Rationality and Potential

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Rationality and potential : Abstract

A 1894 theorem by Émile Borel asserts that power series with integral coefficients that define a meromorphic function on a disk of radius > 1 is the Taylor expansion of a rational function. It has been extended in various directions (Pólya, Dwork, Bertrandias and Robinson) to encompass more complicated shapes than open disks, number fields, and several absolute values. We extend to algebraic curves of arbitrary genus the theorem of Cantor that considers Taylor expansions "at several points".

Our proof runs in two steps. The first step is an algebraicity criterion, which is proved using a method of diophantine approximation. The second step relies on the Hodge index theorem in Arakelov geometry, following an earlier work by Bost and myself.

(Joint work with Camille Noûs) "Potentiel et rationalité", arXiv:2305.17210

Contents

The starting point: a theorem of Émile Borel

Theorem (Émile Borel (1894), "Sur une application d'un théorème de M. Hadamard")

Let $f \in \mathbf{Z}[[T]]$ be a power series with ***integer coefficients.***
Assume that f is the Taylor expansion of a ***meromorphic*** *function ϕ defined on a disk $D(0, R)$ with $R > 1$.*
Then f is the Taylor expansion of a rational function in $\mathbf{Q}(T)$.

Equivalently : the coefficients of f satisfy a linear recurrence relation with integer coefficients.

Borel's theorem: proof

Ingredients of the proof

- Criterium for rationality: vanishing of Hankel determinants
- Consider a "denominator" $P \in \mathbf{C}[T]$ of ϕ on a closed disk $D(0, r)$ with $r > 1$. By linear combinations, one replaces most columns of the determinants by coefficients of $P\phi$.
- By the Cauchy estimates, these coefficients are small, and the other are not too large.
- By the Hadamard theorem, the determinants are small.
- Since these determinants are integers, they ultimately vanish.

Borel's theorem: generalizations and applications

- Pólya (1928), "Über gewisse notwendige Determinantenkriterien für die Fortsetzbarkeit einer Potenzreihe"
  - replaces disks by more general open sets, and radius by transfinite diameter (after inversion)
  - application to the meromorphic continuation of lacunary power series
- Dwork (1960), "On the rationality of the zeta function of an algebraic variety"
- Bertrandias (1963), "Diamètre transfini dans un corps valué. Application au prolongement analytique"
- Cantor (1980), "On an extension of the definition of transfinite diameter and some applications"

Borel's theorem: generalizations and applications

- Pólya (1928), "Über gewisse notwendige Determinantenkriterien für die Fortsetzbarkeit einer Potenzreihe"
- Dwork (1960), "On the rationality of the zeta function of an algebraic variety"
    - also considers the p-adic behavior of f: two radii R_p and R_∞, with condition $R_p R_\infty > 1$
    - in his application to the Weil conjecture, $R_p = +\infty$, $R_\infty > 0$
- Bertrandias (1963), "Diamètre transfini dans un corps valué. Application au prolongement analytique"
- Cantor (1980), "On an extension of the definition of transfinite diameter and some applications"

Borel's theorem: generalizations and applications

- Pólya (1928), "Über gewisse notwendige Determinantenkriterien für die Fortsetzbarkeit einer Potenzreihe"
- Dwork (1960), "On the rationality of the zeta function of an algebraic variety"
- Bertrandias (1963), "Diamètre transfini dans un corps valué. Application au prolongement analytique"
    - number fields, p-adic transfinite diameters, several places
    - Bézivin & Robba (1989), "A new p-adic method for proving irrationality and transcendence results" — new proof of the Lindemann-Weierstrass theorem by way of "Pólya operators"
- Cantor (1980), "On an extension of the definition of transfinite diameter and some applications"

Borel's theorem: generalizations and applications

- Pólya (1928), “Über gewisse notwendige Determinantenkriterien für die Fortsetzbarkeit einer Potenzreihe”
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- Cantor (1980), “On an extension of the definition of transfinite diameter and some applications”
    - introduces expansions at several points
    - the “product of radii > 1” condition is replaced by “the value of some game is strictly positive”

Contents

Formal functions on curves

Let X be a projective, connected, smooth curve over a number field F.
We fix a finite family (p_i) of rational points on F and formal functions $f_i \in \widehat{\mathcal{O}}_{X,p_i}$.

