Fuchs' Theorem, an Exponential Function, and Abel's Problem in Positive Characteristic

joint work with H. Hauser and H. Kawanoue (arXiv:2307.01712 and arXiv:2401.14154)

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Overview

1. Introduction: Local Solution Theory in Characteristic 0
2. Local Solution Theory in Positive Characteristic
3. An Exponential Function in Positive Characteristic
4. Abel's Problem in Positive Characteristic

Definitions

Consider a homogeneous linear ordinary differential equation (ODE) over $\mathbb{C}$

$$a_n y^{(n)} + \ldots + a_1 y' + a_0 y = 0 \qquad (\star)$$

with $a_i \in \mathbb{C}[\![x]\!]$. We can rewrite it in terms of a **differential operator** as $Ly = 0$ with $L = a_n\partial^n + \ldots + a_1\partial + a_0 \in \mathbb{C}[\![x]\!][\partial]$.

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L has a **regular singularity at** 0 if $a_i/a_n \in \mathbb{C}(\!(x)\!)$ has a pole of order at most $n - i$ at 0.

Write $L = \sum_{i=0}^{\infty} \sum_{j=0}^{n} c_{i,j} x^i \partial^j$ and set $L_k = \sum_{i-j=k} c_{i,j} x^i \partial^j$. The minimal τ with $L_\tau \neq 0$ is called the **shift** of L. From now on, we assume w.l.o.g. $\tau = 0$ (multiply L by $x^{-\tau}$).

The operator $L_0 = \sum c_{i,i} x^i \partial^i$ is called the **initial form** of L. It has the same order as L if and only if L is regular singular.

Definitions

For the initial form L_0 we have $L_0(x^k) = \chi_L(k)x^k$, where $\chi_L(k)$ is the **indicial polynomial** of L. Its roots ρ_i for $i = 1, \ldots, k$ of multiplicity m_i are the **local exponents** of L.

A basis of solutions of $L_0 y = 0$ (as $\mathbb{C}$-vector space) is given by $x^{\rho_i} \log(x)^j$ for $1 \leq i \leq k$ and $0 \leq j \leq m_i - 1$.

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Example

The differential operator

$$L = x^5\partial^5 - 2x^4\partial^4 - 2x^3\partial^3 + 16x^2\partial^2 - 16x\partial - x.$$

is regular singular with shift 0. Its normal form is $L_0 = L + x$ and its indicial polynomial is $\chi(s) = s^2(s-2)(s-5)^2$. The local exponents are $\rho_1 = 0$, $\rho_2 = 2$ and $\rho_3 = 5$ with $m_1 = 2$, $m_2 = 1$ and $m_3 = 2$.

Fuchs' Theorem – Local Solution Theory

Theorem (Fuchs 1866)

Let $L \in \mathbb{C}[\![x]\!][\partial]$ be a regular singular differential operator of order n. Then the equation $Ly = 0$ has a basis of n $\mathbb{C}$-linearly independent solutions of the form

$$f_i = x^\rho \left(f_{i,0} + f_{i,1}\log(x) + \ldots + f_{i,n-1}\log(x)^{n-1}\right),$$

where $f_{i,j} \in \mathbb{C}[\![x]\!]$ and ρ ranges over the local exponents (counted with multiplicity).

Fuchs gave a more detailed description on the form of the solution, in particular on the order of $f_{i,j}$ and more precise bounds on the powers of the logarithm appearing.

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Example

For $L = (2x^2 - x^3) + (-4x^2 + 3x^3)\partial + (2x^2 - 3x^3)\partial^2 + x^3\partial^3$ a basis of solutions of $Ly = 0$ is given by e^x, $e^x \log(x)$ and xe^x.

Motivation

Problem (Abel)

When does $y' = ay$ for an algebraic series $a \in \overline{\mathbb{Q}(x)} \cap \mathbb{Q}[\![x]\!]$ admit an **algebraic** solution?

Solved 1970 by Risch algorithmically (although not suitable for implementation).

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Problem (Liouville, Fuchs)

When does

$$a_n y^{(n)} + \ldots + a_1 y' + a_0 y = 0 \qquad (\star)$$

with polynomial coefficients $a_i \in \mathbb{Q}[x]$ admit a **basis of n algebraic solutions?**

Solved algorithmically by Singer 1979 by reducing to Risch's algorithm.

