Klimes, Emmanuel Paul and J.P. R. (KPR), in a study of the linearized equations of Painlevé VI and Painlevé V equations, introduced a notion of *special solution* attached to the data of two singular points, with marked directions, and a simple continuous paths joining them, a *2-points connection problem*.

From special solutions to analytic spectra

Around 2020, Martin Klimes, Emmanuel Paul and J.P. R. (KPR), in a study of the linearized equations of Painlevé VI and Painlevé V equations, introduced a notion of *special solution* attached to the data of two singular points, with marked directions, and a simple continuous paths joining them, a *2-points connection problem*. Such a solution has the remarkable (characteristic) property that its image by the (wild) *Riemann-Hilbert map* belongs to *a line* of the corresponding (wild) character variety, interpreted an affine cubic surface (possibely singular). This property extends to all the Painlevé equations (E. Paul-J.P. R., unpublished).

From special solutions to analytic spectra

Around 2020, Martin Klimes, Emmanuel Paul and J.P. R. (KPR), in a study of the linearized equations of Painlevé VI and Painlevé V equations, introduced a notion of *special solution* attached to the data of two singular points, with marked directions, and a simple continuous paths joining them, a *2-points connection problem*. Such a solution has the remarkable (characteristic) property that its image by the (wild) *Riemann-Hilbert map* belongs to *a line* of the corresponding (wild) character variety, interpreted an affine cubic surface (possibly singular). This property extends to all the Painlevé equations (E. Paul-J.P. R., unpublished).

Taking into account the two previous observations, I got the (simple and natural) idea to define a new notion of spectrum based on the notion of special solution. I called *analytic* the corresponding spectra. Such spectra are by construction *intrinsic spectra*.

Local special solutions

We define *local special solutions* at a *singular* point a.
We fix a direction $d \in S^1$ at the point a and we consider the two dimensional space $Sol_{(a,d)}D$ of sectorial solutions at a on (germs of) sectors bisected by d. There is also a "similar" space of formal solutions.

Local special solutions

We define *local special solutions* at a *singular* point a.
We fix a direction $d \in S^1$ at the point a and we consider the two dimensional space $Sol_{(a,d)}D$ of sectorial solutions at a on (germs of) sectors bisected by d. There is also a "similar" space of formal solutions.

- At a regular-singular point a *special line of solutions* is an *eigenline of the local monodromy* around a (two, one or all according to the cases).
- At an irregular point a *special line of formal solutions* is an *eigenline of the exponential torus*, that is a line of "purely exponential solutions" (two lines). A *special line of sectorial solutions* corresponds by summation in the direction d (one or two lines).

Local special solutions

We define *local special solutions* at a *singular* point a.
We fix a direction $d \in S^1$ at the point a and we consider the two dimensional space $Sol_{(a,d)}D$ of sectorial solutions at a on (germs of) sectors bisected by d. There is also a "similar" space of formal solutions.

- At a regular-singular point a *special line of solutions* is an *eigenline of the local monodromy* around a (two, one or all according to the cases).
- At an irregular point a *special line of formal solutions* is an *eigenline of the exponential torus*, that is a line of "purely exponential solutions" (two lines). A *special line of sectorial solutions* corresponds by summation in the direction d (one or two lines).

In the irregular case, if the formal monodromy is not scalar, then a distinguished line of solutions is an eigenline of the formal monodromy. An important example of local special solution is a (non trivial) *subdominant solution*.

Pure waves and local special solutions

In some applications, under some analyticity hypothesis, the formal eigenlines of the exponential torus correspond to "formal pure wave solutions" ("purely exponential" solutions). But the notion of 'actual pure wave solution" is more delicate.

Pure waves and local special solutions

In some applications, under some analyticity hypothesis, the formal eigenlines of the exponential torus correspond to "formal pure wave solutions" ("purely exponential" solutions). But the notion of 'actual pure wave solution" is more delicate. A rigorous formulation is based on the possibility of summation of a formal solution. In the oscillating case (as the positive energy case in the Schrödinger equation), the two formal pure waves solutions are summable. It can happen (as in the negative energy case in the Schrödinger equation) that only the subdominant formal pure wave solution is summable.
In the applied litterature the difficulty is generally skipped and it can be a source of confusion. There are remarkable exceptions and some physicists gave a variant of our description.

Pure waves and local special solutions

In some applications, under some analyticity hypothesis, the formal eigenlines of the exponential torus correspond to "formal pure wave solutions" ("purely exponential" solutions). But the notion of 'actual pure wave solution" is more delicate. A rigorous formulation is based on the possibility of summation of a formal solution. In the oscillating case (as the positive energy case in the Schrödinger equation), the two formal pure waves solutions are summable. It can happen (as in the negative energy case in the Schrödinger equation) that only the subdominant formal pure wave solution is summable.
In the applied litterature the difficulty is generally skipped and it can be a source of confusion. There are remarkable exceptions and some physicists gave a variant of our description.

When they are defined the pure waves are local special solutions in our sense.

Special solutions

Let $U \subset P^1(\mathbb{C})$ be an open subset. We denote by $\mathcal{M}(U)$ its field of meromorphic functions. Let $D \in \mathcal{M}(U)[d/dx]$. A *special solution* f is the data of:

- two *marked singular points* $(x_1, d_1), (x_2, d_2) \in U \times S^1$;
- a continuous simple path γ joining them "in" the regular subset of U (its image can be a circular arc or not);
- two *local special* sectorial solutions respectively at (x_1, d_1) and (x_2, d_2);
- a solution f *connecting them along* γ.

Special solutions

Let $U \subset P^1(\mathbb{C})$ be an open subset. We denote by $\mathcal{M}(U)$ its field of meromorphic functions. Let $D \in \mathcal{M}(U)[d/dx]$. A *special solution* f is the data of:

- two *marked singular points* $(x_1, d_1), (x_2, d_2) \in U \times S^1$;
- a continuous simple path γ joining them "in" the regular subset of U (its image can be a circular arc or not);
- two *local special* sectorial solutions respectively at (x_1, d_1) and (x_2, d_2);
- a solution f *connecting them along* γ.

Special solutions appear in the theory of special functions, as:

- classical orthogonal polynomials (Jacobi, generalised Laguerre, Hermite); $Qy'' + Ly' - \lambda y = 0$, where L is linear and Q is quadratic;
- Heun functions, confluent Heun functions, in particular spheroidal functions.

Special solutions

In 1975 in his "redbook" Yasutaka Sibuya studied the *Schrödinger equations with a polynomial potential*. He considered some solutions defined by an entire function connecting two subdominant solutions in two non consecutive stokes sectors. Such solutions are special solutions. it is a generalisation of the Hermite case (quadratic potential).

Special solutions

In 1975 in his "redbook" Yasutaka Sibuya studied the *Schrödinger equations with a polynomial potential*. He considered some solutions defined by an entire function connecting two subdominant solutions in two non consecutive stokes sectors. Such solutions are special solutions. it is a generalisation of the Hermite case (quadratic potential).

Special solutions solutions appear also in physical sciences.

- Quantum chemistry: angular and radial equations of the hydrogen atom (in several systems of coordinates), hydrogen molecular ion H_2^+ (an electon orbiting around two protons).
- Schrödinger equation with a *reflectionless potential*;
- Linearized perturbations of black holes: Quasi Normal Modes, algebraically special solutions, Total Transmission Modes.

Three-points connections and 3-special solution

In a 2-point connection, we can replace the path by a connected and simply connected open subset U of the Riemann sphere whose boundary U is homeomorphic to S^1 and contains the two points.

Three-points connections and 3-special solution

In a 2-point connection, we can replace the path by a connected and simply connected open subset U of the Riemann sphere whose boundary U is homeomorphic to S^1 and contains the two points.

We can define similarly a 3-points connection and extends the notion of special solution to this case, defining 3-special solutions. In the case of Heun and confluent Heun equation, a solution is a 3-special solution if and only if it is a generalised polynomial. It is a new result in the confluent case.

Analytic spectrum

Let $D \in \mathcal{M}(U)[d/dx]$. We consider the spectral problem $Dy = \mu y$, $\mu \in \mathbb{C}$. The operator D can depend algebraically of some "natural parameters" as:

- exponents, coefficients of exponential exponents, position of singularities, auxiliary parameters.

Analytic spectrum

Let $D \in \mathcal{M}(U)[d/dx]$. We consider the spectral problem $Dy = \mu y$, $\mu \in \mathbb{C}$. The operator D can depend algebraically of some "natural parameters" as:

- exponents, coefficients of exponential exponents, position of singularities, auxiliary parameters.

We define an *analytic spectrum* from the following data:

- two *marked singular points* $(x_1, d_1), (x_2, d_2) \in U \times S^1$;
- a continuous simple path γ joining (x_1, d_1) to (x_2, d_2) "in" U;
- two *special lines* of sectorial solutions respectively at (x_1, d_1) and (x_2, d_2).

By definition, $\mu \in \mathbb{C}$ is an *eigenvalue* if the operator $D - \mu$ admits a *special solution* that we call a corresponding *eigenfunction*.

Analytic spectrum

Let $D \in \mathcal{M}(U)[d/dx]$. We consider the spectral problem $Dy = \mu y$, $\mu \in \mathbb{C}$. The operator D can depend algebraically of some "natural parameters" as:

- exponents, coefficients of exponential exponents, position of singularities, auxiliary parameters.

We define an *analytic spectrum* from the following data:

- two *marked singular points* $(x_1, d_1), (x_2, d_2) \in U \times S^1$;
- a continuous simple path γ joining (x_1, d_1) to (x_2, d_2) "in" U;
- two *special lines* of sectorial solutions respectively at (x_1, d_1) and (x_2, d_2).

By definition, $\mu \in \mathbb{C}$ is an *eigenvalue* if the operator $D - \mu$ admits a *special solution* that we call a corresponding *eigenfunction*.
Heuristic: under some "mild conditions" the analytic spectra "depends analytically" of the "natural parameters".

Generically, at a regular singular point we can select a line of special solutions using an exponent. It is equivalent to our definition.

I skip some difficulties when the monodromy at a regular singular point is scalar and when there is no logarithm in the formal solutions. In this case all the lines of local solutions are special. However it is possible to select a distinguished line of solutions using one of the exponents. (The largest in the case of an apparent singularity.)

General formulation

In several important physical applications, the "good spectral parameter" is not the "naive" one. Moreover $D - \mu$ is not invariant by multiplication by an invertible $h \in \mathbb{C}(x)$ and by several useful transformations as pull-backs. Therefore, we extend our definition of analytic spectrum, replacing $D - \mu$ by $D(\mu)$, where $D(\mu)$ depends rationally (or more generally analytically) on $\mu \in \mathbb{C}$.

General formulation

In several important physical applications, the "good spectral parameter" is not the "naive" one. Moreover $D - \mu$ is not invariant by multiplication by an invertible $h \in \mathbb{C}(x)$ and by several useful transformations as pull-backs. Therefore, we extend our definition of analytic spectrum, replacing $D - \mu$ by $D(\mu)$, where $D(\mu)$ depends rationally (or more generally analytically) on $\mu \in \mathbb{C}$.

In the case of $D - \mu$, the exponents does not depends on μ. The general formulation allows exponents depending on μ. Notice that $D - \mu$ can be the pull-back by an analytic map of an operator $D_1(\mu)$ such that the exponents of $D_1(\mu)$ depends on $\mu \in \mathbb{C}$.

General formulation

In several important physical applications, the "good spectral parameter" is not the "naive" one. Moreover $D-\mu$ is not invariant by multiplication by an invertible $h \in \mathbb{C}(x)$ and by several useful transformations as pull-backs. Therefore, we extend our definition of analytic spectrum, replacing $D-\mu$ by $D(\mu)$, where $D(\mu)$ depends rationally (or more generally analytically) on $\mu \in \mathbb{C}$.

In the case of $D-\mu$, the exponents does not depends on μ. The general formulation allows exponents depending on μ. Notice that $D-\mu$ can be the pull-back by an analytic map of an operator $D_1(\mu)$ such that the exponents of $D_1(\mu)$ depends on $\mu \in \mathbb{C}$.

An interesting example is the Schrödinger equation defined by the (Darboux-)Pöschl-Teller-Rosen-Morse potential

$$V(x) = -\frac{\ell(\ell+1)}{\cosh^2 x}.$$

(A reflectionless equation whose solutions are KdV solitons.)

The Schrödinger operator $D-\ \mu = -\frac{d^2}{dx^2} + V - \mu$ is the pull-back by $z = \tanh x$ of an hypergeometric operator (an associated Legendre operator) whose exponents depends on μ.

Invariance properties

s-homotopic transformations

For simplicity I consider only the rational case.

Let $D \in \mathbb{C}(x)[d/dx]$. Let g such that $g'/g = \theta \in \mathbb{C}(x)$. We suppose that the poles of θ on $P^1(\mathbb{C})$ are singular points of D. Then $D \mapsto D^{[g]} = D_\theta = g^{-1}Dg$ is the transform of D by the s-homotopic transformation defined by g (or equivalently by the differential equation of order one $y' - \theta y = 0$).

s-homotopic transformations

For simplicity I consider only the rational case.

Let $D \in \mathbb{C}(x)[d/dx]$. Let g such that $g'/g = \theta \in \mathbb{C}(x)$. We suppose that the poles of θ on $P^1(\mathbb{C})$ are singular points of D. Then $D \mapsto D^{[g]} = D_\theta = g^{-1}Dg$ is the transform of D by the s-homotopic transformation defined by g (or equivalently by the differential equation of order one $y' - \theta y = 0$).

A convenient s-homotopic transformation can be used to shift the exponents or the exponential exponents.

An s-homotopic transformation can be interpreted as a tensor product by an equation of order one:

$$D^{[g]} = (d/dx - \theta) \otimes D.$$

If the function g defines an s-homotopic transformation, then the notion of special solution is invariant by multiplication by g^{-1}, therefore *an analytic spectrum is invariant by a s-homotopic transformation*. It is evidently false (in general) for a classical spectrum.

Gauge transformations of differential equations

For simplicity I consider only the rational case.

One can define a notion of (rational) *gauge transformation* of $D \in \mathbb{C}(x)[d/dx]$ using companion systems and gauge transformation of systems.

Gauge transformations of equations can be interpreted using the Ore ring $\mathcal{D} = \mathbb{C}(x)[d/dx]$: if $D, A \in \mathcal{D}$, $A \neq 0$ and if D and A are right relatively prime, then one defines the *transformed of D by A:*

$$T_A(D) = LLCM(A, D)A^{-1}.$$

We denote (as Ore) $T_A(D) = ADA^{-1}$.

Gauge transformations of differential equations

For simplicity I consider only the rational case.

One can define a notion of (rational) *gauge transformation* of $D \in \mathbb{C}(x)[d/dx]$ using companion systems and gauge transformation of systems.

Gauge transformations of equations can be interpreted using the Ore ring $\mathcal{D} = \mathbb{C}(x)[d/dx]$: if $D, A \in \mathcal{D}$, $A \neq 0$ and if D and A are right relatively prime, then one defines the *transformed of D by A:*

$$T_A(D) = LLCM(A, D)A^{-1}.$$

We denote (as Ore) $T_A(D) = ADA^{-1}$.

For D of order two, gauge equivalences of ODEs are also equivalent to transformations introduced by Darboux in 1882 (adding an invertibility condition). The Darboux transformations in today sense are a subclass.

Gauge transformations of differential equations

For simplicity I consider only the rational case.

One can define a notion of (rational) *gauge transformation* of $D \in \mathbb{C}(x)[d/dx]$ using companion systems and gauge transformation of systems.

Gauge transformations of equations can be interpreted using the Ore ring $\mathcal{D} = \mathbb{C}(x)[d/dx]$: if $D, A \in \mathcal{D}$, $A \neq 0$ and if D and A are right relatively prime, then one defines the *transformed of D by A:*

$$T_A(D) = LLCM(A, D)A^{-1}.$$

We denote (as Ore) $T_A(D) = ADA^{-1}$.

For D of order two, gauge equivalences of ODEs are also equivalent to transformations introduced by Darboux in 1882 (adding an invertibility condition). The Darboux transformations in today sense are a subclass.

An analytic spectra is *"nearly invariant" by a rational gauge transformation*. A finite set of eigenvalues can disappear. If D and A are (right) relatively prime, then $A(D - \lambda I)A^{-1} = ADA^{-1} - \lambda I$ if $D - \lambda I$ and A are right relatively prime.

Spectral determinants

Spectral determinants

A spectral determinant F for a spectrum is an *entire function* of the spectral parameter such that the spectrum is the set of its zeros. We suppose a priori nothing about the multiplicities.

Spectral determinants

A spectral determinant F for a spectrum is an *entire function* of the spectral parameter such that the spectrum is the set of its zeros. We suppose a priori nothing about the multiplicities.

It is possible to get a spectral determinant for any spectrum using an Hadamard product but we cannot in general do that with parameters.

Using analytic continuation and k-sums we can define *"natural spectral determinants"* for an analytic spectrum.

Spectral determinants

A spectral determinant F for a spectrum is an *entire function* of the spectral parameter such that the spectrum is the set of its zeros. We suppose a priori nothing about the multiplicities.
It is possible to get a spectral determinant for any spectrum using an Hadamard product but we cannot in general do that with parameters.
Using analytic continuation and k-sums we can define *"natural spectral determinants"* for an analytic spectrum. Under mild hypothesis such spectral determinants depends analytically on "natural parameters". The multiplicities of the zeros can increase for some exceptional values of the parameters. There are interesting phenomena "in Bender-Wu style" if one considers the eigenvalues "up to analytic continuation in a parameter space".

Spectral determinants

A spectral determinant F for a spectrum is an *entire function* of the spectral parameter such that the spectrum is the set of its zeros. We suppose a priori nothing about the multiplicities.

It is possible to get a spectral determinant for any spectrum using an Hadamard product but we cannot in general do that with parameters.

Using analytic continuation and k-sums we can define *"natural spectral determinants"* for an analytic spectrum. Under mild hypothesis such spectral determinants depends analytically on "natural parameters". The multiplicities of the zeros can increase for some exceptional values of the parameters. There are interesting phenomena "in Bender-Wu style" if one considers the eigenvalues "up to analytic continuation in a parameter space".

It is also possible to use the definition of such natural spectral determinants to calculate numerically the eigenvalues by an "analytic matching".

WHAT HAPPENED IN AROSA IN 1925 DURING CHRISTMAS VACATIONS ?

WHAT HAPPENED IN AROSA IN 1925
DURING CHRISTMAS VACATIONS ?

Discovery of the Schrödinger equation
and of its application to the energy levels of the hydrogen atom

WHAT HAPPENED IN AROSA IN 1925
DURING CHRISTMAS VACATIONS ?

Discovery of the Schrödinger equation
and of its application to the energy levels of the hydrogen atom

Emergence of the concept of *intrinsic spectrum*,
a point which was apparently unnoticed.

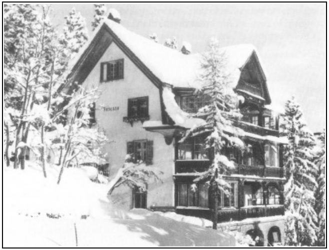
Figure: Erwin Schrödinger

Schrödinger spent Chistmas holidays of 1923 and 1924 with his wife Annemarie in Arosa (a mountain resort) in the Villa Frisia. But in december 1925 *"His marriage with Anny was at a high point of disagreement and tension, with constant talk of breakup and divorce"* (cf. Walter Moore). Annemarie had an affair with Hermann Weyl and Schrödinger had several mistresses.

