On Abel's problem and Gauss congruences
with T. Rivoal

Séminaire différentiel

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Structure of the talk

1. Generalities on Abel's problem
2. The hypergeometric case
3. An arithmetic characterization
4. Sketch of the proof

Structure of the talk

Algebraic solutions of linear differential equations

Let $\mathbb{K}$ be a number field, write $\partial = d/dx$ and consider

$$\mathcal{L} = \partial^n + a_{n-1}(x)\partial^{n-1} + \cdots + a_1(x)\partial + a_0(x),$$

with $a_i(x) \in \mathbb{K}(x)$.

Ricatti equation

There is a nonlinear differential equation $\mathcal{R}(u) = 0$ of order $n-1$ such that y is a non-zero solution of $\mathcal{L}(y) = 0$ if and only if $u = y'/y$ is a solution of $\mathcal{R}(u) = 0$.

Basis of algebraic solutions

If $\mathcal{L}$ is irreducible, then $\mathcal{L}(y) = 0$ has a full basis of algebraic solutions if and only if $\mathcal{R}(u) = 0$ has an algebraic solution η which is the logarithmic derivative of an algebraic function y :

$$y' = \eta y.$$

References : Boulanger, Painlevé (1898).

Algebraic solutions of linear differential equations

Question

When does $\mathcal{L}(y) = 0$ have a full basis of algebraic solutions ?

This question reduces to the following 3 problems that can be solved in a finite number of steps.

1- Reduction to the irreducible case (Singer 1979)

Is $\mathcal{L}$ irreducible ?

2- Algebraic solution of the Ricatti equation (Singer 1979)

Does $\mathcal{R}(u) = 0$ have an algebraic solution η ?

3- Abel's problem (Risch 1969, Baldassari-Dwork 1979)

Let η be an algebraic function. Does $y' = \eta y$ admit a non-trivial algebraic solution ?

Abel's problem $y' = \eta y$: examples

Moderate growth of an algebraic function at ∞

There is $N \in \mathbb{Z}_{\geq 0}$ and an angular sector $\mathcal{S}$ at ∞ such that y is holomorphic on $\mathcal{S}$ and $\lim_{x \to \infty} x^{-N} y(x) = 0$.

$\eta = 1$

$y(x) = \exp(x)$ is transcendental.

$\eta = x$

$y(x) = \exp(x^2/2)$ is transcendental.

$\eta = 1/(1-x)$

$y(x) = 1/(1-x)$ is algebraic.

$\eta = 1/(1-x)^2$

$y(x) = \exp(\sqrt{1-x} - 1)$ is transcendental.

Analytic observation : Puiseux expansions

The Newton (1676) Puiseux (1850) theorem

If y is an algebraic function and $\delta \in \mathbb{P}^1$, then y admits a convergent puiseux expansion in a slit disk of center δ :

- $y(x) = \sum_{n=r}^{\infty} a_n (x-\delta)^{n/d}$ if δ is finite.
- $y(x) = \sum_{n=r}^{\infty} a_n x^{-n/d}$ if δ is infinite.

where $a_n \in \mathbb{C}$, $r \in \mathbb{Z}$ and $d \in \mathbb{Z}_{>0}$.

Application with $\delta = 0$:

$$y(x) = \sum_{n=r}^{\infty} a_n x^{n/d} = x^{r/d} \sum_{n=0}^{\infty} a_{n+r} x^{n/d},$$

$$y'(x) = \sum_{n=r}^{\infty} \frac{n}{d} a_n x^{n/d-1} = x^{r/d-1} \sum_{n=0}^{\infty} \frac{n+r}{d} a_{n+r} x^{n/d},$$

$$\frac{y'(x)}{y(x)} = \sum_{n=0}^{\infty} c_n x^{n/d-1} \text{ : a specific expansion.}$$

Abel's problem : the rational case

- If η is an algebraic function and $y' = \eta y$ has a non-trivial algebraic solution, then the Newton-Puiseux theorem applied to y'/y in each $\delta \in \mathbb{P}^1$ shows that
$$\eta(x) = \sum_{i=1}^{k} \frac{\beta_i}{x - \alpha_i},$$
with $\alpha_i, \beta_i \in \mathbb{C}$.
- A solution is $y(x) = \prod_{i=1}^{k}(x - \alpha_i)^{\beta_i}$. The Newton-Puiseux theorem applied to y gives that y is algebraic if and only if $\beta_i \in \mathbb{Q}$ for all i.