For $X = \mathbf{P}_1$ and the point $p = \infty$, f corresponds to a power series in $F[\![T^{-1}]\!]$.

We have to assume some adelic setup:

- The f_i come from a formal function on an integral model $\mathcal{X}$ along sections $\mathcal{P}_i$ extending p_i;
- For all places v of F, there is a meromorphic function ϕ_v on an open subset Ω_v of the analytic curve X_v containing the points p_i with formal expansion f_i at p_i.
- For almost all nonarchimedean places v, Ω_v is the open subset of X_v consisting of points which have the same reduction as the $\mathcal{P}_i$ mod. v.

Potential theory on Riemann surfaces

Let M be a connected compact Riemann surface
K a compact subset of M with nonempty interior, $\Omega = M - K$

For $p \in \Omega$, the Green function $g_{K,p} \colon \Omega - \{p\} \to \mathbf{R}$ is characterized by the properties:

- It is harmonic;
- If t is a local parameter at p, $g_{K,p}(z) + \log|t(z)|$ has a limit at p.
- It tends to 0 at $\partial(K)$ ("almost surely").

It solves the Laplace equation $\Delta g_{K,p} = \delta_p$ on Ω with Dirichlet condition at infinity.

References:

- Rumely (1989), *Capacity theory on algebraic curves*
- Bost (1999), "Potential theory and Lefschetz theorems for arithmetic surfaces"

Potential theory on non archimedean curves

Let M be a connected proper analytic curve over a nonarchimedean field
K a nonempty affinoid in M, $\Omega = M - K$

For $p \in \Omega$, there is an analogous Green function $g_{K,p} : \Omega - \{p\} \to \mathbf{R}$ which is characterized by the properties:

- It is harmonic (in the sense of non archimedean potential theory);
- If t is a local parameter at p, $g_{K,p}(z) + \log |t(z)|$ has a limit at p.
- It tends to 0 at $\partial(K)$.

References:

- RUMELY (1989), *Capacity theory on algebraic curves*
- THUILLIER (2005), *Théorie du potentiel sur les courbes en géométrie non archimédienne. Applications à la théorie d'Arakelov*

Potential theory — Lemniscates

Both in complex and non archimedean potential theory, **lemniscates** provide important and relevant examples.

f is a rational function with a pole of order $d \geq 1$ at the point p, and no other pole.

$\mathrm{K} = \{z\,;\ |f(z)| \leq 1\}$

$g_{\mathrm{K},p}(z) = \frac{1}{d} \log |f(z)|$

In the nonarchimedean case, an important class of compact sets is given by **affinoid domains**, and a theorem of Rumely (1989) guarantees that they can be viewed as lemniscates provided their complement is connected.

A matrix

We go back to the framework over number fields and fix a local parameter t_i at each p_i.

For every place v of F, we obtain Green functions $g_{\Omega_v,\cdot}$ on the open set Ω_v of the analytic curve X_v, relative to the points p_i and the domains $K_v = X_v - \Omega_v$.

We get local I × I-matrices $G^v = (G^v_{i,j})$:

$$G^v_{i,j} = \begin{cases} g_{K_v,p_i}(p_j) & \text{for } i \neq j \\ \lim_{z\to p_i} g_{K_v,p_i}(z) + \log \left| t_i(z) \right|_v & \text{for } i = j \end{cases}$$

and a global I × I matrix $G = \sum_v G^v$.

These matrices are symmetric.

The matrix G does not depend on the choice of the t_i (product formula).

Value of a matrix

The **value** of the matrix G (von Neumann, 1928) is defined by:

$$\mathrm{val}(G) = \sup_{x \in \Delta_I} \inf_{y \in \Delta_I} \langle x, Gy \rangle,$$

with Δ_I is the simplex $\{\sum x_i = 1\}$ in $\mathbf{R}_+^I$.

In **game theory**, $G_{i,j}$ is the expected gain of Isaac playing move i while Jacob responds with move j.

For $x \in \Delta_I$ and $y \in \Delta_I$, $\langle x, Gy \rangle$ is the expected gain of Isaac if he plays move i with probability x_i and Jacob plays move j with probability y_j, so that the definition of $\mathrm{val}(G)$ says that Isaac wishes to maximize his gain in front of any strategy of Jacob.