Schrödinger spent Chistmas holidays of 1923 and 1924 with his wife Annemarie in Arosa (a mountain resort) in the Villa Frisia. But in december 1925 *"His marriage with Anny was at a high point of disagreement and tension, with constant talk of breakup and divorce"* (cf. Walter Moore). Annemarie had an affair with Hermann Weyl and Schrödinger had several mistresses. Therefore in 1925 Annemarie decided to stay in Zurich during Christmas vacations and Erwin to go to Arosa in the Villa Frisia, but not alone ... *"Erwin wrote to 'an old girlfriend in Vienna' to join him in Arosa, while Anny remained in Zürich. Efforts to establish the identity of this woman have so far been unsuccessful, since Erwin's personal diary for* 1925 *has disappeared"* (cf. Walter Moore).

Schrödinger spent Chistmas holidays of 1923 and 1924 with his wife Annemarie in Arosa (a mountain resort) in the Villa Frisia. But in december 1925 *"His marriage with Anny was at a high point of disagreement and tension, with constant talk of breakup and divorce"* (cf. Walter Moore). Annemarie had an affair with Hermann Weyl and Schrödinger had several mistresses. Therefore in 1925 Annemarie decided to stay in Zurich during Christmas vacations and Erwin to go to Arosa in the Villa Frisia, but not alone ... *"Erwin wrote to 'an old girlfriend in Vienna' to join him in Arosa, while Anny remained in Zürich. Efforts to establish the identity of this woman have so far been unsuccessful, since Erwin's personal diary for* 1925 *has disappeared"* (cf. Walter Moore). According to Hermann Weyl, the mysterious lady of Arosa was the catalyst for the discovery by Schrödinger of the equation that bears his name: *"Hermann Weyl once said that Schrödinger 'did his great work during a late erotic outburst in his life'."* (Cf. Walter Moore.)

FIGURE 1. The Villa Herwig–Frisia (right), Arosa, where it is believed wave mechanics was discovered during the Christmas holidays 1925–26.



Schrödinger equation

The equation in the first communication 1926:

$$i\hbar\dot{\psi} = H\psi$$

$H = -\frac{\hbar^2}{2m}\Delta + V$ is the Hamiltonian.
The stationary equation is $H\psi = 0$.

Figure: Annemarie and Erwin Schrödinger grave in Alpbach

The Arosa letter

Reading Schrödinger's first communication *Quantisierung als Eigenwertproblem* (Quantisation as an eigenvalue problem), I had the feeling that he had the (unformulated) idea of a notion of *"intrinsic spectrum"*. Therefore I was looking for information about his way of thinking during his stay in Arosa. Some weeks ago, from an internet search with the key words "villa Herwig Frisia, Arosa", I found the auction of a Schrödinger's letter written from Arosa on december 27-th.

The Arosa letter

Reading Schrödinger's first communication *Quantisierung als Eigenwertproblem* (Quantisation as an eigenvalue problem), I had the feeling that he had the (unformulated) idea of a notion of *"intrinsic spectrum"*. Therefore I was looking for information about his way of thinking during his stay in Arosa. Some weeks ago, from an internet search with the key words "villa Herwig Frisia, Arosa", I found the auction of a Schrödinger's letter written from Arosa on december 27-th. *Three lines of this letter give a striking positive confirmation of my hypothesis*. It seems that *nobody noticed this important point before*.

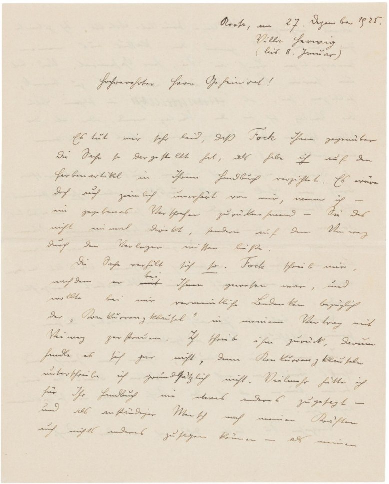
An auction

CHRISTIE'S

ONLINE AUCTION 18958 EUREKA! SCIENTIFIC BREAKTHROUGHS OF THE 20TH CENTURY

LOT 11

+ 2 MORE



*

'A new atomic theory'

Erwin Schrödinger. 27 December 1925

Price Realised
GBP 22,500

Estimate
GBP 3,000 - GBP 5,000

Closed: 16 Jul 2020

FOLLOW

SHARE

DÉTAILS

Erwin Schrödinger (1887-1961).

Autograph letter signed ('E. Schrödinger') to [Wilhelm Wien], Villa Herwig, Arosa, Switzerland, 27 December 1925.

In German. Six pages, 269 × 215mm. *Provenance*: by descent from Wilhelm Wien.

Schrödinger grasping at 'a new atomic theory'. 'Just now a new atomic theory is niggling me. If only I knew more mathematics! I am very optimistic about this thing and hope that if only I can master the calculations, it will be *very* fine. I think I can provide a vibrational system [*ein schwingendes System*] in comparatively natural ways, not through *ad hoc* assumptions', concentrating particularly on the natural frequencies of hydrogen. He provides the mathematical definitions he is working on: 'These frequencies are *very high* against the optical and also against the X-ray frequencies, so have only very small relative differences from each other'; in three further sets of equations, he shows how this provides 'a real *beat frequency*' [*Schwebungsfrequenz*]. He concludes, 'I hope that I will soon be able to report on the thing in a more detailed and comprehensible way', once he has cracked the mathematical challenges.

The letter opens with an attempt to clear up a misunderstanding prompted by the Russian physicist V.A. Fock, concerning Schrödinger's intended contribution of an article on colour theory for Wien's *Handbuch der Experimentalphysik* (1926): Rudolf Tomaschek should have taken his place, but Schrödinger was astonished to discover from Edgar Meyer that there was an objection to Tomaschek 'as a student of [Philipp] Lenard because of their antagonism towards Einstein'. Schrödinger also discusses a small conference of southern German physicists in Carlsruhe, where it appears that physicists from Munich [where Wien was based] are unrepresented; and mentions the fact that for various reasons he has submitted his last three papers not to the *Annalen der Physik* (of which Wien was the editor) but to other journals.

SPECIAL NOTICE

Please note this lot is the property of a consumer. See H1 of the Conditions of Sale.

Brought to you by

C Sophie Hopkins

A Christie's specialist may contact you to discuss this lot or to notify you if the condition changes prior to the sale.

SHOPKINS@CHRISTIES.COM
+44 207 752 3144

A letter from E. Schrödinger to W. Wien

Villa Herwig, Arosa, 27 December 1925

Walter Moore translation (and my own translation of the end part cut by Moore):

"At the moment I am struggling with a new atomic theory. If only I knew more mathematics! I am very optimistic about this thing and expect that if I can only . . . solve it, it will be very beautiful. I think I can specify a vibrating system [ein schwingendes System] *that has as eigenfrequencies the hydrogen term frequencies 'and in a relatively natural way, not through ad hoc assumptions' ... I hope that I will soon be able to report on the thing in a more detailed and comprehensible way.*

A letter from E. Schrödinger to W. Wien

Villa Herwig, Arosa, 27 December 1925

Walter Moore translation (and my own translation of the end part cut by Moore):

"At the moment I am struggling with a new atomic theory. If only I knew more mathematics! I am very optimistic about this thing and expect that if I can only . . . solve it, it will be very beautiful. I think I can specify a vibrating system [ein schwingendes System] *that has as eigenfrequencies the hydrogen term frequencies 'and in a relatively natural way, not through ad hoc assumptions' ... I hope that I will soon be able to report on the thing in a more detailed and comprehensible way. In the meantime I must learn more mathematics, in order to fully master the vibration problem - a linear differential equation, similar to Bessels, but less well known,*

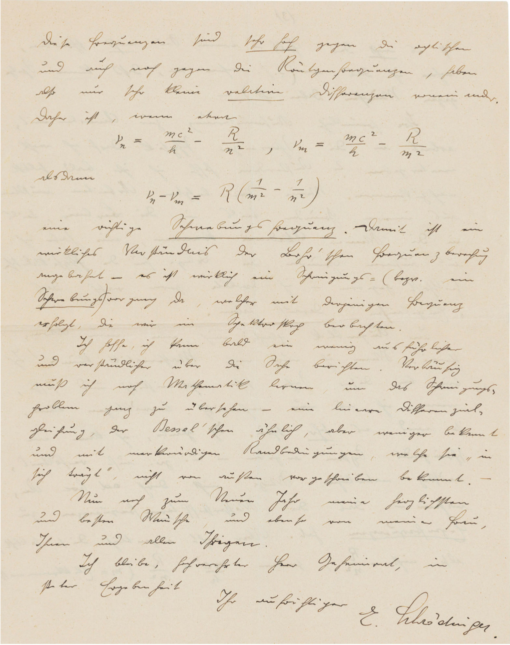
A letter from E. Schrödinger to W. Wien

Villa Herwig, Arosa, 27 December 1925

Walter Moore translation (and my own translation of the end part cut by Moore):

"At the moment I am struggling with a new atomic theory. If only I knew more mathematics! I am very optimistic about this thing and expect that if I can only . . . solve it, it will be very beautiful. I think I can specify a vibrating system [ein schwingendes System] *that has as eigenfrequencies the hydrogen term frequencies 'and in a relatively natural way, not through ad hoc assumptions' ... I hope that I will soon be able to report on the thing in a more detailed and comprehensible way. In the meantime I must learn more mathematics, in order to fully master the vibration problem - a linear differential equation, similar to Bessels, but less well known, and with remarkable boundary conditions that the equation 'carries within itself'* [in sich trägt] *and that are not externally predetermined. ..."*

Arosa's letter to W. Wien, page 6:



diese Frequenzen sind sehr hoch gegen die optischen und auch noch gegen die Röntgenfrequenzen, haben aber nur sehr kleine relative Differenzen untereinander. Daher ist, wenn etwa

$$\nu_n = \frac{mc^2}{h} - \frac{R}{n^2}, \quad \nu_m = \frac{mc^2}{h} - \frac{R}{m^2}$$

also dann

$$\nu_n - \nu_m = R\left(\frac{1}{m^2} - \frac{1}{n^2}\right)$$

eine richtige Schwebungsfrequenz. Damit ist ein wirkliches Verständnis der Bohr'schen Frequenzbedingung angebahnt — es ist wirklich ein Schwingungs= (bezw. ein Schwebungs)vorgang da, welcher mit derjenigen Frequenz erfolgt, die wir im Spektroskop beobachten.

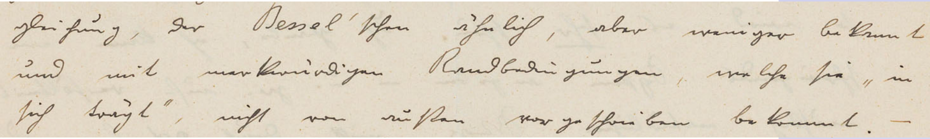
Ich hoffe, ich kann bald ein wenig ausführlicher und verständlicher über die Sache berichten. Vorläufig muss ich noch Mathematik lernen, um das Schwingungsproblem ganz zu übersehen — eine lineare Differentialgleichung, der Bessel'schen ähnlich, aber weniger bekannt und mit merkwürdigen Randbedingungen, welche sie „in sich trägt", nicht von außen vorgeschrieben bekommt. —

Nun noch zum Neuen Jahr meine herzlichsten und besten Wünsche, und ebenso von meiner Frau, Ihnen und allen Ihrigen.

Ich bleibe, hochverehrter Herr Geheimrat, in steter Ergebenheit

Ihr aufrichtiger E. Schrödinger.

Arosa's letter to W. Wien, page 6:


gleichung, der Bessel'schen ähnlich, aber weniger bekannt und mit merkwürdigen Randbedingungen, welche sie „in sich trägt", nicht von außen vorgeschrieben bekommt. —

158 *Helge Kragh*

Im Augenblick plagt mich eine neue Atomtheorie. Wenn ich nur mehr Mathematik könnte! . . . Vorläufig muss ich noch Mathematik lernen, um das Schwingungsproblem ganz zu übersehen – eine lineare Differenzialgleichung, der Bessel'schen ähnlich, aber weniger bekannt und mit merkwürdigen Randbedingungen, welche sie "in sich trägt", nicht von aussen vorgeschrieben bekommt.[11]

" ... a linear differential equation, similar to Bessel's, but less well known, and with remarkable boundary conditions that the equation 'carries within itself' and that are not externally predetermined."

In the original, Schrödinger insist on 'carries within itself': ,,in sich trägt".

The idea of quantification conditions carried by the equation reappears in the second communication (page 511):
"... die Gleichung (18) die Quantenbedingdungen in sich trägt."

The idea of quantification conditions carried by the equation reappears in the second communication (page 511):
"... die Gleichung (18) die Quantenbedingdungen in sich trägt."
Schrödinger suggests a broader scope of application than the hydrogen case, but his formulation is rather obscure. This can explain that this point has hardly ever been noticed. As far as I know, it was noticed and commented only by Peter Enders in 2013 (he apparently ignored the Arosa letter to W. Wien).

The idea of quantification conditions carried by the equation reappears in the second communication (page 511):
"... die Gleichung (18) die Quantenbedingdungen in sich trägt."

Schrödinger suggests a broader scope of application than the hydrogen case, but his formulation is rather obscure. This can explain that this point has hardly ever been noticed. As far as I know, it was noticed and commented only by Peter Enders in 2013 (he apparently ignored the Arosa letter to W. Wien).

I quote Helge Kragh:
"As regards crucial question of how Schrodinger discovered his celebrated equation and subsequently applied it to the hydrogen atom, there are, however, serious lacunae in the historical writings."

I think that the fact that on december 27-th Schrödinger had in mind the notion of intrinsic spectrum and my mathematical analysis of this point implies a revision of the history of the discovery.

Boundary conditions (non intrinsic version)

Quantisierung als Eigenwertproblem (Quantisation as an eigenvalue problem), first communication Published in Annalen der Physik 79 (1926). Submitted on January 26-th 1926, received by W. Wien on January 27-th 1926.

$$K = \hbar, \quad n = \ell \quad \text{(today notation)}.$$

The solutions must be *bounded* (from the "other equation" of Schrödinger).

Boundary conditions (non intrinsic version)

Quantisierung als Eigenwertproblem (Quantisation as an eigenvalue problem), first communication Published in Annalen der Physik 79 (1926). Submitted on January 26-th 1926, received by W. Wien on January 27-th 1926.

$$K = \hbar, \quad n = \ell \quad \text{(today notation).}$$

The solutions must be *bounded* (from the "other equation" of Schrödinger).

QUANTISATION AND PROPER VALUES—I 3

$$\frac{d^2\chi}{dr^2} + \frac{2}{r}\frac{d\chi}{dr} + \left(\frac{2mE}{K^2} + \frac{2me^2}{K^2 r} - \frac{n(n+1)}{r^2}\right)\chi = 0. \tag{7}$$

$$n = 0, 1, 2, 3 \ldots$$

The limitation of n to integral values is *necessary* so that the surface harmonic may be *single-valued*. We require solutions of (7) that will remain finite for all non-negative real values of r. Now[1] equation (7) has *two* singularities in the complex r-plane, at $r=0$ and $r=\infty$, of which the second is an "indefinite point" (essential singularity) of *all* integrals, but the first on the contrary is not (for any integral). These two singularities form exactly the *bounding points of our real interval*. In such a case it is known now that the postulation of the *finiteness* of χ at the bounding points is equivalent to a *boundary condition*. The equation has *in general* no integral which remains finite at *both* end points; such an integral exists only for certain special values of the constants in the equation. It is now a question of defining these special values. This is the *jumping-off* point of the whole investigation.[2]

[1] For guidance in the treatment of (7) I owe thanks to Hermann Weyl.
[2] For unproved propositions in what follows, see L. Schlesinger's *Differential Equations* (Collection Schubert, No. 13, Göschen, 1900, especially chapters 3 and 5).

Intrinsic boundary condition at zero

The non negative exponent at zero.

Let us examine first the singularity at $r=0$. The so-called *indicial* equation which defines the behaviour of the integral at this point, is

$$\rho(\rho-1)+2\rho-n(n+1)=0, \tag{8}$$

with roots

$$\rho_1=n, \quad \rho_2=-(n+1). \tag{8'}$$

The two canonical integrals at this point have therefore the exponents n and $-(n+1)$. Since n is not negative, only the first of these is of use to us. Since it belongs to the greater exponent, it can be represented by an ordinary power series, which begins with r^n. (The other integral, which does not interest us, can contain a logarithm, since the difference between the indices is an integer.) The next singularity is at infinity, so the above power series is always convergent and represents a *transcendental integral function*. We therefore have established that:

The required solution is (except for a constant factor) a single-valued definite transcendental integral function, which at $r=0$ belongs to the exponent n.

We must now investigate the behaviour of this function at infinity on the positive real axis. To that end we simplify equation (7) by the substitution

$$\chi=r^{\alpha}U, \tag{9}$$

where α is so chosen that the term with $1/r^2$ drops out. It is easy to verify that then α must have one of the two values n, $-(n+1)$. Equation (7) then takes the form,

Intrinsic boundary condition at infinity

(7′)
$$\frac{d^2U}{dr^2}+\frac{2(\alpha+1)}{r}\frac{dU}{dr}+\frac{2m}{K^2}\left(E+\frac{e^2}{r}\right)U=0.$$

Its integrals belong at $r=0$ to the exponents 0 and $-2\alpha-1$. For the α-value, $\alpha=n$, the *first* of these integrals, and for the second α-value, $\alpha=-(n+1)$, the *second* of these integrals is an integral function and leads, according to (9), to the desired solution, which is single-valued. We therefore lose nothing if we confine ourselves to *one* of the two α-values. Take, then,

(10)
$$\alpha=n.$$

Our solution U then, at $r=0$, belongs to the exponent 0. Equation (7′) is called Laplace's equation. The general type is

(7″)
$$U''+\left(\delta_0+\frac{\delta_1}{r}\right)U'+\left(\epsilon_0+\frac{\epsilon_1}{r}\right)U=0.$$

Here the constants have the values

(11)
$$\delta_0=0,\quad \delta_1=2(\alpha+1),\quad \epsilon_0=\frac{2mE}{K^2},\quad \epsilon_1=\frac{2me^2}{K^2}.$$

This type of equation is comparatively simple to handle for this reason: The so-called Laplace's transformation, which in general leads *again* to an equation of the *second* order, *here* gives one of the *first*. This allows the solutions of (7″) to be represented by complex integrals. The result [1] only is given here. The integral

(12)
$$U=\int_L e^{zr}(z-c_1)^{\alpha_1-1}(z-c_2)^{\alpha_2-1}dz$$

is a solution of (7″) for a path of integration L, for which

(13)
$$\int_L \frac{d}{dz}\left[e^{zr}(z-c_1)^{\alpha_1}(z-c_2)^{\alpha_2}\right]dz=0.$$

Intrinsic boundary condition at infinity

In his first communication Schrödinger follows a method (due to Poincaré and Horn) described by L. Schlesinger in his 1900 monograph. According to some historians Schrödinger known Schlesinger's book when he was a student. The method works for the so-called Laplace equations whose Laplace transforms are equations of *order one* with two (Fuchsian) singularities $c_1, c_2 \in \mathbb{C}$. Then the integral

$$\int_L e^{-zr}(z-c_1)^{\alpha_1-1}(z-c_2)^{\alpha_2-1}dz$$

gives, for a convenient contour of integration L, a solution of the initial equation.