Abel's problem for rational fractions

Let $\eta(x)$ be a rational fraction. Then $y' = \eta y$ admits a non-trivial algebraic solution if and only if $\eta(x) = \sum_{i=1}^{k} \frac{\beta_i}{x-\alpha_i}$, with $\alpha_i \in \mathbb{C}$ and $\beta_i \in \mathbb{Q}$ for all i.

An algebraic characterization

Let η be an algebraic function over $\mathbb{K}(x)$ and y a non-trivial solution of $y' = \eta y$.

Then y is algebraic over $\mathbb{K}(x)$ if and only if there is $m \in \mathbb{Z}_{>0}$ such that $m\eta = \nu'/\nu$ with $\nu \in \mathbb{K}(x, \eta)$.

Hint : if part : $y = c\nu^{1/m}$; only if part : differentiate the minimal equation of y over $\mathbb{K}(x, \eta)$.

- If y is algebraic, then its minimal equation over $\mathbb{K}(x, \eta)$ is $y^m + \mu = 0$ for some $\mu \in \mathbb{K}(x, \eta)$.
- Hence $y \in \mathbb{K}(x, \eta) \iff m = 1$.
- Rational case : m is the least common denominator of the β_i's.

Baldassari-Dwork algorithm (1979)

Determine in a finite number of steps whether m exists or not by geometric considerations on the curve associated with η.

Structure of the talk

Generalized hypergeometric series

Let $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{Q}^r$, $\beta = (\beta_1, \ldots, \beta_s) \in (\mathbb{Q} \setminus \mathbb{Z}_{\leq 0})^s$ and consider

$$\mathcal{F}_{\alpha,\beta}(x) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_r)_n}{(\beta_1)_n \cdots (\beta_s)_n} x^n,$$

where $(a)_n = a(a+1)\cdots(a+n-1)$ if $n \geq 1$ and $(a)_0 = 1$ denotes the Pocchammer symbol.

$\mathcal{F}_{(\alpha_1),(1)}(x) = (1-x)^{-\alpha_1}$ is algebraic.

$\mathcal{F}_{(\frac{1}{2},\frac{1}{2}),(1,1)}(x) = \frac{2}{\pi} \int_0^x \frac{dt}{\sqrt{(1-t^2)(1-xt^2)}}$ is transcendental.

For a suitable choice of α, β and $C \in \mathbb{Q}$ (Rodriguez Villegas 2007)

$\mathcal{F}_{\alpha,\beta}(Cx) = \sum_{n=0}^{\infty} \frac{(30n)!n!}{(15n)!(10n)!(6n)!} x^n$ is algebraic of degree 483 840.

Factorial hypergeometric series

We say that a hypergeometric series $\mathcal{F}_{\alpha,\beta}$ is factorial if there is $C \in \mathbb{Q}^*$ and positive integers $e_1, \dots, e_u, f_1, \dots, f_v$ such that

$$\mathcal{F}_{\alpha,\beta}(Cx) = \sum_{n=0}^{\infty} \frac{(e_1 n)! \cdots (e_u n)!}{(f_1 n)! \cdots (f_v n)!} x^n.$$

A combinatorial criterion

$\mathcal{F}_{\alpha,\beta}$ is factorial if and only if $\dfrac{(X - e^{i2\pi\alpha_1}) \cdots (X - e^{i2\pi\alpha_r})}{(X - e^{i2\pi\beta_1}) \cdots (X - e^{i2\pi\beta_s})} \in \mathbb{Q}(X)$.

$$\mathcal{F}_{(\frac{1}{2},\frac{1}{2}),(1,1)}(16x) = \frac{2}{\pi} \int_0^{16x} \frac{dt}{\sqrt{(1-t^2)(1-16xt^2)}} = \sum_{n=0}^{\infty} \frac{(2n)!^2}{n!^4} x^n.$$

Golyshev's predictions

According to Zagier (2018), Golyshev made the following prediction using the theory of motives.