Motivation

Grothendieck p-curvature conjecture (1969)

The equation $Ly = 0 \quad (\star)$ with $L \in \mathbb{Q}[x][\partial]$ having polynomial coefficients admits a basis of n algebraic solutions if and only if its reduction $L_p y = 0$ modulo p admits a basis of n $\mathbb{F}_p(\!(x^p)\!)$-linearly independent solutions in $\mathbb{F}_p(\!(x)\!)$ for almost all prime numbers p.

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The reduction $(\star)_p$ of $(\star)$ modulo p is well-defined for almost all prime numbers, $L_p \in \mathbb{F}_p[x][\partial]$.

Rewrite $L_p y = 0$ into a system of n first order ODEs: $Y' = AY$. The **p-curvature** of L_p is the $\mathbb{F}_p[x]$-linear map $(\partial - A)^p : \mathbb{F}_p((x))^n \to \mathbb{F}_p((x))^n$.

Lemma (Cartier)

Equation $L_p y = 0$ admits a basis of n $\mathbb{F}_p((x^p))$-linearly independent solutions in $\mathbb{F}_p((x))$ and if and only if its p-curvature vanishes.

Solution Theory in Characteristic p

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$$\partial x = 1, \quad \partial z_1 = \frac{1}{x}, \quad \partial z_k = \frac{1}{x \cdot z_1 \cdots z_{k-1}} = \frac{\partial z_{k-1}}{z_{k-1}}.$$

Field of **constants:** $\mathcal{C}_p := \mathbb{F}_p(z_1^p, z_2^p, \ldots)(\!(x^p)\!)$. Solutions of differential equations in $\mathcal{R}_p$ form a $\mathcal{C}_p$-vector space of dimension at most n.

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Note: ∂ reduces degree of a non-constant monomial in x by exactly one.

Parallel to **logarithms** from characteristic 0:

$$\log(x)' = \frac{1}{x}, \quad \log^k(x)' = \frac{1}{x \cdot \log(x) \cdots \log^{k-1}(x)}$$

Solution Theory in Characteristic p

Theorem (Honda 1981)

Assume $L_p y = 0$ with polynomial coefficients has nilpotent p-curvature and $n = \operatorname{ord} L_p \leq p$. Then $L_p y = 0$ has a basis of n $\mathbb{F}_p(z_1^p, x^p)$-linearly independent solutions in $\mathbb{F}_p[z_1, x]$.

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Assume $L_p y = 0$ has nilpotent p-curvature. Then $L_p y = 0$ has a basis of n $\mathbb{F}_p(z_1^p, z_2^p, \dots, x^p)$-linearly independent solutions in $\mathbb{F}_p[z_1, z_2, \dots, x]$.

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Theorem (F.–Hauser 2023)

Let $L_p y = 0$ be a regular singular differential equation with polynomial or power series coefficients over $\mathbb{F}_p$, whose local exponents lie in $\mathbb{F}_p$. Then $L_p y = 0$ has a basis of n $\mathcal{C}_p$-linearly independent solutions in $\mathcal{R}_p = \mathbb{F}_p(z_1, z_2, \ldots)((x))$.

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If the local exponents ρ are not in the prime field, but in $\overline{\mathbb{k}}$, we can introduce symbols t^ρ with $t^\rho \cdot t^\sigma = t^{\rho+\sigma}$ and $\partial t^\rho = \rho t^\rho / x$. Then solutions can be found in $\bigoplus t^\rho \mathcal{R}_p$ (group algebra).

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A detailed description of the degree of the monomials appearing in the series expansion of solutions is possible.

Example: $\log(1-x)$

In characteristic 0:

$$y_1 = -\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots \in \mathbb{Q}[\![x]\!]$$

satisfies $Ly = 0$ with $L = x^2\partial^2 - (x^2\partial + x^3\partial^2)$. The second solution $y_2 = 1$ completes a basis.