Intrinsic boundary condition at infinity

In his first communication Schrödinger follows a method (due to Poincaré and Horn) described by L. Schlesinger in his 1900 monograph. According to some historians Schrödinger known Schlesinger's book when he was a student. The method works for the so-called Laplace equations whose Laplace transforms are equations of *order one* with two (Fuchsian) singularities $c_1, c_2 \in \mathbb{C}$. Then the integral

$$\int_L e^{-zr}(z-c_1)^{\alpha_1-1}(z-c_2)^{\alpha_2-1}dz$$

gives, for a convenient contour of integration L, a solution of the initial equation.

Letter from Schrödinger to Sommerfeld:

Finally I wish to add that the discovery of the whole connection [between the wave equation and the quantization of hydrogen atom], goes back to your beautiful quantization method for evaluating the radial quantum integral. ... which suddenly shone out from the exponents α_1 and α_2 like a Holy Grail."

Intrinsic boundary condition at infinity

In his first communication Schrödinger follows a method (due to Poincaré and Horn) described by L. Schlesinger in his 1900 monograph. According to some historians Schrödinger known Schlesinger's book when he was a student. The method works for the so-called Laplace equations whose Laplace transforms are equations of *order one* with two (Fuchsian) singularities $c_1, c_2 \in \mathbb{C}$. Then the integral

$$\int_L e^{-zr}(z-c_1)^{\alpha_1 - 1}(z-c_2)^{\alpha_2 - 1}dz$$

gives, for a convenient contour of integration L, a solution of the initial equation.

Letter from Schrödinger to Sommerfeld:

Finally I wish to add that the discovery of the whole connection [between the wave equation and the quantization of hydrogen atom], goes back to your beautiful quantization method for evaluating the radial quantum integral. ... which suddenly shone out from the exponents α_1 and α_2 like a Holy Grail."

Schrödinger quantification condition at infinity is defined from to the two *Frobenius exponents of the Laplace transform of the equation*. These two exponents must be *integers*.

Laguerre polynomials and hydrogen atom

I quote an article of J. Mawhin and A. Ronveaux (*Archive for History of Exact Sciences*, 2010):

Laguerre polynomials and hydrogen atom

I quote an article of J. Mawhin and A. Ronveaux (*Archive for History of Exact Sciences*, 2010):

> In a footnote of his second (and longer) paper, Schrödinger thanked Erwin Fues (1893–1970), at this time his assistant in Zürich, for identifying the eigenfunctions of Planck's oscillator with Hermite polynomials, and observed that he has identified the polynomials obtained in his first paper with the $(2n+1)^{th}$ derivative of the $(n+l)^{th}$ Laguerre polynomial (Schrödinger 1926_2). Thus, Schrödinger definitely abandoned Schlesinger's approach and, following a suggestion of Weyl and Fues, refered to the recent *Methoden der mathematischen Physik* of Richard Courant (1888–1972) and David Hilbert (1862–1943) published in 1924, with the aim of providing the mathematical foundations of classical physics (Courant–Hilbert 1924). In the French translation of (Schrödinger 1927) published in 1933, Schrödinger advised the reader to forget about his first approach, and suggested it would be better consult recent books on wave mechanics, instead of his papers, for a clearer and simpler approach.

Compared to my notation, one must exchange ℓ and n.

It is surprising that, in january 2016, Hermann Weyl did not recognised Liouville polynomials.

The hydrogen spectrum as an analytic spectrum

The regular singularity at $r = 0$ is *logarithmic* (the possibility was noticed by Schrödinger in his first communication; the difference of exponents $2\ell + 1$ is entire).

The half-line $\mathbb{R}_+$ is a *non-singular* line for the summation at infinity of the formal subdominant solution and a singular line for the other purely exponential solution. (Relation between Borel summation and Laplace transform.)

The hydrogen spectrum as an analytic spectrum

The regular singularity at $r = 0$ is *logarithmic* (the possibility was noticed by Schrödinger in his first communication; the difference of exponents $2\ell + 1$ is entire).

The half-line $\mathbb{R}_+$ is a *non-singular* line for the summation at infinity of the formal subdominant solution and a singular line for the other purely exponential solution. (Relation between Borel summation and Laplace transform.)

It follows that a special solution connecting 0 to ∞ must be *analytic* at 0 and *subdominant* at ∞.

Hence a solution along $\mathbb{R}_+$ is special if and only if it is bounded. The Schrödinger spectrum and the analytic spectrum coincide.

The simplest quantification method

In his first communication Schrödinger give a first caracterisation of the spectrum using the fact that the eigenfunction must be *bounded* at zero and infinity (a "naive spectrum").

The simplest quantification method

In his first communication Schrödinger give a first caracterisation of the spectrum using the fact that the eigenfunction must be *bounded* at zero and infinity (a “naive spectrum”).

This implies that the eigenfunction is analytic at zero.

The simplest quantification method

In his first communication Schrödinger give a first caracterisation of the spectrum using the fact that the eigenfunction must be *bounded* at zero and infinity (a "naive spectrum").
This implies that the eigenfunction is analytic at zero.
Using convenient coordinates, the radial function $R(x)$ can be written $R(x) = x^{\ell} e^{-x/2} u(x)$, where u is a solution of the generalised Laguerre equation

$$x^2 u'' + (1 + 2\ell - x)u' - \lambda u = 0 \ (\ell \in \mathbb{N}).$$

The simplest quantification method

In his first communication Schrödinger give a first caracterisation of the spectrum using the fact that the eigenfunction must be *bounded* at zero and infinity (a “naive spectrum”).
This implies that the eigenfunction is analytic at zero.
Using convenient coordinates, the radial function $R(x)$ can be written $R(x) = x^{\ell} e^{-x/2} u(x)$, where u is a solution of the generalised Laguerre equation

$$x^2 u'' + (1 + 2\ell - x)u' - \lambda u = 0 \ (\ell \in \mathbb{N}).$$

The function u is entire and $u = \sum u_h x^h$, the coefficients satisfying the recursion relation

$$(h+1)(h+2\ell+2)u_{h+1} = (h+\lambda)u_h.$$

If u_h *never vanishes*, that is if $\lambda \notin -\mathbb{N}$, one can prove that there exists $x_0 > 0$ such that $R(x) > \frac{1}{2} x^{\ell} e^{x/4}$ for $x > x_0$. It follows that R is not bounded. Therefore the series *must terminate*, the spectrum is given by $\lambda = -n$, $n \in \mathbb{N}$ and the corresponding u are *generalised Laguerre polynomials*.
Refs: Schiff, Quantum Mechanics, Ramis-Warusfel, Licence L3.

The simplest quantification method

The proof is very simple. Therefore we can follow Schrödinger 1933 advice and forget about his initial proof. Almost everybody does that today. But my feeling is that doing so, we, in some sense, throw the baby out with the bathwater.

It is not a good idea to forget the approach by a study of solutions in the complex domain.

15 YEARS LATTER

15 YEARS LATTER

Intrinsic spectra of the harmonic oscillator and of the non-relativistic hydrogen atom by *factorisation of operators*

Schrödinger, *A method of determining quantum-mechanical eigenvalues and eigenfunctions*, 1940.

II.

A METHOD OF DETERMINING QUANTUM-MECHANICAL EIGENVALUES AND EIGENFUNCTIONS.

By E. SCHRÖDINGER.

[Read 11 December, 1939. Published 12 February, 1940.]

§ 1. *Planck's Oscillator.*

The attractive feature of the method to be described here is that it avoids cumbersome transformations, recourse to the ready-made equipment of the mathematical warehouse or expansion into power-series. It yields all the eigenfunctions of the line spectrum by *one* quite elementary quadrature. The method originates from a, virtually, well-known treatment of the oscillator. Its amplitude equation

$$\frac{d^2\psi}{dx^2} - x^2\psi + \lambda\psi = 0 \qquad (1, 1)$$

$$(\lambda = \text{eigenvalue})$$

can be written in either of the two following ways :—

$$\text{(I)} \quad \left(\frac{d}{dx} - x\right)\left(\frac{d}{dx} + x\right)\psi + (\lambda - 1)\psi = 0,$$

$$\text{(II)} \quad \left(\frac{d}{dx} + x\right)\left(\frac{d}{dx} - x\right)\psi + (\lambda + 1)\psi = 0. \qquad (1, 2)$$

§ 2. *The (non-relativistic) hydrogen atom (Kepler motion).*

After a spherical harmonic $P_l(\theta, \phi)$, of given integral order l has been split off and convenient units have been introduced, the equation for the radial variable x reads

$$\frac{d^2\psi}{dx^2} + \frac{2}{x}\frac{d\psi}{dx} - \frac{l(l+1)}{x^2}\psi + \left(\lambda + \frac{2}{x}\right)\psi = 0. \qquad (2, 1)$$

(x is the radius vector r, with the radius of Bohr's first orbit as unit of length; λ is the energy E, in the Rydberg energy unit; thus

$$x = \frac{4\pi^2 me^2}{h^2} r, \quad \lambda = \frac{h^2}{2\pi^2 me^4} E\Big). \qquad (2, 2)$$

We propose to investigate *negative* eigenvalues and put

$$\lambda = -\frac{1}{a^2}, \qquad (2, 3)$$

taking for a the positive value of $(-\lambda)^{-\frac{1}{2}}$. The equation (2, 1), multiplied by x^2, can be written in either of the two following ways:—

$$\text{(I)} \quad \left(x\frac{d}{dx} + a - \frac{x}{a}\right)\left(x\frac{d}{dx} - a + 1 + \frac{x}{a}\right)\psi + [(a-1)a - l(l+1)]\psi = 0,$$

$$\text{(II)} \quad \left(x\frac{d}{dx} - a + \frac{x}{a}\right)\left(x\frac{d}{dx} + a + 1 - \frac{x}{a}\right)\psi + [a(a+1) - l(l+1)]\psi = 0.$$

(2, 4)

We recall that if $D \in \mathbb{C}(x)[d/dx]$ is of order two and $\lambda \in \mathbb{C}$, then the following "integrability conditions" are equivalent:

(i) the operator $D - \lambda$ *factorises* into the Ore ring $\mathbb{C}(x)[d/dx]$ into two operators of order one;

(ii) there exists a solution f of $D - \lambda$ such $f'/f \in \mathbb{C}(x)$;

(iii) the operator $D - \lambda$ admits a *generalised polynomial solution*;

(iv) Kovacic's algorithm for the operator $D - \lambda$ stops at the $n = 1$ step;

(iv) the projective differential Galois group of the operator $D - \lambda$ is *triangulable*.

Quantification by differential Galois groups

At the beginning of the eighties, I noticed that the spectrum of the Schrödinger equation with a Coulomb potential corresponds to the *vanishing of a Stokes multiplier at infinity*; equivalently to the fact that *the differential Galois group is triangulable*. (J.P. R., Lecture at the Dolbeaux-Lelong seminar in Paris. Unpublished.)

Quantification by differential Galois groups

At the beginning of the eighties, I noticed that the spectrum of the Schrödinger equation with a Coulomb potential corresponds to the *vanishing of a Stokes multiplier at infinity*; equivalently to the fact that *the differential Galois group is triangulable*. (J.P. R., Lecture at the Dolbeaux-Lelong seminar in Paris. Unpublished.)

In 1989, Jean Martinet and J.P. R calculated the Stokes multipliers of the confluent hypergeometric equation

$$y'' + (c - x)y' - ay = 0$$

(using Gamma functions) and the differential Galois group G. The group G is triangulable if and only if a Stokes multiplier vanishes, that is:

$a \in -\mathbb{N}$, or $-c + a + 1 \in -\mathbb{N}$, or $1 - a \in -\mathbb{N}$, or $c - a \in -\mathbb{N}$.

Algebraic spectrum

In his PhD thesis 2009, Primitivo Acosta-Humanez got the hydrogen spectrum using the first step ($n = 1$) of Kovacic's algorithm.

Later Primitivo Acosta-Humanez, Juan J. Morales-Ruiz and Jacques-Arthur Weyl introduced a notion of *algebraic spectrum* 2010. By definition the spectral values correspond to the Liouvillian cases.

Algebraic spectrum

In his PhD thesis 2009, Primitivo Acosta-Humanez got the hydrogen spectrum using the first step ($n = 1$) of Kovacic's algorithm.

Later Primitivo Acosta-Humanez, Juan J. Morales-Ruiz and Jacques-Arthur Weyl introduced a notion of *algebraic spectrum* 2010. By definition the spectral values correspond to the Liouvillian cases.
Today I use a variant: the eigenfunctions correspond to the solutions f of $D - \lambda$ such $\partial f/f \in K$ (the step one of Kovacic's algorithm when $K = \mathbb{C}(x)$).
When $K = \mathbb{C}(x)$, an algebraic spectrum appears as a subset of an analytic spectrum (equality in the hypergeometric case).
In the CHE case *an algebraic spectrum corresponds to a 3-point connection* and *the analytic spectrum correspond to a 2-points connection* (two points among the three).

SPECTRA OF THE HEUN CLASS

SPECTRA OF THE HEUN CLASS

and lines on affine cubic surfaces

Heun class and Painlevé equations

In 1888 Karl Heun published an article on the *general second order Fuchsian linear differential equations with four singularities*. These equations are called today *Heun equations*. At that time the case of *three* singularities, the *Riemann equations* (hypergeometric case), was well known with numerous applications and the purpose of Heun was to explore the next step.

In 1888 Karl Heun published an article on the *general second order Fuchsian linear differential equations with four singularities*. These equations are called today *Heun equations*. At that time the case of *three* singularities, the *Riemann equations* (hypergeometric case), was well known with numerous applications and the purpose of Heun was to explore the next step. Unfortunately the occurence of an extra singularity introduce a completely new phenomenon, the equation is no longer "determined" by the local exponents and it appears an *auxiliary parameter*. It is a source of considerable difficulties, in particular it is no longer possible to calculate "explicitely" the global monodromy. Heun did not overcome the new difficulties and, in a first step, his work was not appreciated.

In 1888 Karl Heun published an article on the *general second order Fuchsian linear differential equations with four singularities*. These equations are called today *Heun equations*. At that time the case of *three* singularities, the *Riemann equations* (hypergeometric case), was well known with numerous applications and the purpose of Heun was to explore the next step. Unfortunately the occurence of an extra singularity introduce a completely new phenomenon, the equation is no longer "determined" by the local exponents and it appears an *auxiliary parameter*. It is a source of considerable difficulties, in particular it is no longer possible to calculate "explicitely" the global monodromy. Heun did not overcome the new difficulties and, in a first step, his work was not appreciated.

Later it was discovered that Heun equations give new interesting equations by various *confluence* processes, generalizing what happens in the hypergeometric case. In this way one gets a rich family of equations, the *Heun class* (confluent, biconfluent, double confluent and triconfluent and degenerated forms). The confluent equations in the class have one or two *irregular singularities* (with an integer or half-integer slope.

Figure: Karl Heun

Separation of variables and Heun class

The story continues with the study of the Laplace, Helmoltz and Klein-Gordon equations in various dimensions ($d = 2$, 3 or 4). In some cases these partial diffferential equations *separate* in coordinate systems (11 possibilities if $d = 3$) and give rise to d linear second order equations.

Separation of variables and Heun class

The story continues with the study of the Laplace, Helmoltz and Klein-Gordon equations in various dimensions ($d = 2$, 3 or 4). In some cases these partial diffferential equations *separate* in coordinate systems (11 possibilities if $d = 3$) and give rise to d linear second order equations. Algebraic forms of these equations are in the most usual cases 'trivial' ($y'' + \omega^2 y = 0$), hypergeometric (possibly confluent) or *belongs to the Heun class*. (This observation is relatively recent in some cases, in particular in the black holes theory. During the eighties in this last case.). Mathieu's, Lamé's, Ince's and the spheroidal wave equation appeared in this way.

Analytic spectra and Riemann-Hilbert map

The (wild) Riemann-Hilbert map RH *algebraize* the analytic spectra of the CHE. They can be interpreted using pull-backs of (explicitely computable) algebraic curves on the (wild) character varieties, which are (possibly singular) affine cubic surfaces.

Analytic spectra and Riemann-Hilbert map

The (wild) Riemann-Hilbert map RH *algebraize* the analytic spectra of the CHE. They can be interpreted using pull-backs of (explicitely computable) algebraic curves on the (wild) character varieties, which are (possibly singular) affine cubic surfaces.

In several cases the algebraic curves are lines. The image by RH of a generalised polynomial solutions is the intersection of two lines.

Analytic spectra and Riemann-Hilbert map

The (wild) Riemann-Hilbert map RH *algebraize* the analytic spectra of the CHE. They can be interpreted using pull-backs of (explicitely computable) algebraic curves on the (wild) character varieties, which are (possibly singular) affine cubic surfaces.
In several cases the algebraic curves are lines. The image by RH of a generalised polynomial solutions is the intersection of two lines.

The transcendence of the spectra is "not worse" than the transcendence of RH. This last transcendence can be understood from recent results of Lisovyy and al. (irregular conformal blocks).

Analytic spectra and Riemann-Hilbert map

The (wild) Riemann-Hilbert map RH *algebraize* the analytic spectra of the CHE. They can be interpreted using pull-backs of (explicitely computable) algebraic curves on the (wild) character varieties, which are (possibly singular) affine cubic surfaces.

In several cases the algebraic curves are lines. The image by RH of a generalised polynomial solutions is the intersection of two lines.

The transcendence of the spectra is "not worse" than the transcendence of RH. This last transcendence can be understood from recent results of Lisovyy and al. (irregular conformal blocks).

A similar idea for the calculation of the QNM of black-holes appears in articles of two Brazilian physicists, B. C. da Cunha and J. P. Cavalcante 2020, RH being 'calculated' by *confluent conformal blocks*.

Isomonodromic deformations

18 special solutions of Painlevé V

System of rank two with regular singular points at 0, ∞ and an irregular singular point at 1 of slope one. Generically, there are 18 lines on the character variety, a cubic affine surface $\mathcal{S}_V$.

Special solutions correspond to pairings of local distinguished subspaces by a simple continuous path.

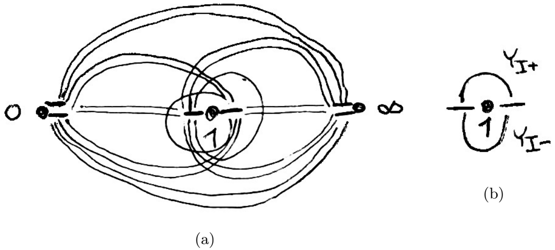


Figure 2: (a) Pairing on the distinguished subspaces of the solution space corresponding to the 18 lines on $\mathcal{S}_V$.
(b) The two pairs of subspaces of the solution space (each attached to one anti-Stokes direction) corresponding to the sectoral bases $Y_{I\pm}$.