Golyshev's predictions

If $x\eta(x)$ is an algebraic factorial hypergeometric function, then

- $y' = \eta y$ has a non-trivial algebraic solution,
- We have even more : $y \in \mathbb{Q}(x, \eta)$.

In other words, if $F_{e,f}(x) = \sum_{n=0}^{\infty} Q_{e,f}(n)x^n$ is algebraic, with

$$Q_{e,f}(n) = \frac{(e_1 n)! \cdots (e_u n)!}{(f_1 n)! \cdots (f_v n)!},$$

then

$$y_{e,f}(x) = \exp \int \frac{F_{e,f}(x)}{x} dx = x \exp \left(\sum_{n=1}^{\infty} \frac{Q_{e,f}(n)}{n} x^n \right)$$

is also algebraic and $y_{e,f} \in \mathbb{Q}(x, F_{e,f})$.

Golyshev's predictions

By *ad hoc* explicit computations, Zagier (2018) proved the first part of Golyshev's predictions when $Q_{e,f}(n)$ is one of

$$\binom{en}{fn}, \quad \frac{(6n)!n!}{(3n)!(2n)!^2} \quad \text{or} \quad \frac{(10n)!n!}{(5n)!(4n)!(2n)!},$$

with $e \geq f \geq 1$.

Abel's problem for hypergeometric series (D., Rivoal, 2022)

Let $x\eta(x)$ be an algebraic hypergeometric function. The following assertions are equivalent.

- $y' = \eta y$ has a non-trivial algebraic solution.
- $x\eta(x)$ is factorial.

Structure of the talk

Gauss congruences

We write $\mathbb{Z}_{(p)}$ for the localization of $\mathbb{Z}$ at the prime p : the ring of rational numbers a/b with b not divisible by p.

We say that $(a_n)_{n\in\mathbb{Z}} \in \mathbb{Q}^{\mathbb{Z}}$ satisfies Gauss congruences for the prime p if

$$\forall n \in \mathbb{Z}, \quad a_{np} - a_n \in np\mathbb{Z}_{(p)}.$$

- An equivalent congruence :

$$\forall s \in \mathbb{Z}_{\geq 0}, \forall m \in \mathbb{Z}, \quad a_{mp^{s+1}} - a_{mp^s} \in p^{s+1}\mathbb{Z}_{(p)}.$$

- If $(a_n)_{n\geq 0} \in \mathbb{Z}^{\mathbb{N}}$, then it satisfies Gauss congruences for all p if and only if

$$\forall n \in \mathbb{Z}_{\geq 0}, \quad \sum_{d|n} \mu(n/d)a_d \equiv 0 \mod n,$$

where μ is the Möbius function.

- The latter congruence was proved by Gauss (1863) for $a_n = r^n$, r prime, and later for $r \in \mathbb{Z}$ by Kantor, Weyr, Lucas, Grandi, Pellet, Thué (1880-1883 independently).

An arithmetic characterization

We say that $(a_n)_{n\in\mathbb{Z}}$ has the Gauss property if it satisfies Gauss congruences for all large enough prime p.

Rational Puiseux expansion

Let η be an algebraic function over $\mathbb{C}(x)$ and $\delta \in \mathbb{C}$. Write

$$\eta(x) = \sum_{n=r}^{\infty} p_n (x-\delta)^{n/d}.$$

We say that this Puiseux expansion is rational if $p_n \in \mathbb{Q}$ for all n.

Abel's problem for rational Puiseux expansions (D., Rivoal, 2022)

Let η be an algebraic function over $\mathbb{C}(x)$ which has a rational Puiseux expansion at $\delta \in \mathbb{C}$ with coefficients $(p_n)_{n\in\mathbb{Z}}$. Then the following assertions are equivalent.

- $y' = \eta y$ has a non-trivial algebraic solution.
- $(p_{n-1})_{n\in\mathbb{Z}}$ has the Gauss property.