For all prime numbers p a basis of solutions of $L_p y = 0$ is given by

$$y_1 = x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots + \frac{x^{p-1}}{p-1} + x^p z_1 \quad \text{and} \quad y_2 = 1.$$

This is an example for an equation with nilpotent p-curvature for all prime numbers p.

Example: Exponential Function

The exponential differential equation $y' = y$ admits a solution $\exp_p$ in $\mathcal{R}_p$. For $p = 3$ one obtains:

$$\begin{aligned}\exp_3 = 1 &+ x + 2x^2 + 2x^3 z_1 + x^4(1 + 2z_1) + x^5 z_1 + 2x^6 z_1^2 + x^7(1 + 2z_1 + 2z_1^2) \\ &+ x^8(2 + z_1^2) + x^9(2z_1 + z_1^3 z_2) + \dots\end{aligned}$$

This solution is unique up to multiplication with constants. Here the solution is chosen, such that 1 is the only monomial in the series expansion that is constant.

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One checks for example:

$$(x^7(1 + 2z_1 + 2z_1^2))' = x^6(1 + 2z_1 + 2z_1^2)) + x^7 \cdot \left(\frac{2}{x} + \frac{z_1}{x}\right) = 2x^6 z_1^2$$

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Observation: Setting $z_1 = z_2 = \ldots = 0$ in $\exp_p$ gives power series in $\mathbb{F}_p[\![x]\!]$. Computer experiments (with A. Bostan) suggest that this series is **algebraic** over $\mathbb{F}_p(x)$.

A Different Approach

Proposition (F.–Hauser–Kawanoue, 2024)

Define $w_i := x^{p^i} z_1^{p^{i-1}} \cdots z_{i-1}^{p^1} z_i$. Then $w_i^{(p^i - p^{i-1}+1)} = -w_{i-1}'$. Thus,

$$\widetilde{\exp}_p := \sum_{i=0}^{\infty} \sum_{k=1}^{p^i - p^{i-1}} (-1)^i w_i^{(k)}$$

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$\widetilde{\exp}_p$ up to order $p^i - 1$ is given by $\sum_{k=1}^{p^i} (-1)^i w_i^{(k)}$.

$\widetilde{\exp}_p$ differs from $\exp_p$ by a multiplicative constant in $\mathcal{C}_p$.

Yet Another Approach

Proposition (F.–Hauser–Kawanoue, 2024)

Define

$$\sigma : \mathbb{F}_p[\![s]\!] \to \mathbb{F}_p[\![s]\!], s \mapsto s + s^p + s^{p^2} + \dots .$$

Define $g_0 := \sigma(x)$ and recursively $g_i := \sigma(g_{i-1}^p z_i)$. Set

$$H(t) := \prod_{k=1}^{p-1} \left(1 - \frac{t}{k}\right)^k \quad \text{and} \quad \widehat{\exp}_p := \prod_{i=0}^{\infty} H\left((-1)^i g_i\right).$$

Then $\widehat{\exp}_p$ solves $y' = y$.

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Lemma

$$\widehat{\exp}_p = \widetilde{\exp}_p.$$

Algebraicity of Projections

$$g_i := \sigma(g_{i-1}^p z_i), \qquad \widehat{\exp}_p := \prod_{i=0}^{\infty} H\left((-1)^i g_i\right).$$

σ is algebraic, as $\sigma(s) = \sigma(s)^p + s$. Thus, inductively, g_i is algebraic over $\mathbb{F}_p(x, z_1, \ldots, z_i)$.

Note: $g_i \in 1 + z_i \cdot \mathbb{F}_p[z_1, \ldots, z_i][\![x]\!]$. Thus, for the **projection** $\pi_j(\widehat{\exp}_p)$ we have

$$\pi_j(\widehat{\exp}_p) := \widehat{\exp}_p|_{z_{j+1}=z_{j+2}=\ldots=0} = \prod_{i=0}^{j} H\left((-1)^i g_i\right),$$

which is algebraic over $\mathbb{F}_p(x, z_1, \ldots, z_j)$.

In particular: $\widehat{\exp}_p|_{z_1=z_2=\ldots=0}$ is algebraic over $\mathbb{F}_p(x)$. The same holds true for $\exp_p$.