Figure: From Martin Klimes

24 special solutions of Painlevé VI

In the Painlevé VI there are (generically) 24 special solutions corresponding to the 24 lines on the affine surface. there are $27 = 24 + 3$ lines on the (smooth) complete cubic surface.

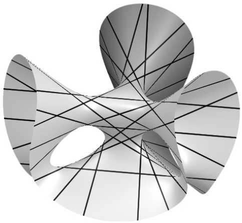


Figure: Smooth cubic surface: 27 lines

24 special solutions of Painlevé VI

In the Painlevé VI there are (generically) 24 special solutions corresponding to the 24 lines on the affine surface. there are $27 = 24 + 3$ lines on the (smooth) complete cubic surface.

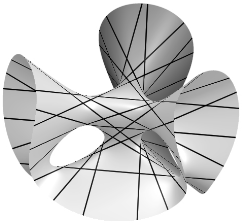


Figure: Smooth cubic surface: 27 lines

There is a similar mechanism for all the Painlevé equations. Spectra for equations of the Heun Class correspond to lines on affine cubic surfaces. The (wild) Riemann-Hilbert maps *algebraize the spectra*.

GENERALISED POLYNOMIAL SOLUTIONS

of

CONFLUENT HEUN EQUATIONS

GENERALISED POLYNOMIAL SOLUTIONS

of

CONFLUENT HEUN EQUATIONS

A work in progress,

Anne Duval, Michèle Loday-Richaud and J.P. R.

Canonical form of the Confluent Heun Equation

The Confluent Heun Equation (CHE) is a second order rational linear differential equation with *two regular singular points* and an *irregular singular point*, that we can suppose to be located at infinity (up to a Möbius transformation). A *canonical form* is:

$$\mathrm{D}_{a,c,d,\varepsilon,\lambda}\, w \equiv \frac{d^2 w}{dx^2} + \left(\frac{c}{x} + \frac{d}{x-1} + \varepsilon\right)\frac{dw}{dx} + \frac{ax-\lambda}{x(x-1)} w = 0,$$

where $a, c, d \in \mathbb{C}$, $\varepsilon \in \mathbb{C}^*$. This equation has *three* singular points. The points 0 and 1 are regular singular. The point ∞ is irregular with Katz rank 1. If $a = \varepsilon = 0$, then the equation degenerates into a *hypergeometric equation*.

The monodromy exponents at 0 and 1 are respectively

$$(0, 1-c),\ (0, 1-d).$$

The generalised exponents at ∞ are

$$(a/\varepsilon, 0) \text{ and } (c + d - a/\varepsilon\,,\ -\varepsilon x).$$

Canonical form of the Confluent Heun Equation

A rational second order linear differential equation with two regular singular points and an irregular singular point of Katz rank 1 can be transformed by a Möbius transformation and an s-homotopic transformation into a CHE in canonical form. The proof is easy. (There are 8 s-homotopic transformations.) *Be careful:* the canonical form is *not unique*, the differences of the exponents at the regular singular points and of the exponential exponents at infinity are defined only up to sign.

Bôcher Symmetrical Form of CHE

The following form of the CHE is called *Bôcher symmetrical form*. It appears frequently in applications, In particular in quantum chemistry and in the study of linear perturbations of black holes (angular equations):

$$\mathcal{S}_{\tau,\mu,\beta,\xi}\, y - \lambda y = 0$$

where $\mu, \beta, \xi, \lambda \in \mathbb{C}$, $\tau \in \mathbb{C}^*$ and

$$\mathcal{S}_{\tau,\mu,\beta,\xi}\, y(x) = \frac{d}{dx}\left((x^2-1)\frac{d}{dx}\right) y(x) + \left(\tau^2(x^2-1) + 2i\tau\beta x - \frac{\mu^2+\xi^2+2\mu\xi x}{x^2-1}\right) y(x).$$

For $\beta = \xi = 0$, we get a spheroidal equation (SE).
For $\xi = 0, \beta \neq 0$, we get a Coulomb spheroidal equation (CSE).

For any choice of the parameters the Bôcher symmetrical equation has three singular points: the points $+1$ and -1 (symmetrical with respect to 0) are regular singular; infinity is irregular with a Katz rank equal to 1. (For $\tau = 0$, ∞ is a regular singular point and the equation degenerates into a hypergeometric equation.)

For any choice of the parameters the Bôcher symmetrical equation has three singular points: the points $+1$ and -1 (symmetrical with respect to 0) are regular singular; infinity is irregular with a Katz rank equal to 1. (For $\tau = 0$, ∞ is a regular singular point and the equation degenerates into a hypergeometric equation.)

The exponents at $x = 1$ are $\pm(\mu + \xi)/2$.
Those at $x = -1$ are $\pm(\mu - \xi)/2$.
The generalised exponents at infinity are $(1 + \varepsilon\beta\,,\, i\varepsilon\tau)$ for $\varepsilon = \pm$.

For any choice of the parameters the Bôcher symmetrical equation has three singular points: the points $+1$ and -1 (symmetrical with respect to 0) are regular singular; infinity is irregular with a Katz rank equal to 1. (For $\tau = 0$, ∞ is a regular singular point and the equation degenerates into a hypergeometric equation.)

The exponents at $x = 1$ are $\pm(\mu + \xi)/2$.
Those at $x = -1$ are $\pm(\mu - \xi)/2$.
The generalised exponents at infinity are $(1 + \varepsilon\beta \, , \, i\varepsilon\tau)$ for $\varepsilon = \pm$.
The *differences of exponents and exponential exponents* are (up to a sign):

$$\mu + \xi = J + 1 \quad \text{at} \quad x = 1, \quad \mu - \xi = K + 1 \quad \text{at} \quad x = -1,$$

$$(2\beta, 2i\tau x) \quad \text{at } \infty.$$

These differences are *invariant by an s-homotopic transformation*. This allows an easy comparison with the parameters of a corresponding canonical form

Generalised polynomial solution

A generalised polynomial solution of a rational linear second order equation is a solution f such that f'/f is rational. It corresponds to a polynomial solution of an equation obtained by a a convenient s-homotopical transformation.

A generalised polynomial solution is the product of "elementary functions" and of a polynomial. In the CHE case:

$$f = (x - x_1)^{\gamma_1}(x - x_2)^{\gamma_2} e^{\rho x} P = gP.$$

An operator $D \in \mathbb{C}(x)[d/dx]$ admits a generalised polynomial solution if and only if it factorizes as a product of two operators of order one in $\mathbb{C}(x)[d/dx]$.

The first step of the Kovacic algorithm

The first step of the Kovacic algorithm

Let D be a GSE (or a GSE in reduced form). There are 8 s-homotopic transformations transforming D into an operator D_θ in canonical form. If an operator in canonical form admits a polynomial solution P, then one of its generalised exponents at ∞ is $(-N, 0)$, where $N \in \mathbb{N}$. This gives a *necessary* condition on the exponents for the existence of a generalised polynomial solution for D.

The first step of the Kovacic algorithm

Let D be a GSE (or a GSE in reduced form). There are 8 s-homotopic transformations transforming D into an operator D_θ in canonical form. If an operator in canonical form admits a polynomial solution P, then one of its generalised exponents at ∞ is $(-N, 0)$, where $N \in \mathbb{N}$. This gives a *necessary* condition on the exponents for the existence of a generalised polynomial solution for D.

This condition appears, independently of the Kovacic algorithm, in a series of articles of P. Fiziev in black holes mathematical theory.

The first step of the Kovacic algorithm

The degree of a polynomial involved in a generalised polynomial solution of a GSE in reduced form may be chosen among one of the 8 following possible values:

$$N = -\eta_1 - \eta_{-1} - \epsilon_\infty \beta$$

where η_1, η_{-1} are exponents respectively in ± 1 and $\epsilon_\infty = \pm 1$.

This implies the *necessary conditions* $N \in \mathbb{N}$ and

$$\mu \pm \beta \in \mathbb{Z}^* \quad \text{or} \quad \xi \pm \beta \in \mathbb{Z}^*$$

The first step of the Kovacic algorithm

The degree of a polynomial involved in a generalised polynomial solution of a GSE in reduced form may be chosen among one of the 8 following possible values:

$$N = -\eta_1 - \eta_{-1} - \epsilon_\infty \beta$$

where η_1, η_{-1} are exponents respectively in ± 1 and $\epsilon_\infty = \pm 1$.

This implies the *necessary conditions* $N \in \mathbb{N}$ and

$$\mu \pm \beta \in \mathbb{Z}^* \quad \text{or} \quad \xi \pm \beta \in \mathbb{Z}^*$$

The s-homotopic transformation

$$v(x) = (x-1)^{\eta_1}(x+1)^{\eta_{-1}} \exp(\epsilon_\infty i \tau x) w(x) = g(x) w(x)$$

changes the equation into a new one

$$D_\theta y = 0 \quad (\theta = g'/g),$$

which is in canonical form.

The first step of the Kovacic algorithm

We denote by $\mathcal{M}^{\epsilon} = \left(U_{i,j}\right)_{0 \leq i,j \in \mathbb{N}}$ the infinite *tridiagonal* matrix associated with the action of

$$(x^2 - 1)D_{\theta} \in \mathbb{C}[x][d/dx]$$

on $\mathbb{C}[x] = \mathbb{C}[x - \epsilon]$ endowed with the monomial basis

$$\left((x - \epsilon)^k\right)_{k \in \mathbb{N}}.$$

The first step of the Kovacic algorithm

We denote by $\mathcal{M}^\epsilon = \left(U_{i,j}\right)_{0\leq i,j\in\mathbb{N}}$ the infinite *tridiagonal* matrix associated with the action of

$$(x^2-1)D_\theta \in \mathbb{C}[x][d/dx]$$

on $\mathbb{C}[x]=\mathbb{C}[x-\epsilon]$ endowed with the monomial basis

$$\left((x-\epsilon)^k\right)_{k\in\mathbb{N}}.$$

The *subdiagonal* has one and only one null entry: $R_{N+1}=0$. Therefore the differential operator $(x^2-1)D_\theta$ defines an endomorphism of $\mathbb{C}_N[x]$. We denote by $\mathcal{M}^\epsilon_{N+1}$ its matrix and by Δ_{N+1} the determinant of this matrix.

The first step of the Kovacic algorithm

We denote by $\mathcal{M}^\epsilon = \left(U_{i,j}\right)_{0\leq i,j\in\mathbb{N}}$ the infinite *tridiagonal* matrix associated with the action of

$$(x^2-1)D_\theta \in \mathbb{C}[x][d/dx]$$

on $\mathbb{C}[x]=\mathbb{C}[x-\epsilon]$ endowed with the monomial basis

$$\left((x-\epsilon)^k\right)_{k\in\mathbb{N}}.$$

The *subdiagonal* has one and only one null entry: $R_{N+1}=0$. Therefore the differential operator $(x^2-1)D_\theta$ defines an endomorphism of $\mathbb{C}_N[x]$. We denote by $\mathcal{M}^\epsilon_{N+1}$ its matrix and by Δ_{N+1} the determinant of this matrix.

The equation $D_\theta y=0$ admits a polynomial solution P if and only if the determinant Δ_{N+1} (of dimension $N+1$) vanishes.

The resulting solution of the initial GSE is

$$y(x)=(x-1)^{\eta_1-1/2}(x+1)^{\eta_{-1}-1/2}\,P_N(x)\,\exp(\epsilon_\infty i\tau x)$$

Given a GSE $Dy = 0$ and an integer $N \geq 0$ as above, we consider the following facts:

1. one (or two) superdiagonal entry of a tridiagonal matrix vanishes;
2. a determinant Δ_{N+1} *factorises* (into two or three terms);
3. at (at least) one of the regular singular points the difference of exponents of D is *entire* ≥ 2 and there is *no logarithm* in the corresponding local solutions (an *apparent singularity* of D_θ);
4. a polynomial solution of degree N is a linear combination of $k \leq N$ *generalised Laguerre polynomials*.

Given a GSE $Dy = 0$ and an integer $N \geq 0$ as above, we consider the following facts:

1. one (or two) superdiagonal entry of a tridiagonal matrix vanishes;
2. a determinant Δ_{N+1} *factorises* (into two or three terms);
3. at (at least) one of the regular singular points the difference of exponents of D is *entire* ≥ 2 and there is *no logarithm* in the corresponding local solutions (an *apparent singularity* of D_θ);
4. a polynomial solution of degree N is a linear combination of $k \leq N$ *generalised Laguerre polynomials*.

We got rather complete relations between such facts. It is quite technical and I give only some indications.

Given a GSE $Dy = 0$ and an integer $N \geq 0$ as above, we consider the following facts:

1. one (or two) superdiagonal entry of a tridiagonal matrix vanishes;
2. a determinant Δ_{N+1} *factorises* (into two or three terms);
3. at (at least) one of the regular singular points the difference of exponents of D is *entire* ≥ 2 and there is *no logarithm* in the corresponding local solutions (an *apparent singularity* of D_θ);
4. a polynomial solution of degree N is a linear combination of $k \leq N$ *generalised Laguerre polynomials*.

We got rather complete relations between such facts. It is quite technical and I give only some indications.
This question was studied by Antoine Hautot around 1970. We followed some of his ideas.

Generalised Laguerre polynomials

We already met generalised Laguerre polynomials in the expression of the Schrödinger solutions of the radial equation of the hydrogen atom.

For $k \in \mathbb{N}$ and $\gamma \in \mathbb{C}$, the *generalised Laguerre polynomial* of degree k is :

$$L_k^{(\gamma)}(x) = \sum_{j=0}^{k} \binom{k+\gamma}{k-j} (-1)^j \frac{x^j}{j!}.$$

Generalised Laguerre polynomials

We already met generalised Laguerre polynomials in the expression of the Schrödinger solutions of the radial equation of the hydrogen atom.

For $k \in \mathbb{N}$ and $\gamma \in \mathbb{C}$, the *generalised Laguerre polynomial* of degree k is :

$$L_k^{(\gamma)}(x) = \sum_{j=0}^{k} \begin{pmatrix} k+\gamma \\ k-j \end{pmatrix} (-1)^j \frac{x^j}{j!}.$$

For any $\gamma \in \mathbb{C}$, and any $N \in \mathbb{N}$:

$$\left(L_0^{(\gamma)} = 1, L_1^{(\gamma)}(x), \cdots, L_N^{(\gamma)}(x)\right)$$

is a *basis* of the vector space of polynomials of degree not greater than N.

The polynomial $L_k^{(\gamma)}$ satisfies the *confluent hypergeometric differential equation* :

$$x\left(L_k^{(\gamma)}\right)''(x) + (\gamma+1-x)\left(L_k^{(\gamma)}\right)'(x) + kL_k^{(\gamma)}(x) = 0;$$

$E_{c,a}$ where $c = \gamma + 1$, $a = -k$.

Generalised Laguerre polynomials

If $j, k \in \mathbb{N}$, and $j \leq k$, then the polynomial $L_k^{(-j)}(x)$ has valuation j and more precisely :

$$L_k^{(-j)}(x) = (-x)^j \, \frac{(k-j)!}{k!} \, L_{k-j}^{(j)}(x).$$

If $j \in \mathbb{N}$, then

$$\left(L_k^{(-j)}\right)_{k \in \mathbb{N}, k \geq j}$$

is a *basis* of the vector space $x^j \mathbb{C}[x]$.

Factorisation of the determinant Δ_{N+1} and apparent singularities

We denote $\mathcal{M}_{k+1}^{(\epsilon)} = \left(U_{i,j}\right)_{0 \le i,j \le k}$ and by $\Theta_{k+1}^{(\epsilon)}$ its determinant.

We suppose $J \in \mathbb{N}^*$, $\eta_\epsilon = -J/2$ and $J < N$.

Then the determinant $\Delta_{N+1} = \Theta_{N+1}^{(\epsilon)}$ *factorises*:

$$\Delta_{N+1} = \Theta_{J+1}^{(\epsilon)} \Phi_{N-J}^{(\epsilon)}.$$

This corresponds to the vanishing of a *superdiagonal* entry of the tridiagonal matrix $\mathcal{M}_{N+1}^{\epsilon}$.

Factorisation of the determinant Δ_{N+1} and apparent singularities

We denote $\mathcal{M}_{k+1}^{(\epsilon)} = \left(U_{i,j}\right)_{0 \le i,j \le k}$ and by $\Theta_{k+1}^{(\epsilon)}$ its determinant.

We suppose $J \in \mathbb{N}^*$, $\eta_\epsilon = -J/2$ and $J < N$.

Then the determinant $\Delta_{N+1} = \Theta_{N+1}^{(\epsilon)}$ *factorises*:

$$\Delta_{N+1} = \Theta_{J+1}^{(\epsilon)} \Phi_{N-J}^{(\epsilon)}.$$

This corresponds to the vanishing of a *superdiagonal* entry of the tridiagonal matrix $\mathcal{M}_{N+1}^{\epsilon}$.

If the difference $J+1$ of the exponents at $x = \epsilon$ is an *integer* ≥ 2 and if *there is no logarithm* in the local solutions, we get an *apparent singularity* up to an s-homotopic transformation.

Factorisation of the determinant Δ_{N+1} and apparent singularities

We denote $\mathcal{M}_{k+1}^{(\epsilon)} = \left(U_{i,j}\right)_{0\le i,j\le k}$ and by $\Theta_{k+1}^{(\epsilon)}$ its determinant.

We suppose $J \in \mathbb{N}^*$, $\eta_\epsilon = -J/2$ and $J < N$.

Then the determinant $\Delta_{N+1} = \Theta_{N+1}^{(\epsilon)}$ *factorises*:

$$\Delta_{N+1} = \Theta_{J+1}^{(\epsilon)} \Phi_{N-J}^{(\epsilon)}.$$

This corresponds to the vanishing of a *superdiagonal* entry of the tridiagonal matrix $\mathcal{M}_{N+1}^{\epsilon}$.

If the difference $J+1$ of the exponents at $x=\epsilon$ is an *integer* ≥ 2 and if *there is no logarithm* in the local solutions, we get an *apparent singularity* up to an s-homotopic transformation.

The point $x=\epsilon$ is an apparent singularity of D_θ if and only if $\Theta_{J+1}^{(\epsilon)} = 0$ (*log free condition* in [MS] style).

Factorisation of determinants and Laguerre polynomials

We can “simplify” the GSE by suppression of one of the regular singular points. We get a confluent hypergeometric equation. Then it is natural to use one of the two bases of “Laguerre polynomials” $\mathcal{L}_k^{(\pm 1)}(x)$, $k = 0, 1, \cdots, N$, where

$$\mathcal{L}_k^{(\epsilon)}(x) = L_k^{(-1+2\eta_{-\epsilon})}(-2\epsilon_\infty i\tau(x+\epsilon)).$$

Factorisation of determinants and Laguerre polynomials

We can "simplify" the GSE by suppression of one of the regular singular points. We get a confluent hypergeometric equation. Then it is natural to use one of the two bases of "Laguerre polynomials" $\mathcal{L}_k^{(\pm 1)}(x)$, $k = 0, 1, \cdots, N$, where

$$\mathcal{L}_k^{(\epsilon)}(x) = L_k^{(-1+2\eta_{-\epsilon})}(-2\epsilon_\infty i\tau(x+\epsilon)).$$

The matrices $\mathcal{M}_{N+1}^{L,(-\epsilon)}$ and $\mathcal{M}_{N+1}^{L,(\epsilon)}$ of the operator D_θ in these bases are tridiagonal, as in the bases of polynomials in $x-1$ or $x+1$.