Von Neumann's equilibrium theorem says that

$$\mathrm{val}(G) = \inf_{y \in \Delta_I} \sup_{x \in \Delta_I} \langle x, Gy \rangle,$$

confirming this interpretation.

Rationality

Theorem

If the value

$$\operatorname{val}(G) = \sup_{x\in\Delta_I} \inf_{y\in\Delta_I} \langle x, Gy\rangle,$$

of the matrix G is strictly positive, then the f_i are Taylor expansions of a rational function on M.

Generalizes the theorems of Borel,... Cantor when $X = \mathbf{P}_1$, and of Bost & C-L (2009, “Analytic Curves in Algebraic Varieties over Number Fields”) when there is a single point.

If the Ω_v are enlarged, the Green functions increase (maximum principle), so the matrix G increases coefficientwise, and hence val(G) increases. Knowing that the functions f_i exist on larger domains makes them more likely to come from a rational function.

Contents

Two steps

The proof follows the method used to prove the case of a single point (Bost & C-L, 2009, "Analytic Curves in Algebraic Varieties over Number Fields").

1. One first proves that **the f_i are algebraic.**
   The method is a diophantine technique ("polynomial method") involving Bost's slope method. It is inspired by the approaches by Chudnovski and André Grothendieck's p-curvature conjecture, and its nonlinear version by Bost.

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2. One then proves that **the f_i are rational.**
   This is an application of the arithmetic Hodge index theorem on the finite covering of X defined by the f_i.

Algebraicity

One considers the surface $Y = X \times \mathbf{P}_1$ and the formal graphs of the formal functions f_i — formal subschemes $\hat{C}_i$ at points $o_i = (p_i, f_i(p_i))$ of Y.

The question is reformulated as the algebraicity of these formal subschemes, and its proof applies in the more general context of formal/analytic germs of curves in a projective variety.

Proposition

The Zariski closure in Y *of the union* $\bigcup \hat{C}_i$ *has dimension* 1.

Technique of diophantine approximation.

We replace Y by the Zariski closure, let $d = \dim(Y)$.

The goal is to prove $d = 1$.

Algebraicity — diophantine approximation

Let R be the ring of integers of the number field F.

Fix an ample line bundle $\mathcal{L}$ on Y (plus integral model and hermitian metrics) and consider $E_n = \Gamma(Y, \mathcal{L}^n)$ for large n: projective R-module with norms at archimedean places.

Numerical invariants:

- $\operatorname{rank}(E_n) \approx n^d$ (Hilbert-Samuel)
- generated by elements of norms $\ll c^n$
- "arithmetic degree" $\gg -cn^{d+1}$

We have to replace the norms by canonical hermitian norms (John ellipsoids).

Algebraicity — evaluation

The polynomial method "evaluates" the elements of E_n at the points p_i, one at a time.

Filtration $(F^k\mathrm{E}_n)$ of E_n: at step k, one adds a vanishing condition at a point p_{i_k} at order one more than what was required before.

Zariski density: this is an exhaustive filtration

We will need to choose the "evaluation speeds" of p_i: $\omega_i(k) = \mathrm{Card}\{m < k\,;\, i_m = i\}$. (Herblot, 2011, *Sur le théorème de Schneider Lang*)

Evaluation morphisms: $\phi_n^k : F^{k-1}\mathrm{E}_n / F^k\mathrm{E}_n \hookrightarrow \mathrm{T}^*\widehat{C_{i_k}}^{\omega_{i_k}(k)} \otimes \mathrm{L}^n(o_{i_k})$.