Abel's Problem in Characteristic p

Does the same hold true for any differential equation? More precisely:

Question

Let $L \in \mathbb{F}_p[x][\partial]$ be a regular singular differential operator of order n and assume its local exponents lie in the prime field $\mathbb{F}_p$. Does there exist a basis of solutions $y_1, \ldots, y_n$ in $\mathbb{F}_p[z_1, z_2, \ldots][\![x]\!]$, such that its projections $\pi_j(y_k) = y_k|_{z_{j+1}=z_{j+2}=\ldots=0} \in \mathbb{F}_p[z_1, \ldots, z_j][\![x]\!]$ are algebraic over $\mathbb{F}_p(x, z_1, \ldots, z_j)$ for all j, k?

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Partial answer:

Theorem (F.–Hauser–Kawanoue, 2024)

Let $y' = ay$ be an order one regular singular differential equation with rational or algebraic coefficient $a \in \mathbb{F}_p(\!(x)\!)$ and local exponent $\rho \in \mathbb{F}_p$. Then there is a solution y such that $\pi_j(y)$ is algebraic over $\mathbb{F}_p(z_1, \ldots, z_j, x)$ for all j.

Ideas of Proof for π_0

The p-curvature of $y' = ay$ is given by $(\partial - a)^p y = a_p y$, where $a_p = -a^{(p-1)} - a^p$.

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$$a^{(p-1)} + a^p + \frac{g}{x^p} - \frac{g^p}{x^p} = 0$$

implicitly to obtain an algebraic series $g \in \mathbb{F}_p[\![x^p]\!]$. Then the p-curvature of $y' = (a - g/x)\, y$ vanishes, and by a variant of Cartier's Lemma this equation has an **algebraic** solution q.

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The equation $y' = (g/x)y = (a - q'/q)y$ is equivalent to $(qy)' = aqy$. Because $g \in \mathbb{F}_p[\![x^p]\!]$, its solutions lie in $\mathbb{F}_p[z_1, z_2, \dots][\![x^p]\!]$ and from this it follows that it has a solution $y_0 \in 1 + z_1\mathbb{F}_p[z_1, z_2, \dots][\![x^p]\!]$. Thus $y = qy_0$ satisfies $y' = ay$ and $\pi_0(y) = q$ is algebraic.

Product Representations

Iterating this construction leads to a more precise statement, generalizing the product representation of $\widehat{\exp}_p$:

Theorem (F.–Hauser–Kawanoue, 2024)

Let $L = \partial + a$ be a first order regular singular linear differential operator with rational function coefficient $a \in \mathbb{F}_p(x)$ (or algebraic coefficient $a \in \mathbb{F}_p[\![x]\!]$) and local exponent $\rho = 0$. Then for all $i \in \mathbb{N}$ there exist series $h_i \in 1 + z_i\mathbb{F}_p[z_1, \dots, z_i][\![x]\!]$, which are algebraic over $\mathbb{F}_p(z_1, z_2, \dots, z_i, x)$ and $P = \prod_{i=0}^{\infty} h_i$ satisfies $LP = 0$. In particular, $\pi_j(P) = \prod_{i=0}^{j} h_i$ is algebraic over $\mathbb{F}_p(x, z_1, \dots, z_j)$ for all j.

Further Questions

Does this generalizes to higher order differential equations? Idea: Factorisation of differential operators in $\overline{\mathbb{Q}(x)}[\partial]$ into linear factors.

Consider a (first order) differential equation $Ly = 0$ with $L \in \mathbb{Q}[x][\partial]$. Let $y_p \in \mathcal{R}_p$ be a (basis of) solution(s) of $L_p y = 0$. Do the **Galois groups** of $\pi_j(y_p)$ relate to the **differential Galois group** of $Ly = 0$? Is there a variant of the differential Galois Group in characteristic p?

Is there a "canonical" basis of solutions of the n-dimensional $\mathcal{C}_p$-vector space of solutions of $L_p y = 0$?

The Artin-Hasse exponential function is defined as $\exp(x + x^p/p + x^{p^2}/p^2 + \ldots) \in \mathbb{Z}_{(p)}[\![x]\!]$. Is there a connection to $\exp_p$?

The End

Thank you for your attention!