We have a *miraculous* result.

Lemma

The matrices $\mathcal{M}_{N+1}^{L,(-\epsilon)}$ and $\mathcal{M}_{N+1}^{L,(\epsilon)}$ are symmetric from each other with respect to the second diagonal . Hence, for all $k \in \{1, \cdots, N+1\}$:

$$\Theta_k^{L,(\epsilon)} = \Phi_k^{L,(-\epsilon)}.$$

This result is the (mysterious) key of our main theorem and of other results.

With the same hypotheses and notation as above on J and N, we get factorisations:

$$\Delta_{N+1} = \Theta_{J+1}^{L,(\epsilon)} \times \Phi_{N-J}^{L,(\epsilon)} = \Phi_{J+1}^{L,(-\epsilon)} \times \Theta_{N-J}^{L,(-\epsilon)}.$$

and equalities:

$$\Theta_{J+1}^{(\epsilon)} = \Theta_{J+1}^{L,(\epsilon)} = \Phi_{J+1}^{L,(-\epsilon)}, \quad \Phi_{N-J}^{(\epsilon)} = \Phi_{N-J}^{L,(\epsilon)} = \Theta_{N-J}^{L,(-\epsilon)}.$$

With the same hypotheses and notation as above on J and N, we get factorisations:

$$\Delta_{N+1} = \Theta_{J+1}^{L,(\epsilon)} \times \Phi_{N-J}^{L,(\epsilon)} = \Phi_{J+1}^{L,(-\epsilon)} \times \Theta_{N-J}^{L,(-\epsilon)}.$$

and equalities:

$$\Theta_{J+1}^{(\epsilon)} = \Theta_{J+1}^{L,(\epsilon)} = \Phi_{J+1}^{L,(-\epsilon)}, \quad \Phi_{N-J}^{(\epsilon)} = \Phi_{N-J}^{L,(\epsilon)} = \Theta_{N-J}^{L,(-\epsilon)}.$$

If J, K are integers such that $1 \leq J, K \leq N-1$, then there is a factorisation of Δ_{N+1} in *three* factors (vanishing of two superdiagonal terms).

With the same hypotheses and notation as above on J and N, we get factorisations:

$$\Delta_{N+1} = \Theta_{J+1}^{L,(\epsilon)} \times \Phi_{N-J}^{L,(\epsilon)} = \Phi_{J+1}^{L,(-\epsilon)} \times \Theta_{N-J}^{L,(-\epsilon)}.$$

and equalities:

$$\Theta_{J+1}^{(\epsilon)} = \Theta_{J+1}^{L,(\epsilon)} = \Phi_{J+1}^{L,(-\epsilon)}, \quad \Phi_{N-J}^{(\epsilon)} = \Phi_{N-J}^{L,(\epsilon)} = \Theta_{N-J}^{L,(-\epsilon)}.$$

If J, K are integers such that $1 \leq J, K \leq N-1$, then there is a factorisation of Δ_{N+1} in *three* factors (vanishing of two superdiagonal terms).

We use this fact to prove that the Regge-Wheeler equation (with spin $\sigma = 2$) appearing in the perturbation theory of the Schwarzschild black hole admits a generalised polynomial solution with a corresponding polynomial P_{2s+1} of degree $2s+1$ which is the sum of *only* 4 generalised Laguerre polynomials.

With the same hypotheses and notation as above on J and N, we get factorisations:

$$\Delta_{N+1} = \Theta^{L,(\epsilon)}_{J+1} \times \Phi^{L,(\epsilon)}_{N-J} = \Phi^{L,(-\epsilon)}_{J+1} \times \Theta^{L,(-\epsilon)}_{N-J}.$$

and equalities:

$$\Theta^{(\epsilon)}_{J+1} = \Theta^{L,(\epsilon)}_{J+1} = \Phi^{L,(-\epsilon)}_{J+1}, \quad \Phi^{(\epsilon)}_{N-J} = \Phi^{L,(\epsilon)}_{N-J} = \Theta^{L,(-\epsilon)}_{N-J}.$$

If J, K are integers such that $1 \leq J, K \leq N-1$, then there is a factorisation of Δ_{N+1} in *three* factors (vanishing of two superdiagonal terms).

We use this fact to prove that the Regge-Wheeler equation (with spin $\sigma = 2$) appearing in the perturbation theory of the Schwarzschild black hole admits a generalised polynomial solution with a corresponding polynomial P_{2s+1} of degree $2s+1$ which is the sum of *only* 4 generalised Laguerre polynomials.

This proves the remarkable efficiency of the Laguerre bases. These bases are also the key of the proof of our main theorem.

When the first step of Kovacic's algorithm finds a basis of solutions

According to Meixner and Schäfke, in the SE case ($\beta = \xi = 0$), the Kovacic algorithm finds a basis of generalised polynomial solutions if and only if the order $|\mu|$ is a *positive integer* and if $A_\mu = 0$, where $A_\mu \in \mathbb{C}[\lambda, \tau]$ is a polynomial obtained from the *non existence of a logarithm* in the local solutions at ± 1. If there is a generalised polynomial solution, then there is a basis of such solutions because of the symmetry $x \mapsto -x$ of the equation.

The Kovacic algorithm gives a determinant proportional to A_m.

When the first step of Kovacic's algorithm finds a basis of solutions

According to Meixner and Schäfke, in the SE case ($\beta = \xi = 0$), the Kovacic algorithm finds a basis of generalised polynomial solutions if and only if the order $|\mu|$ is a *positive integer* and if $A_\mu = 0$, where $A_\mu \in \mathbb{C}[\lambda, \tau]$ is a polynomial obtained from the *non existence of a logarithm* in the local solutions at ± 1. If there is a generalised polynomial solution, then there is a basis of such solutions because of the symmetry $x \mapsto -x$ of the equation.

The Kovacic algorithm gives a determinant proportional to A_m.

The case $\beta = 0$, $\xi \neq 0$ is open

When the first step of the Kovacic algorithm finds a basis of solutions

We consider a CHE with the irregular singular point at ∞.
We denote by $(2\beta, 2\tau x)$ the difference of the generalised exponents at ∞.
We suppose that the three diffferences of exponents

$$J+1, \quad K+1, \quad 2\beta$$

are *positive integers* and that $J, K \geq 1$. Without loss of generality we suppose $J > K$.

When the first step of the Kovacic algorithm finds a basis of solutions

We consider a CHE with the irregular singular point at ∞.
We denote by $(2\beta, 2\tau x)$ the difference of the generalised exponents at ∞.
We suppose that the three diffferences of exponents

$$J+1, \quad K+1, \quad 2\beta$$

are *positive integers* and that $J, K \geq 1$. Without loss of generality we suppose $J > K$.

The possible values for N are

$$N_1 = \frac{J+K}{2} - \beta, \quad N_2 = \frac{J+K}{2} + \beta.$$

We have $N_1 < N_2$.

Let $\Delta_{N_1+1} = 0$ and $\Delta'_{N_2+1} = 0$ be the corresponding necessary and sufficient conditions for the existence of suitable polynomials.

In the GSE case, for J, K, β fixed, these two determinants are polynomials in (λ, τ).

When the first step of the Kovacic algorithm finds a basis of solutions

Theorem

1. If $\beta \geq \frac{J-K}{2}$, for all $\tau \in \mathbb{C}^*$, $\lambda \in \mathbb{C}$ such that $\Delta_{N_1+1} = 0$, then the equation admits a basis of generalised polynomial solutions of the form

$$\begin{aligned} v_1(x) &= (x-1)^{-J/2}(x+1)^{-K/2}P_{N_1}(x)e^{-i\tau x} \\ v_2(x) &= (x-1)^{-J/2}(x+1)^{-K/2}P_{N_2}(x)e^{i\tau x}. \end{aligned}$$

   with polynomials P_{N_1} and P_{N_2} of degree N_1 and N_2 respectively.

   Moreover, P_{N_2} is a linear combination of J suitable generalised Laguerre polynomials.

2. If $\beta < \frac{J-K}{2}$, then there exists $\Phi_{K+1} \in \mathbb{C}[\lambda, \tau]$ (a determinant of order $K+1$) which is a common factor of Δ_{N_1+1} and Δ'_{N_2+1}.
   For all $\tau \in \mathbb{C}^*$ and $\lambda \in \mathbb{C}$ such that this factor vanishes, there is a basis (v_1, v_2) of solutions as above.

One parameter algebraic families of CHE with a diagonalisable differential Galois group

We return to the GSE case. The existence of families of GSE with a diagonalisable Galois group is a consequence of the factorisation of the determinants. The phenomenon emerged with the use of Laguerre bases.

One parameter algebraic families of CHE with a diagonalisable differential Galois group

We return to the GSE case. The existence of families of GSE with a diagonalisable Galois group is a consequence of the factorisation of the determinants. The phenomenon emerged with the use of Laguerre bases.

We suppose $\beta \neq 0$. Then a necessary condition for the existence of one parameter algebraic families of CHE with a basis of generalised algebraic solutions is that *the three differences of exponants $J+1$, $K+1$, 2β are positive integers, $J, K \geq 1$*. We have $J \neq K$ and without loss of generality we suppose $J > K$. We suppose $J,\ K,\ \beta$ *fixed*

One parameter algebraic families of CHE with a diagonalisable differential Galois group

We return to the GSE case. The existence of families of GSE with a diagonalisable Galois group is a consequence of the factorisation of the determinants. The phenomenon emerged with the use of Laguerre bases.

We suppose $\beta \neq 0$. Then a necessary condition for the existence of one parameter algebraic families of CHE with a basis of generalised algebraic solutions is that *the three differences of exponants $J+1$, $K+1$, 2β are positive integers, $J, K \geq 1$*. We have $J \neq K$ and without loss of generality we suppose $J > K$. We suppose J, K, β *fixed*

Using the theorem, we get the (only) two cases for which the GSE admits *a basis of generalised polynomial solutions all over an algebraic curve of the (τ, λ) plane*.

In each case one equation is calculable, $\Delta_{N_1+1} = 0$ in the first case and $\Phi_{K+1} = 0$ in the second. In the first case all the equations admitting a generalised polynomial solution belongs to the curve.

One parameter algebraic families of CHE with a diagonalisable differential Galois group

In the second case of the theorem, the intersection

$$\{\Delta_{N_1+1} = 0\} \cap \{\Delta_{N_2+1} = 0\}$$

is the union of the algebraic curve $\{\Phi_{K+1} = 0\}$ and of an algebraic set X. Generically X is a finite set but, a priori, it could in some cases contain an algebraic curve.

We proved the theorem using bases of Laguerre polynomials and corresponding factorisations.

Transcendental approach

3-points connections

We can connect the 3 singular points $x_1,\ x_2,\ x_3$ of a CHE (or 3 singular points of a Heun equation) by an open disc U of the Riemann sphere, that is $x_1, x_2, x_3 \in \partial U \approx S^1$ (and in the Heun case $x_4 \notin U$).

Proposition

A 3-special solution of a Heun equation or of a CHE is a generalised polynomial.

Generalised polynomial solutions and 3-points connections

We can connect the 3 singular points $x_1,\ x_2,\ x_3$ of a CHE (or 3 singular points of a Heun equation) by an open disc U of the Riemann sphere, that is $x_1, x_2, x_3 \in \partial U \approx S^1$ (and in the Heun case $x_4 \notin U$).

Proposition

A 3-special solution of a Heun equation or of a CHE is a generalised polynomial.

This is well known in the Heun case. After reduction of the equation to a convenient canonical form, the result follows immediately from Liouville theorem.
For the CHE, we reduce also the equation to a convenient canonical form. Then we must prove that an entire function which is a local special solution at infinity is a polynomial. This follows from an argument of summability followed by the application of the Liouville theorem.

THE CONNES-MOSCOVICI PROLATE SPECTRUM

THE CONNES-MOSCOVICI PROLATE SPECTRUM

A work in progress,

J.P. R., Françoise Richard-Jung, Jean Thomann

Figure: Alain Connes

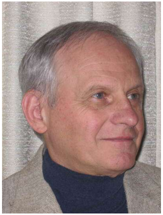


Figure: Henri Moscovici

In 2021, Alain Connes and Henri Moscovici discovered a new spectrum for the prolate spheroidal operator of order zero $W_\Lambda = -\frac{d}{dx}(\Lambda^2 - x^2)\frac{d}{dx} + (2\pi\Lambda x)^2$.

In 2021, Alain Connes and Henri Moscovici discovered a new spectrum for the prolate spheroidal operator of order zero $W_\Lambda = -\frac{d}{dx}(\Lambda^2 - x^2)\frac{d}{dx} + (2\pi\Lambda x)^2$.
This (formally) Sturm-Liouville operator admits two (logarithmic) regular-singular points at $\pm\Lambda$ and an irregular singular point at infinity. For CM, $\Lambda > 0$.

In 2021, Alain Connes and Henri Moscovici discovered a new spectrum for the prolate spheroidal operator of order zero $W_\Lambda = -\frac{d}{dx}(\Lambda^2 - x^2)\frac{d}{dx} + (2\pi\Lambda x)^2$.
This (formally) Sturm-Liouville operator admits two (logarithmic) regular-singular points at $\pm\Lambda$ and an irregular singular point at infinity. For CM, $\Lambda > 0$.
The classical spectrum is very well known. It is related to $[-\Lambda, \Lambda]$. The eigenvalues are ≥ 0.
The CM-spectrum is the spectrum of a self-ajoint extension $W_{\Lambda,sa}$ to $\mathbb{R}$ of W_Λ introduced by A. Connes in 1998.
CM consider mainly the *even* spectrum.

In 2021, Alain Connes and Henri Moscovici discovered a new spectrum for the prolate spheroidal operator of order zero $W_\Lambda = -\frac{d}{dx}(\Lambda^2 - x^2)\frac{d}{dx} + (2\pi\Lambda x)^2$.

This (formally) Sturm-Liouville operator admits two (logarithmic) regular-singular points at $\pm\Lambda$ and an irregular singular point at infinity. For CM, $\Lambda > 0$.

The classical spectrum is very well known. It is related to $[-\Lambda, \Lambda]$. The eigenvalues are ≥ 0.

The CM-spectrum is the spectrum of a self-ajoint extension $W_{\Lambda,sa}$ to $\mathbb{R}$ of W_Λ introduced by A. Connes in 1998.

CM consider mainly the *even* spectrum.

This spectrum is discrete. It contains a replica of the classical even spectrum and *new negative eigenvalues* μ_n.

CM uncovered a "miracle": for $\Lambda = \sqrt{2}$, these negative eigenvalues reproduce the UV behaviour of the squares of the imaginary parts of the non trivial zeroes of the *Riemann zeta function*.

In 2021, Alain Connes and Henri Moscovici discovered a new spectrum for the prolate spheroidal operator of order zero $W_\Lambda = -\frac{d}{dx}(\Lambda^2 - x^2)\frac{d}{dx} + (2\pi\Lambda x)^2$.

This (formally) Sturm-Liouville operator admits two (logarithmic) regular-singular points at $\pm\Lambda$ and an irregular singular point at infinity. For CM, $\Lambda > 0$.

The classical spectrum is very well known. It is related to $[-\Lambda, \Lambda]$. The eigenvalues are ≥ 0.

The CM-spectrum is the spectrum of a self-ajoint extension $W_{\Lambda,sa}$ to $\mathbb{R}$ of W_Λ introduced by A. Connes in 1998.

CM consider mainly the *even* spectrum.

This spectrum is discrete. It contains a replica of the classical even spectrum and *new negative eigenvalues* μ_n.

CM uncovered a "miracle": for $\Lambda = \sqrt{2}$, these negative eigenvalues reproduce the UV behaviour of the squares of the imaginary parts of the non trivial zeroes of the *Riemann zeta function*.

The UV prolate spectrum matches the zeros of zeta.

Fourier transform and limitation operators

Fourier transform $\mathbb{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, defined by

$$f(x) \mapsto \mathbb{F}f = \tilde{f}(y) = \int_{\mathbb{R}} f(x)e^{-2i\pi xy}dx$$

Space-limitation operator $P_\Lambda : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, defined by

$$P_\Lambda f = 1_{[-\Lambda,\Lambda]}f.$$

Multiplication by the characteristic function $1_{[-\Lambda,\lambda]}$.

We set $\widetilde{P}_\Lambda = \mathbb{F}P_\Lambda\mathbb{F}^{-1}$, *Band-limitation operator*.

The prolate operator W_Λ is *formally* selfadjoint. A. Connes introduced in 1998 a self-adjoint extension of W_Λ.

We have a symmetric operator $W_{\Lambda,min} : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ defined on the Schwarz space. The Connes self-adjoint extension $W_{\Lambda,sa}$ is *the unique self-adjoint extension commuting with P_Λ and $\widetilde{P}_\Lambda$*.
Moreover it *commutes with the Fourier transform $\mathbb{F}$*.

The prolate operator W_Λ is *formally* selfadjoint. A. Connes introduced in 1998 a self-adjoint extension of W_Λ.

We have a symmetric operator $W_{\Lambda,min} : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ defined on the Schwarz space. The Connes self-adjoint extension $W_{\Lambda,sa}$ is *the unique self-adjoint extension commuting with P_Λ and $\widetilde{P}_\Lambda$*.

Moreover it *commutes with the Fourier transform $\mathbb{F}$*.

Therefore $W_{\Lambda,sa}$ *commutes with the angle operator $P_\Lambda \widetilde{P}_\Lambda P_\Lambda$*.

This miraculous commutation property, *"the lucky accident"* was (re)discovered by a Bell Labs group, around D. Slepian, in relation with the study of transmission of signals.

The spectrum of $W_{\Lambda,sa}$ is discrete and unbounded on both sides; its negative eigenvalues are *simple*, while its positive eigenvalues (with possibly finitely many exceptions) have multiplicity *two*.

The non classical eigenfunctions are identically zero on $]-1,1[$. They belong to the Sonin space.

The spectrum of $W_{\Lambda,sa}$ is discrete and unbounded on both sides; its negative eigenvalues are *simple*, while its positive eigenvalues (with possibly finitely many exceptions) have multiplicity *two*.
The non classical eigenfunctions are identically zero on $]-1,1[$. They belong to the Sonin space.

Conjecture (in CM). The non classical eigenvalues are negative.
This was proved later by F. Richard-Jung, J. Thomann and J.P. R.

The spectrum of $W_{\Lambda,sa}$ is discrete and unbounded on both sides; its negative eigenvalues are *simple*, while its positive eigenvalues (with possibly finitely many exceptions) have multiplicity *two*.

The non classical eigenfunctions are identically zero on $]-1,1[$. They belong to the Sonin space.

Conjecture (in CM). The non classical eigenvalues are negative.

This was proved later by F. Richard-Jung, J. Thomann and J.P. R.