An arithmetic characterization : remarks

$$\eta(x) = \sum_{n=r}^{\infty} p_n (x-\delta)^{n/d}, \quad (p_n \in \mathbb{Q},\, \delta \in \mathbb{C}).$$

Non-zero algebraic sol. to $y' = \eta y \iff (p_{n-1})_{n\in\mathbb{Z}}$ has the G.P.

- A characterization at $\delta = \infty$ exists.
- The case $\delta = 0$ is sufficient since $\overline{\eta}(x) = \eta(x+\delta)$ is algebraic and leads to an equivalent problem of Abel.
- When $d = 1$, prime numbers p for which $(p_{n-1})_{n\in\mathbb{Z}}$ satisfies Gauss congruences can be guessed from the Eisenstein constant of y.
- Useful to prove algebraicity *via* congruences, *e.g.* Golyshev's prediction.
- Useful to prove Gauss congruences *via* algebraicity :

$$\eta(x) = \frac{r}{1-rx} \quad \text{and} \quad y(x) = \frac{1}{1-rx} \in \mathbb{Z}[[x]]$$

yield Gauss congruences for all p for $(r^n)_{n\geq 0}$, $r \in \mathbb{Z}$.

Application to the hypergeometric case

A hypergeometric series $\mathcal{F}_{\alpha,\beta}$ is globally bounded if it has a positive radius of convergence and if there is $C \in \mathbb{Q}^*$ such that $\mathcal{F}_{\alpha,\beta}(Cx) \in \mathbb{Z}[[x]]$.

Gauss congruences for hypergeometric series (D., Rivoal, 2022)

Let $\mathcal{F}_{\alpha,\beta}$ be a globally bounded hypergeometric series. Then the following assertions are equivalent.

- The sequence of coefficients of $\mathcal{F}_{\alpha,\beta}$ has the Gauss property.
- $\mathcal{F}_{\alpha,\beta}$ is factorial.

In addition, if $C \in \mathbb{Q}^*$ is such that $\mathcal{F}_{\alpha,\beta}(Cx) = F_{e,f}(x)$, then $F_{e,f}$ satisfies Gauss congruences for all prime p.

Every algebraic $\mathcal{F}_{\alpha,\beta}$ is globally bounded, but this criterion also applies to transcendental hypergeometric functions. Hence the first part of Golyshev's predictions follows from this criterion.

The hypergeometric case : sketch of the proof

Assume that $\mathcal{F}_{\alpha,\beta}$ is globally bounded, write

$$\mathcal{Q}_{\alpha,\beta}(n) = \frac{(\alpha_1)_n \cdots (\alpha_r)_n}{(\beta_1)_n \cdots (\beta_r)_n}$$

and d the least common denominator of the rational numbers α_i and β_j.

An almost Gauss congruence for $p > d$

For all $s \in \mathbb{Z}_{\geq 0}$ and $m \in \mathbb{Z}$, we have

$$\mathcal{Q}_{\alpha,\beta}(mp^{s+1}) - \mathcal{Q}_{\langle k\alpha \rangle, \langle k\beta \rangle}(mp^s) \in p^{s+1}\mathbb{Z}_{(p)},$$

with $kp \equiv 1 \mod d$ and $\langle x \rangle = \{x\}$ if $x \notin \mathbb{Z}$, $\langle x \rangle = 1$ otherwise.

$\alpha = (1/3, 1/3)$, $\beta = (1/2, 1)$ and $p = 11$

We have $d = 6$ and $p \equiv 5 \mod 6$ so $k = 5$:

$$\mathcal{Q}_{(\frac{1}{3},\frac{1}{3}),(\frac{1}{2},1)}(mp^{s+1}) - \mathcal{Q}_{(\frac{2}{3},\frac{2}{3}),(\frac{1}{2},1)}(mp^s) \in p^{s+1}\mathbb{Z}_{(p)}.$$

The hypergeometric case : sketch of the proof

An almost Gauss congruence for $p > d$

For all $s \in \mathbb{Z}_{\geq 0}$ and $m \in \mathbb{Z}$, we have

$$\mathcal{Q}_{\alpha,\beta}(mp^{s+1}) - \mathcal{Q}_{\langle k\alpha\rangle,\langle k\beta\rangle}(mp^s) \in p^{s+1}\mathbb{Z}_{(p)},$$

with $kp \equiv 1 \mod d$ and $\langle x\rangle = \{x\}$ if $x \notin \mathbb{Z}$, $\langle x\rangle = 1$ otherwise.