Inequalities

$$\widehat{\deg}(F^{k-1}\mathrm{E}_n / F^k\mathrm{E}_n) \leq (-\omega_{i_k}(k)\widehat{\deg}(\mathrm{T}\widehat{C_{i_k}}) + nh_{\mathrm{L}}(o_{i_k}) + h(\phi_n^k)) \cdot \mathrm{rank}(F^{k-1}\mathrm{E}_n / F^k\mathrm{E}_n).$$

Algebraicity — estimates and conclusion

- Using a variant of the Schwarz lemma, $h(\phi_n^k)$ can be controlled by the Green functions
- The definition of $\mathrm{val}(G)$ furnishes a point $x \in \Delta_I$ such that $\sum_i x_i G_{ij} \geq \mathrm{val}(G)$ for all j. We can *choose* the evaluation speeds so that $\omega_i(k) - kx_i$ remains bounded when k varies; more precisely:

$$\sum_i \omega_i(k) G_{ij} \geq k\,\mathrm{val}(G) - c$$

  for all k.
- Ranks: $\mathrm{rank}(F^{k-1}\mathrm{E}_n / F^k\mathrm{E}_n) \leq 1$, one has $\mathrm{rank}(\mathrm{E}_n^k) \geq \mathrm{rank}(\mathrm{E}_n) - k$,
- Sum manipulations lead to $n^{2d} \ll n^{d+1}$, hence $d = 1$.

Rationality

Once we know that the formal functions f_i are algebraic, the normalization X' of Y in $X \times \mathbf{P}_1$ is a smooth curve with a finite morphism $\pi\colon X' \to X$.

It suffices to prove that π is an isomorphism.

The formal functions f_i furnish a formal lift of π on the union of the formal neighborhoods of the points p_i.

Rationality: geometric analogue

Proposition

Let $f\colon S' \to S$ be a generically finite morphism of projective smooth connected surfaces over a field k. Let D and D' be effective divisors on S and S'.
Assume that f induces an isomorphism on formal neighborhoods: $\hat{f}\colon \hat{S}_D \xrightarrow{\sim} \widehat{S'}_D$.
If D is big and nef, then f is birational.

Observe that f^*D is big and nef.

As a consequence of the Hodge index theorem (Ramanujam, 1972), it is connected.

We can write $f^*D = D' + D''$, where D'' is effective and disjoint from D'.

Consequently, $D'' = 0$.

Rationality: Ramanujam's connectedness criterion

Proposition (Ramanujam (1972), "Remarks on the Kodaira vanishing theorem")

Let X be a projective smooth surface over a field k and let D be an effective divisor on X which is big and nef. Then D is connected.

Write $D = A + B$ where A and B are effective and non zero divisors. We shall prove that $A \cdot B > 0$. Assume otherwise.

Since $D = A + B$ is nef, one has $D \cdot A \geq 0$, hence $A \cdot A \geq -A \cdot B$; similarly, $B \cdot B \geq -A \cdot B$. Consequently, $(A \cdot A)(B \cdot B) \geq (A \cdot B)^2$.

Using that $D \cdot D > 0$, we see that A and B are not collinear in the Picard group. The Hodge index theorem implies that the quadratic form on $\langle A, B \rangle$ is nondegenerate, and has at most one + sign, and it has one because $D \cdot D = (A + B) \cdot (A + B) > 0$. Consequently, the determinant

$$\begin{vmatrix} A{\cdot}A & A{\cdot}B \\ A{\cdot}B & B{\cdot}B \end{vmatrix} < 0,$$

that is, $(A \cdot B)^2 < (A \cdot A)(B \cdot B)$. This is the desired contradiction.

Rationality: conclusion

Arakelov geometry allows to formulate a similar statement as the preceding geometric proposition.

The most basic statement involves a generically finite morphism of arithmetic surfaces, divisors equiped with hermitian metrics given by potential theory, and the arithmetic Hodge index theorem, as in (Bost, 1999). An analogue of Ramanujam's connectedness criterion is proved there (Lemma 2).

As shown in (Bost & C-L, 2019), this framework can be expanded using metrics at all places and gives precisely the result needed to conclude the proof of the theorem.

References

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J.-B. Bost & A. Chambert-Loir, "Analytic Curves in Algebraic Varieties over Number Fields", *Algebra, Arithmetic, and Geometry*, vol. 1, p. 69–124, Birkhäuser

Camille Noûs

Camille Noûs is an allegorical polymath French researcher, of indefinite gender, born in 2020 in the minds of the research advocacy group RogueESR.

Their presence as a co-author of this work is both

- a protest against the ever developing short-term financing policies of higher education and research, in particular their increasing dependence to uncertain calls for proposition, as opposed to a (notably state-) sustained financing of research, and
- the acknowledgment that this work is just one link in a long chain of research works to which I meant to add some comments.