If φ is an eigenfunction of $W_{\Lambda,sa}$, then the leading term of its "asymptotic expansion" at infinity is proportional to

$$\frac{\sin(2\pi\Lambda x)}{x}$$

if φ is even and is proportional to

$$\frac{\cos(2\pi\Lambda x)}{x}$$

if φ is odd.

Spheroidal operators of arbitrary order

Setting $\tau = 2\pi\Lambda^2$ and $x = \Lambda\tilde{x}$ we rescale the band $[-\Lambda, \Lambda]$ into $[-1, 1]$. The operator W_Λ is transformed into $\mathcal{D}_\tau$ $(\tilde{x} \leftarrow x)$:

$$\mathcal{D}_\tau = -\frac{d}{dx}(1-x^2)\frac{d}{dx} + \tau^2 x^2 \quad \tau \in \mathbb{C}$$

Spectral version $\mathcal{D}_\tau - \mu y = 0$, $\mu \in \mathbb{C}$.
The spheroidal operators $\mathcal{D}_\tau$ are a subfamily of the operators

$$\mathcal{D}_{\tau,m} := (x^2-1)\left(\frac{d}{dx}\right)^2 + 2x\frac{d}{dx} + \left(\tau^2 x^2 - \frac{m^2}{x^2-1}\right), \quad \tau, m \in \mathbb{C}.$$

The parameter m is called the *order*. The differential operator $\mathcal{D}_{\tau,m}$ is called (abusively) a *spheroïdal wave operator*. If τ^2 is a positive real, then $\mathcal{D}_{\tau,m}$ is called *prolate*. If τ^2 is a negative real, then $\mathcal{D}_{\tau,m}$ is called *oblate*.

The traditional spheroidal operators appearing in the study of Helmoltz equations are slightly different:

$$\mathcal{S}_{\tau,m} := (x^2 - 1)\left(\frac{d}{dx}\right)^2 + 2x\frac{d}{dx} + \left(\tau^2(x^2 - 1) - \frac{m^2}{x^2 - 1}\right).$$

Then $\mathcal{D}_{\tau,m} - \mu = \mathcal{S}_{\tau,m} - \lambda$, where $\mu = \lambda + \tau^2$. *Be careful* with the spectral shift.

The classical prolate spectrum. Eigenfunctions

We consider only the order zero case ($m = 0$).

By definition (the naive definition) a function solution φ on $]-\Lambda, \Lambda[$ is an eigenfunction of W_Λ on $]-\Lambda, \Lambda[$ if it is *bounded* at $x = \pm\Lambda$.
This definition was extended to Λ *complex* by Meixner and Schäfke 1954.

The eigenfunctions are *even* or *odd*.

The classical prolate spectrum. Eigenfunctions

We consider only the order zero case ($m = 0$).

By definition (the naive definition) a function solution φ on $]-\Lambda, \Lambda[$ is an eigenfunction of W_Λ on $]-\Lambda, \Lambda[$ if it is *bounded* at $x = \pm\Lambda$.
This definition was extended to Λ *complex* by Meixner and Schäfke 1954.

The eigenfunctions are *even* or *odd*.

Equivalently φ is *analytic* at $-\Lambda$ and Λ [MS]. This function connects two local analytic solutions (invariant by the local monodromy) along $[-\Lambda, \Lambda]$.

Analytic definition of the spectrum.

Symmetry

Using the symmetry $x \mapsto -x$, we can generalize our notions of special solution and of analytic spectrum.

We consider the fixed point 0 as a “singular point” and the lines of even or odd solutions at this point as distinguished lines.

Infinity is also a fixed point and we consider the lines of formal even or odd solutions as formal distinguished lines and their sums as distinguished lines.

Symmetry

The equation $W_\lambda y = 0$ is *invariant by the symmetry* $x \mapsto -x$, which fixes 0 and ∞. Then 0 is a "false singular point". An eigenfunction φ connects an analytic solution at Λ and an even (or odd) solution at 0 (which is necessarily analytic).

Another analytic definition of the spectrum.

Symmetry

The equation $W_\lambda y = 0$ is *invariant by the symmetry* $x \mapsto -x$, which fixes 0 and ∞. Then 0 is a "false singular point". An eigenfunction φ connects an analytic solution at Λ and an even (or odd) solution at 0 (which is necessarily analytic).

Another analytic definition of the spectrum.

If $y_l = \mathfrak{P}(\Lambda - x)$ is the analytic solution at $x = \Lambda$, then

- the even spectrum is given by $y_l'(0) = 0$
- and the odd spectrum by $y_l(0) = 0$.

Symmetry

The equation $W_\lambda y = 0$ is *invariant by the symmetry* $x \mapsto -x$, which fixes 0 and ∞. Then 0 is a "false singular point". An eigenfunction φ connects an analytic solution at Λ and an even (or odd) solution at 0 (which is necessarily analytic).

Another analytic definition of the spectrum.

If $y_l = \mathfrak{P}(\Lambda - x)$ is the analytic solution at $x = \Lambda$, then

- the even spectrum is given by $y_l'(0) = 0$
- and the odd spectrum by $y_l(0) = 0$.

The functions $y_l(\tau^2, \mu)(0)$ and $y_l'(\tau^2, \mu)(0)$ are *entire of exponential order* $\leq 1/2$ in $(\tau^2, \mu) \in \mathbb{C}^2$. Cf. [MS].

Then $y_l(\tau^2, \mu)(0)$ and $y_l'(\tau^2, \mu)(0)$ are *functional determinants*. This allows efficient numerical computations of the eigenvalues.

There are infinite product formulas if $\Lambda \in \mathbb{R}$ [MS]. (The eigenvalues are simple zeros.)

The CM spectrum. Eigenfunctions

We consider the restrictions of the CM eigenfunctions, even or odd, to $]\Lambda, +\infty[$. One dimensional subspaces.

A function φ on $]\Lambda, +\infty[$ is the restriction of a CM-eigenfunction if it is a solution of W_Λ, it is bounded at $x = \Lambda$ and

- $\varphi = c\frac{sin(2\pi\Lambda x)}{x} + O(1/x^2)$ in the even case
- $\frac{cos(2\pi\Lambda x)}{x}$ in the odd case.

We denote (abusively) $\quad \varphi \sim c\frac{sin(2\pi\Lambda x)}{x}$.

The CM spectrum. Eigenfunctions

We consider the restrictions of the CM eigenfunctions, even or odd, to $]\Lambda, +\infty[$. One dimensional subspaces.

A function φ on $]\Lambda, +\infty[$ is the restriction of a CM-eigen-function if it is a solution of W_Λ, it is bounded at $x = \Lambda$ and

- $\varphi = c\frac{sin(2\pi\Lambda x)}{x} + O(1/x^2)$ in the even case
- $\frac{cos(2\pi\Lambda x)}{x}$ in the odd case.

We denote (abusively) $\varphi \sim c\frac{sin(2\pi\Lambda x)}{x}$.

Equivalent definition: φ is analytic at $x = \Lambda$ and at ∞ it is the 1-sum (Borel-sum) in the direction $\mathbb{R}^+$ of an even (resp. odd) formal solution.

Analytic definition of the spectrum

by summation and analytic connection.

Sonin space

Sonin space: functions of $L^2(\mathbb{R})$ which vanishes identically near the origin as well as their Fourier transform.

Sonin space

Sonin space: functions of $L^2(\mathbb{R})$ which vanishes identically near the origin as well as their Fourier transform.

Introduced by L. de Branges. Non triviality is not evident.
A. Connes and H. Moscovici proved:

The non classical CM eigenfunctions belong to the Sonin space.

Sonin space

Sonin space: functions of $L^2(\mathbb{R})$ which vanishes identically near the origin as well as their Fourier transform.

Introduced by L. de Branges. Non triviality is not evident.
A. Connes and H. Moscovici proved:

The non classical CM eigenfunctions belong to the Sonin space.

Before [CM], A. Connes and C. Consani used Sonin space for their study of Weil's positivity.

Functional determinant

In spectral theory, a *functional determinant* is an entire function $F(\mu)$ whose zeros are the eigenvalues. In some works functional determinants are obtained using *analytic continuation* (Gelfand-Yaglom, Volkmer, ...).

We use a similar idea.

Functional determinant by analytical matching

We work with D_τ. Let $\psi \sim -\frac{\sin \tau x}{x}$ *interpreted as a Borel sum of a formal solution*.

Basis of solutions at $x = 1$: (y_I, y_{II}), y_I is analytic, $y_I(1) = 1$, $W(y_I, y_{II}) = \frac{1}{1-x^2}$: [MS] normalization.

Functional determinant by analytical matching

We work with D_τ. Let $\psi \sim -\frac{\sin \tau x}{x}$ *interpreted as a Borel sum of a formal solution*.

Basis of solutions at $x = 1$: (y_I, y_{II}), y_I is analytic, $y_I(1) = 1$, $W(y_I, y_{II}) = \frac{1}{1-x^2}$: [MS] normalization.

We have $\psi = \alpha y_I + \beta y_{II}$: connection formula for $1 \leftrightarrow +\infty$. Then

$$\alpha = W(\psi, y_{II})/W(y_I, y_{II}) = (1 - x^2)W(\psi, y_{II}),$$
$$\beta = W(\psi, y_I)/W(y_I, y_{II}) = (1 - x^2)W(\psi, y_I))$$

Functional determinant by analytical matching

We work with D_τ. Let $\psi \sim -\frac{\sin \tau x}{x}$ *interpreted as a Borel sum of a formal solution*.

Basis of solutions at $x = 1$: (y_I, y_{II}), y_I is analytic, $y_I(1) = 1$, $W(y_I, y_{II}) = \frac{1}{1-x^2}$: [MS] normalization.

We have $\psi = \alpha y_I + \beta y_{II}$: connection formula for $1 \leftrightarrow +\infty$. Then

$$\alpha = W(\psi, y_{II})/W(y_I, y_{II}) = (1 - x^2)W(\psi, y_{II}),$$
$$\beta = W(\psi, y_I)/W(y_I, y_{II}) = (1 - x^2)W(\psi, y_I))$$

For $\tau = \tau_0$ fixed, y_I and ψ are entire functions of the spectral parameter μ, of order $\leq 1/2$. Therefore

$$F(\tau_0, \mu) = -W(\psi, y_I)(\sqrt{2})(\tau_0, \mu) = \beta$$

and F is a functional determinant for the CM spectrum. This allows a numerical computation of the CM eigenvalues. Moreover $F(\tau_0, \mu)$ *is entire of order* $\leq 1/2$. (Proved with Reinhard Schäfke's help for ψ; [MS] for y_I.).

New eigenfunctions

for

the negative prolate eigenvalues

A new interpretation of the *negative* CM spectrum

New eigenfunctions

for

the negative prolate eigenvalues

A new interpretation of the *negative* CM spectrum

The charms of bispectrality

Some progresses

We discovered *new eigenfunctions* for the non classical part of the CM spectrum. These new eigenfunctions allow a very easy and efficient computation of the eigenvalues.

Some progresses

We discovered *new eigenfunctions* for the non classical part of the CM spectrum. These new eigenfunctions allow a very easy and efficient computation of the eigenvalues.

Using our new very elementary definition of the non trivial part of the CM spectrum, we can prove that the non classical CM eigenvalues are *negative*. This was conjectured by CM. After their work it remained the possibility of a finite set of positive new eigenvalues.

Some progresses

We discovered *new eigenfunctions* for the non classical part of the CM spectrum. These new eigenfunctions allow a very easy and efficient computation of the eigenvalues.

Using our new very elementary definition of the non trivial part of the CM spectrum, we can prove that the non classical CM eigenvalues are *negative*. This was conjectured by CM. After their work it remained the possibility of a finite set of positive new eigenvalues.

We guessed the new eigenfunctions from the fact that the Connes-Moscovici eigenfunctions belongs to the Sonin space, thinking to the counterpart of this fact for the complex analytic solutions.

Naive eigenvalues

The classical prolate eigenvalues are the $\mu \in \mathbb{C}$ such that $\mathcal{D}_\tau - \mu$ admits a *bounded* solution on $]-1,1[$. If $\tau \in \mathbb{R}^*$, they are *real positive*.

Naive eigenvalues

The classical prolate eigenvalues are the $\mu \in \mathbb{C}$ such that $\mathcal{D}_\tau - \mu$ admits a *bounded* solution on $]-1, 1[$. If $\tau \in \mathbb{R}^*$, they are *real positive*.

We consider now, for $\tau > 0$, the naive eigenvalues on the *imaginary axis*, that is the $\mu \in \mathbb{C}$ such that $\mathcal{D}_\tau - \mu$ admits a *bounded* solution on $\mathbb{R}i =]-i\infty, +i\infty[$.

Naive eigenvalues

The classical prolate eigenvalues are the $\mu \in \mathbb{C}$ such that $\mathcal{D}_\tau - \mu$ admits a *bounded* solution on $]-1,1[$. If $\tau \in \mathbb{R}^*$, they are *real positive*.

We consider now, for $\tau > 0$, the naive eigenvalues on the *imaginary axis*, that is the $\mu \in \mathbb{C}$ such that $\mathcal{D}_\tau - \mu$ admits a *bounded* solution on $\mathbb{R}i =]-i\infty, +i\infty[$.

We proved that if μ is a naive eigenvalue on $\mathbb{R}i$, then

- μ is real and $\mu < 0$;
- the eigenspaces are of dimension one;
- the eigenfunctions are *even* or *odd* and they admit an exponential decay at infinity on $\mathbb{R}i$ (at $\pm i\infty$).

Naive eigenvalues

The classical prolate eigenvalues are the $\mu \in \mathbb{C}$ such that $\mathcal{D}_\tau - \mu$ admits a *bounded* solution on $]-1, 1[$. If $\tau \in \mathbb{R}^*$, they are *real positive*.

We consider now, for $\tau > 0$, the naive eigenvalues on the *imaginary axis*, that is the $\mu \in \mathbb{C}$ such that $\mathcal{D}_\tau - \mu$ admits a *bounded* solution on $\mathbb{R}i =]-i\infty, +i\infty[$.
We proved that if μ is a naive eigenvalue on $\mathbb{R}i$, then

- μ is real and $\mu < 0$;
- the eigenspaces are of dimension one;
- the eigenfunctions are *even* or *odd* and they admit an exponential decay at infinity on $\mathbb{R}i$ (at $\pm i\infty$).

We also proved that μ is a naive eigenvalue on $\mathbb{R}i$ if and only if it is a non-classical CM eigenvalue. This implies that the naive spectrum is *infinite*, in particular non void ! (No direct proof.) This implies also that the non classical CM eigenvalues are negative.

The CM eigenfunctions as boundary values of the naive eigenfunctions

How to recover the (non-classical) CM-eigenfunctions from the new eigenfunctions ?

The CM eigenfunctions as boundary values of the naive eigenfunctions

How to recover the (non-classical) CM-eigenfunctions from the new eigenfunctions ?

We recall that if $U \subset \mathbb{C}$ is open and if f is holomorphic on U, then f admits a boundary value at $t \in U \cap \mathbb{R}$ if the limit

$$\lim_{\varepsilon \to 0, \varepsilon > 0} f(t - i\varepsilon) - f(t + i\varepsilon)$$

exists. By definition, this limit is the boundary value $\mathrm{BV} f(t)$.

The CM eigenfunctions as boundary values of the naive eigenfunctions

How to recover the (non-classical) CM-eigenfunctions from the new eigenfunctions ?

We recall that if $U \subset \mathbb{C}$ is open and if f is holomorphic on U, then f admits a boundary value at $t \in U \cap \mathbb{R}$ if the limit

$$\lim_{\varepsilon \to 0, \varepsilon > 0} f(t - i\varepsilon) - f(t + i\varepsilon)$$

exists. By definition, this limit is the boundary value $\mathrm{BV}f(t)$.

If μ is a naive eigenvalue on $i\mathbb{R}$, then the naive eigenfunction is the sum of a (even or odd) power series solution at 0 of $\mathcal{D}_\tau - \mu$. This solution extends analytically into an holomorphic function f on the cut plane $\mathbb{C} \setminus ([0, +\infty[\cup] - \infty, -1[)$. Then the boundary value $\mathrm{BV}f$ is a CM eigenfunction. If f is even (resp. odd), then $\mathrm{BV}f$ is odd (resp. even).

Numerical computation of the CM-eigenvalues

Connes and Moscovici used a quite rough computation of their eigenvalues based on a matching of oscillating functions (it suffices for their purposes in relation with zeta zeros). We introduced an efficient method ("analytical matching"), based on analytic continuation *and* Borel summation. However the computations are rather complicated and lengthy, due mainly to problems with numerical resommation. Using our new eigenfunctions we got a *very simple, quick and efficient method* that I will explain.

Numerical computation of the CM-eigenvalues

Connes and Moscovici used a quite rough computation of their eigenvalues based on a matching of oscillating functions (it suffices for their purposes in relation with zeta zeros). We introduced an efficient method ("analytical matching"), based on analytic continuation *and* Borel summation. However the computations are rather complicated and lengthy, due mainly to problems with numerical resommation. Using our new eigenfunctions we got a *very simple, quick and efficient method* that I will explain.

In 1981, in his PhD thesis in Strasbourg, Jean-Louis Callot proposed a very similar method for a description of the Hermite (and similar) spectrum: "opening the curtains". In fact, for τ big, prolate and Hermite spectrums are "similar" as I will explain later.

Opening the curtains

We start from an odd (resp. even) power series solution at $x = 0$ and we search μ such that its sum remains *bounded* on $i\mathbb{R}_+$.
We use a dichotomy method. It is similar to the case of the *"chasse au canard"* (duck hunting) studied by the Reeb school.
If $\tilde{\mu}$ is close to a naive eigenvalue $\mu < 0$, with $\tilde{\mu} \neq \mu$ then the corresponding eigenfunction "explodes at $\pm i\infty$". We rotate the picture, replacing x by ix and we consider the graphs of the functions on $[0, +\infty[$. Then, we observe an explosion at $\approx x_0$: the graph looks like a vertical half-line going up or down. Before we observe a domain of oscillations on $[0, x_0[$. We move $\tilde{\mu}$ (upward or lowward). If the explosion occures later (x_0 bigger), then we are nearer μ. If a upper vertical half-line becomes a lower vertical half-line (or conversely), then we have crossed μ.

Opening the curtain

Numerical experiments

We used *SageMath* with a great certified precision for our curves (Marc Mezzarobba's package). The choice between upper or lower explosion on our pictures is "sure" even if not formally certified. Our intervals for the eigenvalues are sure.

Our conjectures on the number of zeroes allow to guess the waited sign for the explosion.

Numerical experiments

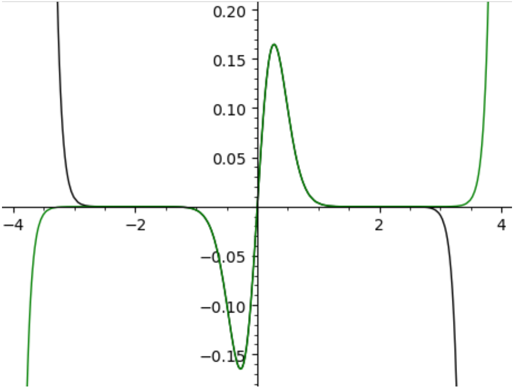
First even negative eigenvalue. CM value: -39.