Those congruences give the equivalence of the following assertions.

- $\mathcal{Q}_{\alpha,\beta}$ has the Gauss property.
- $\alpha = \langle k\alpha\rangle$ and $\beta = \langle k\beta\rangle$ for all k coprime to d.
- $\mathcal{F}_{\alpha,\beta}$ is factorial.

Structure of the talk

Step 1 ($\delta = 0$) : $d = 1$ and $p_n \in \mathbb{Z}$

$$\eta(x) = \sum_{n=-1}^{\infty} p_n x^n \quad \text{and} \quad y(x) = x^{p_{-1}} \exp\left(\sum_{n=1}^{\infty} \frac{p_{n-1}}{n} x^n\right).$$

Chudnovsky and Chudnovsky (1985)

Let $y(x) \in \mathbb{Q}[[x]]$ be such that $y(\lambda x) \in \mathbb{Z}[[x]]$ for some $\lambda \in \mathbb{Q}^*$. If y'/y is algebraic over $\mathbb{Q}(x)$, then y is algebraic over $\mathbb{Q}(x)$.

For simplicity we assume that $p_{-1} \geq 0$ so that $y(x) \in \mathbb{Q}[[x]]$. Since $y'/y = \eta$ is algebraic we have to show that the following assertions are equivalent.

- There is $\lambda \in \mathbb{Q}^*$ such that $y(\lambda x) \in \mathbb{Z}[[x]]$.
- $(p_{n-1})_{n \geq 0}$ has the Gauss property.

Step 1 : $d = 1$ and $p_n \in \mathbb{Z}$

$$(p_{n-1})_{n\geq 0} \in \mathbb{Z}^{\mathbb{N}} \Longrightarrow \forall p \text{ prime}, \quad y(p^2 x) \in \mathbb{Z}_{(p)}[[x]].$$

Hence we have the equivalent assertions :

- There is $\lambda \in \mathbb{Q}^*$ such that $y(\lambda x) \in \mathbb{Z}[[x]]$.
- $y(x) \in \mathbb{Z}_{(p)}[[x]]$ for all large enough prime p.

Dieudonné (1957), Dwork (1958)

Let $s(x) \in x\mathbb{Q}[[x]]$ and p be a prime number. Then the following assertions are equivalent.

- $e^{s(x)} \in \mathbb{Z}_{(p)}[[x]]$.
- $s(x^p) - ps(x) \in px\mathbb{Z}_{(p)}[[x]]$.

$$y(x) = x^{p-1} \exp\left(\sum_{n=1}^{\infty} \frac{p_{n-1}}{n} x^n\right)$$

Step 1 : $d = 1$ and $p_n \in \mathbb{Z}$

Dieudonné (1957), Dwork (1958)

Let $s(x) \in x\mathbb{Q}[[x]]$ and p be a prime number. Then the following assertions are equivalent.

- $e^{s(x)} \in \mathbb{Z}_{(p)}[[x]]$.
- $s(x^p) - ps(x) \in px\mathbb{Z}_{(p)}[[x]]$.

By the Dieudonné-Dwork lemma, the following assertions are equivalent.

- $\exp\left(\sum_{n=1}^{\infty} \frac{p_{n-1}}{n} x^n\right) \in \mathbb{Z}_{(p)}[[x]]$.
- $(p_{n-1})_{n\geq 1}$ satisfies Gauss congruences for the prime p.

This proves Step 1. □

This result generalizes itself twice !

- Step 2 : $d = 1$ and $p_n \in \mathbb{Q}$.
- Step 3 : $d \geq 1$ and $p_n \in \mathbb{Q}$.

Thank you for your attention !