Our analytic matching value: -39.383216574(4)

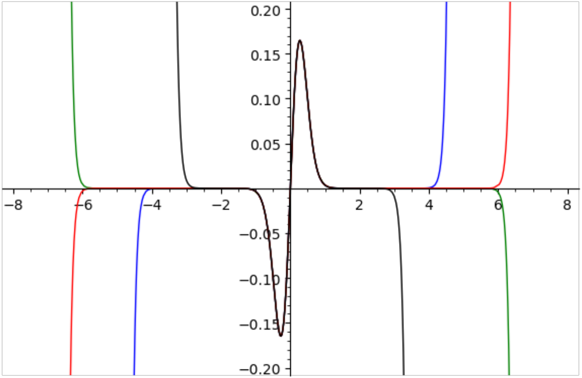
Curtain method:

DOWN: -39.3832165745

UP: -39.383216574261; BEST CURVE.



Numerical experiments



Numerical experiments

$$-39.383216574261539476150563 22 < \mu$$
$$\mu < -39.383216574261539476150563 17.$$

On the right, first even negative eigenvalue:

$\mu = -39.3832165742615394761505 63$

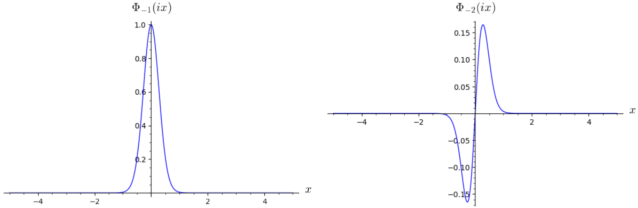


FIGURE 15 – On the left, $\mu = -13.30274638329156253742394584086 9$ is the first negative eigenvalue, it is an odd eigenvalue. The function $\Phi_{-1}(ix)$ is even.
On the right, $\mu = -39.3832165742615394761505 63$ is the second negative eigenvalue, it is an even eigenvalue. The function $\Phi_{-2}(ix)$ is odd.

Numerical experiments

Negative eigenvalue of rank $n = 148$. CM value: -9100.

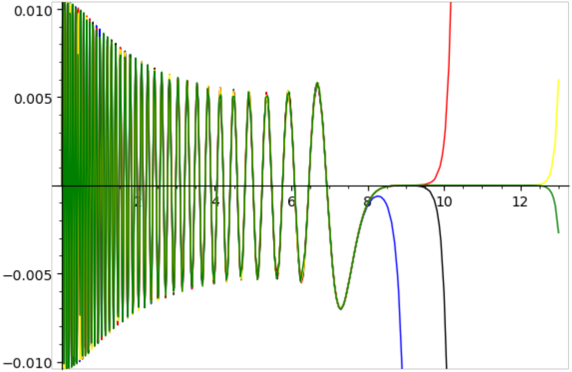
Our analytic matching value: -9104.3331714128495040.

Curtain method:

DOWN: -9104, -9104.3331, -9104.3331714128495039

UP: -9104.3332, -9104.3331714128495040

BEST VALUE, BEST CURVE



$$-9104.3331714128495039 < \mu < -9104.3331714128495040$$

Our best upper and lower bounds (it is easy to improve):

$$-9104.333171412849503985 < \mu < -9104.333171412849503994.$$

LINEAR PERTURBATIONS OF BLACK-HOLES

LINEAR PERTURBATIONS OF BLACK-HOLES

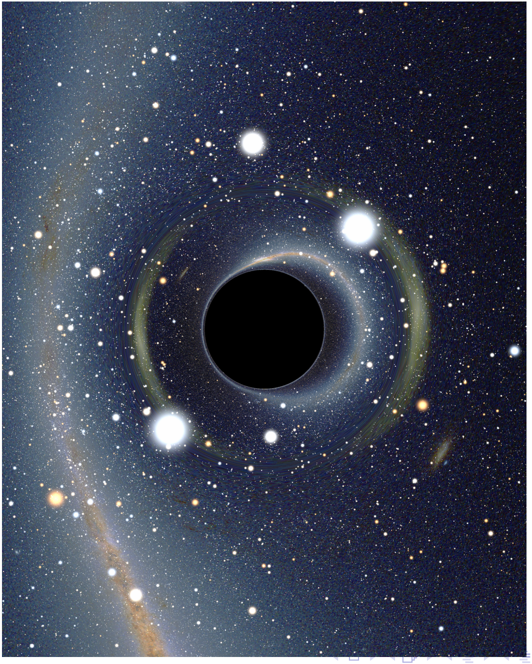
Algebraically special solutions and Quasi-Normal-Modes

LINEAR PERTURBATIONS OF BLACK-HOLES

Algebraically special solutions and Quasi-Normal-Modes

A work in progress,

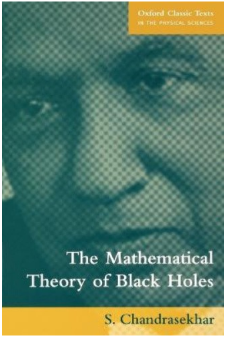
Anne Duval, Michèle Loday-Richaud and J.P. R.



Black holes

The mathematical theory of black holes

Figure: Subrahmanyan Chandrasekhar



Black holes

In 1915, Albert Einstein developed his theory of general relativity. A few months later, Karl Schwarzschild found a solution to the Einstein field equations that describes the gravitational field of a punctual mass. It was later interpreted as a black hole.

Black holes

In 1915, Albert Einstein developed his theory of general relativity. A few months later, Karl Schwarzschild found a solution to the Einstein field equations that describes the gravitational field of a punctual mass. It was later interpreted as a black hole.

PROLOGUE

The black holes of nature are the most perfect macroscopic objects there are in the universe: the only elements in their construction are our concepts of space and time. And since the general theory of relativity provides only a single unique family of solutions for their descriptions, they are the simplest objects as well.

The unique two-parameter family of solutions which describes the space-time around black holes is the Kerr family discovered by Roy Patrick Kerr in July, 1963. The two parameters are the mass of the black hole and the angular momentum of the black hole. The static solution, with zero angular momentum, was discovered .by Karl Schwarzschild in December, 1915. A study of the black holes of nature is then a study of these solutions. It is to this study that this book is devoted.

$$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right) dt^2 - \frac{dr^2}{\left(1 - \frac{2GM}{c^2 r}\right)} - r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

This metric is *singular* at the *Schwarzschild radius* (the horizon)

$$r_S = \frac{2GM}{c^2}.$$

Figure: Karl Schwarzschild 1873-1916

Linear perturbations of black holes

Linear perturbations of black holes

Perturbations of stars and black holes have been one of the main topics of relativistic astrophysics for the last few decades. They are of particular importance today, because of their relevance to gravitational wave astronomy.

Linear perturbations of black holes

Perturbations of stars and black holes have been one of the main topics of relativistic astrophysics for the last few decades. They are of particular importance today, because of their relevance to gravitational wave astronomy.

A perturbation of a black hole can be either gravitational or electromagnetic or scalar (or by neutrinos ...). For example the field that accompanies a particle falling along a geodesic of the geometry, produced by the black hole, can be considered as a perturbation on the background geometry. The mathematical treatment of the black hole perturbation theory (in the Schwarzschild case) was first originated by Regge and Wheeler and later was continued by Zerilli. Their motivation was the problem of stability of black holes.

Linear perturbations of black holes

I quote Chandrasekhar (The mathematical theory of BH):

. *"The treatment of perturbations of Kerr spacetime has been prolixious in its complexity. Perhaps at a later time the complexity will be unraveled by deeper insights. But meantime the analysis has led into a realm of the rococo, splendorous, joyful and immensely ornate."*

Linear perturbations of black holes

I quote Chandrasekhar (The mathematical theory of BH):

. *"The treatment of perturbations of Kerr spacetime has been prolixious in its complexity. Perhaps at a later time the complexity will be unraveled by deeper insights. But meantime the analysis has led into a realm of the rococo, splendorous, joyful and immensely ornate."*

The exuberant decoration of baroque churches is carried by a simple and beautiful architecture. I think that similarly the rococo and immensely ornate world of black-holes linear perturbations is carried by a simple (!) and beautiful 'geometry'.

Linear perturbations of black holes

I quote Chandrasekhar (The mathematical theory of BH):

. *"The treatment of perturbations of Kerr spacetime has been prolixious in its complexity. Perhaps at a later time the complexity will be unraveled by deeper insights. But meantime the analysis has led into a realm of the rococo, splendorous, joyful and immensely ornate."*

The exuberant decoration of baroque churches is carried by a simple and beautiful architecture. I think that similarly the rococo and immensely ornate world of black-holes linear perturbations is carried by a simple (!) and beautiful 'geometry'. A candidate for this geometry is the theory of *wild linear representations* and wild *Riemann-Hilbert map*. *Painlevé five, its (wild) fundamental group and its wild character variety, an affine cubic surface, seem central* (as ellipses in baroque architecture).

Linear perturbations of black holes

I quote Chandrasekhar (The mathematical theory of BH):

. *"The treatment of perturbations of Kerr spacetime has been prolixious in its complexity. Perhaps at a later time the complexity will be unraveled by deeper insights. But meantime the analysis has led into a realm of the rococo, splendorous, joyful and immensely ornate."*

The exuberant decoration of baroque churches is carried by a simple and beautiful architecture. I think that similarly the rococo and immensely ornate world of black-holes linear perturbations is carried by a simple (!) and beautiful 'geometry'. A candidate for this geometry is the theory of *wild linear representations* and wild *Riemann-Hilbert map*. *Painlevé five, its (wild) fundamental group and its wild character variety, an affine cubic surface, seem central* (as ellipses in baroque architecture).

A similar idea appears in articles of two Brazilian physicists, B. C. da Cunha and J. P. Cavalcante 2020, RH being 'calculated' by *confluent conformal blocks*.

San Andrea al Quirinale in Roma.

Architect and decorations drawings: Gian Lorenzo Bernini.

Regge-Wheeler and Zerilli equations

The equations governing the evolution of external sourceless test fields (be it scalar, electromagnetic, or a perturbation of the gravitational field itself) on the background geometry of a Schwarzschild black hole can be reduced to a single second order linear ordinary differential equation (the *radial* equation obtained after a separation of variables for the Klein-Gordon equation corresponding to the linearisation of the problem):

$$\psi = e^{i(\omega t+m\phi)}S(\theta)R(r).$$

Regge-Wheeler and Zerilli equations

The equations governing the evolution of external sourceless test fields (be it scalar, electromagnetic, or a perturbation of the gravitational field itself) on the background geometry of a Schwarzschild black hole can be reduced to a single second order linear ordinary differential equation (the *radial* equation obtained after a separation of variables for the Klein-Gordon equation corresponding to the linearisation of the problem):

$$\psi = e^{i(\omega t + m\phi)} S(\theta) R(r).$$

The perturbations of a Schwarzchild metric are classified into two types, namely, axial (odd) and polar (even) perturbations, a terminology introduced by S. Chandrasekhar.
The equation for *axial perturbations* is called the *Regge-Wheeler equation* (1957) and the equation governing *polar perturbations* is called the *Zerilli equation* (1970).

These equations are named after Tullio Regge, John Archibald Wheeler and Frank J. Zerilli.

Regge-Wheeler equation

We choose units such that $G = c = M = 1$. In these units the Schwarzschild horizon is at $r = 2$. We introduce the *tortoise coordinate* r_* defined by

$$r_* = r + \log(r - 2).$$

In this coordinate the horizon $r = 2$ corresponds to $r_* = -\infty$. The coordinate r_* take its values in $\mathbb{R}$.

Regge-Wheeler equation

We choose units such that $G = c = M = 1$. In these units the Schwarzschild horizon is at $r = 2$. We introduce the *tortoise coordinate* r_* defined by

$$r_* = r + \log(r - 2).$$

In this coordinate the horizon $r = 2$ corresponds to $r_* = -\infty$. The coordinate r_* take its values in $\mathbb{R}$.
The Regge-Wheeler equation is

$$\left(-\left(\frac{d}{dr_*}\right)^2 + V\left(r(r_*)\right) - \omega^2\right)\psi = 0.$$

with the *Regge-Wheeler potential*

$$V(r) = \left(1 - \frac{2}{r}\right)\left(\frac{\ell(\ell+1)}{r^2} + \frac{2(1-\sigma^2)}{r^3}\right),$$

There are solutions on the real line behaving as $\psi \sim e^{\pm i\omega r_*}$ at infinity (*outgoing and ingoing waves at $r_* = \pm\infty$*).

A CHE form of the RW equation

The RW equation describes the radial dependence of a spin-σ perturbation of the background, with orbital angular momentum $\ell \in \mathbb{N}$ and time dependence $e^{-i\omega t}$.

We set $s^2 = -4\omega^2$. By changing the independent variable from r_* to r and using the s-homotopic transformation defined by $g = \left(1 - \frac{2}{r}\right)^{-1/2}$, we get the equation

$$\frac{d^2\phi}{d^2 r} - R\phi = 0,$$

where $R(r) = \frac{r^2}{(r-2)^2}\left(V(r) + \frac{s^2}{4} - \frac{2}{r^3} + \frac{3}{r^4}\right)$

Using the change of independent variable by translation $x = r - 1$ we get an equation in reduced form with regular singular points at $x = \pm 1$.

A GSE reduced form of the RW equation

Using the change of independent variable $x = r - 1$, we get a GSE in reduced form with parameters satisfying the two conditions

$$\mu + \xi + 4i\tau = \beta + 2i\tau = 0.$$

We have

$$\tau = is/2, \quad \mu = s + \sigma, \quad \xi = s - \sigma, \quad \beta = s, \quad \lambda = 2s^2 + \ell(\ell + 1).$$

A GSE reduced form of the RW equation

Using the change of independent variable $x = r - 1$, we get a GSE in reduced form with parameters satisfying the two conditions

$$\mu + \xi + 4i\tau = \beta + 2i\tau = 0.$$

We have

$$\tau = is/2, \quad \mu = s + \sigma, \quad \xi = s - \sigma, \quad \beta = s, \quad \lambda = 2s^2 + \ell(\ell + 1).$$

We have

$$e^{-i\omega r_*} = e^{-\frac{s}{2}(x+1)}(x-1)^{\frac{s}{2}},$$

giving asymptotics corresponding to a pure wave at $x = 1$ (i.e. $r = 1$, $r_* = -\infty$) and

$$e^{-i\omega r_*} \sim e^{-\frac{s}{2}x} x^{\frac{s}{2}}$$

at $x = +\infty$ (i.e. $r = +\infty$, $r_* = +\infty$).

We get an interpretation of the *local special solutions* at $x = 1$ and $x = \infty$ as *pure waves* and the physical explanation of the two conditions.

Quasi Normal Modes

Quasi Normal Modes

Natural oscillation frequencies of BHs, their "song"

Quasi Normal Modes

Natural oscillation frequencies of BHs, their “song”

Analogy with a ringing bell or a striken wine glass

Quasi Normal Modes (QNM)

The fingerprint of a black hole

The *quasinormal modes* of black holes were discovered by Vishveshwara in 1970 and popularized by Chandrasekhar.

Quasi Normal Modes (QNM)

The fingerprint of a black hole

The *quasinormal modes* of black holes were discovered by Vishveshwara in 1970 and popularized by Chandrasekhar. The gravitational waves emitted by a perturbed black hole are described by a superposition of exponentially decaying sinusoidal modes, called quasinormal modes (QNMs), whose *complex* frequencies depend only on the properties of the black hole (mass, kinetic moment and charge).

Quasi Normal Modes (QNM)

The fingerprint of a black hole

The *quasinormal modes* of black holes were discovered by Vishveshwara in 1970 and popularized by Chandrasekhar.

The gravitational waves emitted by a perturbed black hole are described by a superposition of exponentially decaying sinusoidal modes, called quasinormal modes (QNMs), whose *complex* frequencies depend only on the properties of the black hole (mass, kinetic moment and charge).

The real part of these frequencies corresponds to the frequency of vibration, and the imaginary part corresponds to the rate at which each mode is damped as a result of the emission of radiation.

Quasi Normal Modes (QNM)

The fingerprint of a black hole

The *quasinormal modes* of black holes were discovered by Vishveshwara in 1970 and popularized by Chandrasekhar. The gravitational waves emitted by a perturbed black hole are described by a superposition of exponentially decaying sinusoidal modes, called quasinormal modes (QNMs), whose *complex* frequencies depend only on the properties of the black hole (mass, kinetic moment and charge).
The real part of these frequencies corresponds to the frequency of vibration, and the imaginary part corresponds to the rate at which each mode is damped as a result of the emission of radiation.
QNM correspond to solutions of the perturbation equations satisfying *purely ingoing conditions* at the event horizon and *purely outgoing conditions* at infinity.

Gravitational waves and black holes

Observations a century after the fundamental predictions of Einstein (in 1916) and Schwarzschild (in december 1915) about GW and BH.

Gravitational waves and black holes

Observations a century after the fundamental predictions of Einstein (in 1916) and Schwarzschild (in december 1915) about GW and BH.

I quote the conclusion of B. P. Abbott and al. announcement (11 February 2016):

The LIGO detectors have observed gravitational waves from the merger of two stellar-mass black holes. The detected waveform matches the predictions of general relativity for the inspiral and merger of a pair of black holes and the ringdown of the resulting single black hole. These observations demonstrate the existence of binary stellar-mass black hole systems. This is the first direct detection of gravitational waves and the first observation of a binary black hole merger.

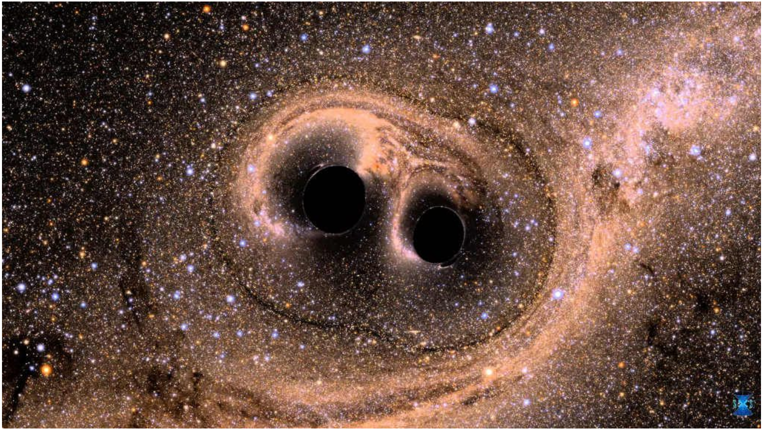
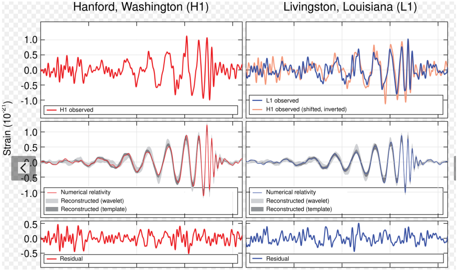


Figure: Binary system of coalescing BHs

Observation of Gravitational Waves from a Binary Black Hole Merger. First detection of GWs and observation of the BH QNMs

The first direct observation of gravitational waves was made on 14 September 2015 and was announced by the LIGO and Virgo collaborations on 11 February 2016. (The names of authors cover three pages.) It was also the first direct observation of a binary black hole merger.



The signal ... matches the waveform predicted by general relativity for the inspiral and merger of a pair of black holes and the ringdown of the resulting single black hole.

BH ringdown

The BH ringdown is the final stage of the GW signal emitted by a binary system of coalescing BHs. As the name suggests, the remnant BH, rings for a short time after the merger, as it settles down to an equilibrium configuration. The physics of the ringdown can be well understood within the framework of BH perturbation theory; QNM.

Our mathematical interpretation of pure waves

Physicists gave characterisations of purely ingoing (resp. outgoing) conditions in terms of particular solutions of the radial equations at the singular points. Recently some authors proposed a precise description in terms of *monodromy* and *Stokes phenomenon*. I quote A. Castro and al. 2013 (they consider the Kerr case, the Schwarzschild case is similar but easier):

Our mathematical interpretation of pure waves

Physicists gave characterisations of purely ingoing (resp. outgoing) conditions in terms of particular solutions of the radial equations at the singular points. Recently some authors proposed a precise description in terms of *monodromy* and *Stokes phenomenon*. I quote A. Castro and al. 2013 (they consider the Kerr case, the Schwarzschild case is similar but easier):
Therefore, our first task is to define ingoing and outgoing modes at the outer horizon and infinity, keeping the role of the monodromy as explicit as possible throughout the process.

Our mathematical interpretation of pure waves

Physicists gave characterisations of purely ingoing (resp. outgoing) conditions in terms of particular solutions of the radial equations at the singular points. Recently some authors proposed a precise description in terms of *monodromy* and *Stokes phenomenon*. I quote A. Castro and al. 2013 (they consider the Kerr case, the Schwarzschild case is similar but easier):

Therefore, our first task is to define ingoing and outgoing modes at the outer horizon and infinity, keeping the role of the monodromy as explicit as possible throughout the process.

In this line, we define locally (at a singular point, corresponding to the horizon or at infinity) a *pure wave solution* as a local special solution in KPR sense, that is

- a solution invariant by the local monodromy at a regular singular point (horizon);
- a 1-sum of a purely exponential formal solution at infinity.

The QNM spectrum as an analytic spectrum

Our definition of a pure wave solution at a regular singular point, corresponding to the Schwarzschild horizon, is natural if one uses the *tortoise coordinate*: $r_* = r + \log(r - 2)$.

The QNM spectrum as an analytic spectrum

Our definition of a pure wave solution at a regular singular point, corresponding to the Schwarzschild horizon, is natural if one uses the *tortoise coordinate*: $r_* = r + \log(r - 2)$.

Then the QNM spectrum is defined as an *analytic spectrum* associated to the connection between a regular singular point and an irregular singular point.

The QNM spectrum as an analytic spectrum

Our definition of a pure wave solution at a regular singular point, corresponding to the Schwarzschild horizon, is natural if one uses the *tortoise coordinate*: $r_* = r + \log(r-2)$.

Then the QNM spectrum is defined as an *analytic spectrum* associated to the connection between a regular singular point and an irregular singular point.

This allows a possible calculation by analytic matching, using numerical summability and analytic continuation (Marc Mezzarobba's package in Sagemath). We used this method for one of our calculations of the Connes-Moscovici spectrum.
One of the classical methods of calculations of QNM is based on an argument of continued fractions derived from a similar case in quantum chemistry [Leaver].
Our approach could give more precise and faster calculations. An efficient method of numerical summability is waited.

Algebraically Special Solutions

Algebraically Special Solutions

In black holes theory, some authors considered *algebraically special perturbations*. Such perturbations excite gravitational waves that are all either *purely ingoing* or *purely outgoing*.

Algebraically Special Solutions

In black holes theory, some authors considered *algebraically special perturbations*. Such perturbations excite gravitational waves that are all either *purely ingoing* or *purely outgoing*. S. Chandrasekhar calculated the algebraically special solutions in terms of "elementary functions" and a *quadrature* (generalised polynomial). Later Evangelos Melas noticed that this corresponds to the case $n = 1$ of the Kovacic algorithm (already used by Couch and Holder for similar problems), that is to *generalised polynomial solutions*. In his proof, Chandrasekhar used various approaches and in particular the fact that a sufficient condition for the existence of algebraically special perturbations is the vanishing of the Starobinsky constant. We propose a different approach and a new and simple explanation of the Chandrasekhar's result:

Algebraically Special Solutions

In black holes theory, some authors considered *algebraically special perturbations*. Such perturbations excite gravitational waves that are all either *purely ingoing* or *purely outgoing*. S. Chandrasekhar calculated the algebraically special solutions in terms of "elementary functions" and a *quadrature* (generalised polynomial). Later Evangelos Melas noticed that this corresponds to the case $n = 1$ of the Kovacic algorithm (already used by Couch and Holder for similar problems), that is to *generalised polynomial solutions*. In his proof, Chandrasekhar used various approaches and in particular the fact that a sufficient condition for the existence of algebraically special perturbations is the vanishing of the Starobinsky constant. We propose a different approach and a new and simple explanation of the Chandrasekhar's result:

Algebraically special solutions correspond to 3-points connections,

they are *3-special solutions*.

Algebraically Special Solutions

In the algebraically special case for the Schwarschild black hole it happens a ‘miracle’ at the center of the black hole, its singularity:
all the solutions of the radial Regge-Wheeler equation are *meromorphic* at $r = 0$.

Algebraically Special Solutions

In the algebraically special case for the Schwarschild black hole it happens a ‘miracle’ at the center of the black hole, its singularity:
all the solutions of the radial Regge-Wheeler equation are *meromorphic* at $r = 0$. This was noticed by some physicists. We give a rigorous mathematical proof using our results (differential algebra) and another proof using the computation by Chandrasekhar of the Starobinsky constant $|\mathcal{C}|^2$.

Differential algebraic approach

Algebraically special solutions from our main results

If 2σ is an integer (in particular $\sigma = 0, 1, 2$) and ℓ is an integer, then the difference of exponents at $x = \pm 1$ are *integers* and we can apply our main results. If a convenient determinant Δ_σ vanishes, then then there exists a *basis of generalised polynomial solutions*.

Algebraically special solutions in the gravitational case

In the gravitational case $\sigma = 2$. We define $s = -2i\tau$. Then $\Delta_\sigma = 6s - (\ell-1)\ell(\ell+1)(\ell+2))$. The roots of $\Delta_\sigma = 0$ are given by

$$s = \frac{(\ell-1)\ell(\ell+1)(\ell+2))}{6}.$$

Algebraically special solutions in the gravitational case

In the gravitational case $\sigma = 2$. We define $s = -2i\tau$. Then $\Delta_\sigma = 6s - (\ell - 1)\ell(\ell + 1)(\ell + 2))$. The roots of $\Delta_\sigma = 0$ are given by

$$s = \frac{(\ell-1)\ell(\ell+1)(\ell+2))}{6}.$$

We suppose that ℓ is an integer, $\ell > 1$. Then s is a positive *integer* and there is a polynomial $P_{2s+1}(x)$ of degree $2s + 1$ such that the RW equation (in GSE form) with parameters (s, σ, ℓ), admits the basis of solutions

$$y_1(x) = (x-1)^{\frac{1}{2}-s}(x+1)^{-\frac{3}{2}} \left(x + 1 + \frac{6}{(\ell-1)(\ell+2)} \right) e^{-sx/2}.$$

$$y_2(x) = (x-1)^{\frac{1}{2}-s}(x+1)^{-\frac{3}{2}} P_{2s+1}(x) e^{sx/2}.$$

Algebraically special solutions in the gravitational case

In the gravitational case $\sigma = 2$. We define $s = -2i\tau$. Then $\Delta_\sigma = 6s - (\ell-1)\ell(\ell+1)(\ell+2))$. The roots of $\Delta_\sigma = 0$ are given by

$$s = \frac{(\ell-1)\ell(\ell+1)(\ell+2))}{6}.$$

We suppose that ℓ is an integer, $\ell > 1$. Then s is a positive *integer* and there is a polynomial $P_{2s+1}(x)$ of degree $2s+1$ such that the RW equation (in GSE form) with parameters (s, σ, ℓ), admits the basis of solutions

$$y_1(x) = (x-1)^{\frac{1}{2}-s}(x+1)^{-\frac{3}{2}}\left(x+1+\frac{6}{(\ell-1)(\ell+2)}\right)e^{-sx/2}.$$

$$y_2(x) = (x-1)^{\frac{1}{2}-s}(x+1)^{-\frac{3}{2}}P_{2s+1}(x)e^{sx/2}.$$

The generalised polynomial solution y_1 was discovered by Chandrasekhar. He derived y_2 in integral form. The generalised polynomial expression of y_2 was discovered by Melas using the Kovacic algorithm. For $\ell = 2$, Melas calculated the polynomial $P_{2s+1} = P_9$ of degree 9.

Algebraically special solutions in the gravitational case

Using our main results, we proved that P_{2s+1} is a linear combination of *four* generalised Laguerre polynomials.

This fact was proved before by E. Melas in 2017 (unpublished preprint) by another method.

It is important to notice that *four* has a *physical significance*, in relation with the spin of the perturbation.

Using our main results, we proved that P_{2s+1} is a linear combination of *four* generalised Laguerre polynomials.

This fact was proved before by E. Melas in 2017 (unpublished preprint) by another method.

It is important to notice that *four* has a *physical significance*, in relation with the spin of the perturbation.

For mathematical and physical reasons it is very important to express the solutions of the CHE as (formal) series of Laguerre polynomials. In 2016, Alberto Grunbaum and al. proposed a good formalism: the *trigonalisation*. This also gives a new light on the *"lucky accident"* of Slepian, which is the source of Alain Connes' approach of the Riemann hypothesis using prolate spheroidal functions.

Transcendental approach

3-point connection

Algebraically special perturbations of black holes

Using the Norman-Penrose formalism of the General Relativity, perturbations of a black hole can be expressed in terms of five Weyl 'scalars' (complex functions) Φ_i, $i = 0, \ldots, 4$. We can arrange that Φ_1 and Φ_3 continue to vanish in the perturbed space-time, while Φ_2 retains its original value unchanged.

Then perturbations in which we have *only incoming ($\Phi_4 = 0$, $\Phi_0 \neq 0$) or only outgoing ($\Phi_4 \neq 0$, $\Phi_0 = 0$) waves* are called *algebraically special perturbations* of the black hole considered.

Algebraically special perturbations of black holes

Using the Norman-Penrose formalism of the General Relativity, perturbations of a black hole can be expressed in terms of five Weyl 'scalars' (complex functions) Φ_i, $i = 0, \ldots, 4$. We can arrange that Φ_1 and Φ_3 continue to vanish in the perturbed space-time, while Φ_2 retains its original value unchanged.

Then perturbations in which we have *only incoming* ($\Phi_4 = 0$, $\Phi_0 \neq 0$) *or only outgoing* ($\Phi_4 \neq 0$, $\Phi_0 = 0$) *waves* are called *algebraically special perturbations* of the black hole considered.

The Teukolsky-Starobinsky identites relate the two Weyl scalars Φ_0 and Φ_4. They involve two constants $\mathcal{C}_1$, $\mathcal{C}_2$ and one defines the Starobinski constant by $|\mathcal{C}|^2 = \mathcal{C}_1 \mathcal{C}_2$.

Algebraically special perturbations of black holes

Using the Norman-Penrose formalism of the General Relativity, perturbations of a black hole can be expressed in terms of five Weyl 'scalars' (complex functions) Φ_i, $i = 0, \ldots, 4$. We can arrange that Φ_1 and Φ_3 continue to vanish in the perturbed space-time, while Φ_2 retains its original value unchanged.

Then perturbations in which we have *only incoming ($\Phi_4 = 0$, $\Phi_0 \neq 0$) or only outgoing ($\Phi_4 \neq 0$, $\Phi_0 = 0$) waves* are called *algebraically special perturbations* of the black hole considered.

The Teukolsky-Starobinsky identites relate the two Weyl scalars Φ_0 and Φ_4. They involve two constants $\mathcal{C}_1$, $\mathcal{C}_2$ and one defines the Starobinski constant by $|\mathcal{C}|^2 = \mathcal{C}_1\mathcal{C}_2$.

Following S. Chandrasekhar, the vanishing of the Starobinski constant is equivalent to the existence of an algebraically special mode.

Starobinski constant and no log condition

Schwarzschild case, gravitational perturbation $\sigma = 2$, $\ell \in \mathbb{N}$, $\ell \geq 2$.

A new result

The determinant expressing the Starobinski constant $|\mathcal{C}|^2$ is equal to the determinant $\Theta_{2\sigma}^{(-1)} = \Theta_4^{(-1)}$ which expresses the no log condition at $x = -1$ $(r = 0)$.

Starobinski constant and no log condition

Schwarzschild case, gravitational perturbation $\sigma = 2$, $\ell \in \mathbb{N}$, $\ell \geq 2$.

A new result

The determinant expressing the Starobinski constant $|\mathcal{C}|^2$ is equal to the determinant $\Theta_{2\sigma}^{(-1)} = \Theta_4^{(-1)}$ which expresses the no log condition at $x = -1$ ($r = 0$).

The vanishing of the Starobinski constant is equivalent to the fact that all the solutions of the Regge-Wheeler equation are meromorphic at $r = 0$.

Starobinski constant and no log condition

Schwarzschild case, gravitational perturbation $\sigma = 2$, $\ell \in \mathbb{N}$, $\ell \geq 2$.

A new result

The determinant expressing the Starobinski constant $|\mathcal{C}|^2$ is equal to the determinant $\Theta_{2\sigma}^{(-1)} = \Theta_4^{(-1)}$ which expresses the no log condition at $x = -1$ ($r = 0$).

The vanishing of the Starobinski constant is equivalent to the fact that all the solutions of the Regge-Wheeler equation are meromorphic at $r = 0$.

We have a factorisation $\Theta_{2\sigma}^{-1} = \Delta_\sigma \Delta_\sigma^*$, where Δ_σ^* is the determinant of dimension two obtained by the change $s \mapsto -s$ in Δ_σ.

$$|\mathcal{C}|^2 = \left(\ell(\ell-1)(\ell+1)(\ell+2) - 6s\right)\left(\ell(\ell-1)(\ell+1)(\ell+2) + 6s\right)$$
$$= \Delta_\sigma \Delta_\sigma^* = \Theta_{2\sigma}^{(-1)}.$$

Algebraically special solutions are 3-special solutions

An algebraically special solution of equation RW is purely ingoing (or outgoing) at the horizon and at infinity, therefore it is a *special solution* associated to the 2-connection between $x = 1$ (horizon) and ∞.
We connect by $U = \Im x > 0$. This half-plane connects the 3 points $x \pm 1$ and ∞.
The monodromy at $x = -1$ is trivial, then all the local solutions are special at $x = -1$ $(r = 0)$. Therefore our 2-special solution is a 3-special solution.

Algebraically special solutions are 3-special solutions

An algebraically special solution of equation RW is purely ingoing (or outgoing) at the horizon and at infinity, therefore it is a *special solution* associated to the 2-connection between $x = 1$ (horizon) and ∞.

We connect by $U = \Im x > 0$. This half-plane connects the 3 points $x \pm 1$ and ∞.

The monodromy at $x = -1$ is trivial, then all the local solutions are special at $x = -1$ $(r = 0)$. Therefore our 2-special solution is a 3-special solution.

We recall.

Proposition

A 3 special solution of a CHE is a generalised polynomial.

Therefore an algebraically special solution of equation RW is a generalised polynomial.

Kerr black holes, 3-special solutions

In the Kerr case of black holes (rotating BH), the *Teukolsky radial equation* can be reduced to a CHE. The regular points are at the *outer and inner horizon* and the irregular singular point at infinity.

Kerr black holes, 3-special solutions

In the Kerr case of black holes (rotating BH), the *Teukolsky radial equation* can be reduced to a CHE. The regular points are at the *outer and inner horizon* and the irregular singular point at infinity.

If a solution of the Teukolsky radial equation, in CHE form, is *purely ingoing (or outgoing) at the 3 singular points*, then it is locally special at these points. It is a 3-special solution and therefore a *generalised polynomial solution*. In particular *an algebraically special solution is a generalised polynomial solution*.

Algebraically special modes, QNMs, TTMs, towards a clarification

The relations between algebraically special modes, QNMs and total-transmission modes (TTM) are controversial (Maassen van den Brink, 2000).

Algebraically special modes, QNMs, TTMs, towards a clarification

The relations between algebraically special modes, QNMs and total-transmission modes (TTM) are controversial (Maassen van den Brink, 2000).

Using our approach, we see that:

All these modes correspond to special solutions.

Moreover, in the Schwarzschild case, the singularity at $r = 0$ of the radial equation is generically *logarithmic*. We saw that this last point is important in relation with the algebraically special solutions. (This was noticed before by Maassen van den Brink.)

We think that our results could be used for a complete clarification of the controverses.

Darboux transformations and black-holes

It is possible to transform the Regge-Wheeler equation into the Zerilli equation using a *Darboux transformation* defined by an algebraically special solution as a seed function. The two potentials are *SUSY partners*:

$$V_{RW} = W^2 + \frac{dW}{dr_*} + \omega, \quad V_Z = W^2 - \frac{dW}{dr_*} + \omega,$$

where W is the *superpotential* (or SUSY potential); $W = -\phi'/\phi$, where ϕ is the seed function.

It is possible to transform the Regge-Wheeler equation into the Zerilli equation using a *Darboux transformation* defined by an algebraically special solution as a seed function. The two potentials are *SUSY partners*:

$$V_{RW} = W^2 + \frac{dW}{dr_*} + \omega, \quad V_Z = W^2 - \frac{dW}{dr_*} + \omega,$$

where W is the *superpotential* (or SUSY potential);
$W = -\phi'/\phi$, where ϕ is the seed function.

This is a reformulation of a "tour de force" calculation of Chandrasekhar (cf. Maassen van den Brink 2000, Glampedakis and al 2017). A Darboux transformation is a *gauge equivalence* and therefore *the differential Galois groups of the two equations are isomorphic*. Chandrasekhar proved by "brute calculations" that the Regge-Wheeler and Zerilli equations have the same QNMs. Using our analytic spectra this follows easily from the gauge equivalence.

It is possible to transform the Regge-Wheeler equation into the Zerilli equation using a *Darboux transformation* defined by an algebraically special solution as a seed function. The two potentials are *SUSY partners*:

$$V_{RW} = W^2 + \frac{dW}{dr_*} + \omega, \quad V_Z = W^2 - \frac{dW}{dr_*} + \omega,$$

where W is the *superpotential* (or SUSY potential);
$W = -\phi'/\phi$, where ϕ is the seed function.

This is a reformulation of a "tour de force" calculation of Chandrasekhar (cf. Maassen van den Brink 2000, Glampedakis and al 2017). A Darboux transformation is a *gauge equivalence* and therefore *the differential Galois groups of the two equations are isomorphic*. Chandrasekhar proved by "brute calculations" that the Regge-Wheeler and Zerilli equations have the same QNMs. Using our analytic spectra this follows easily from the gauge equivalence.

The Zerilli equation is not a CHE, there is *an added apparent singular point*. (It is a linearized equation of Painlevé five).

THE END

THANK YOU FOR YOUR ATTENTION