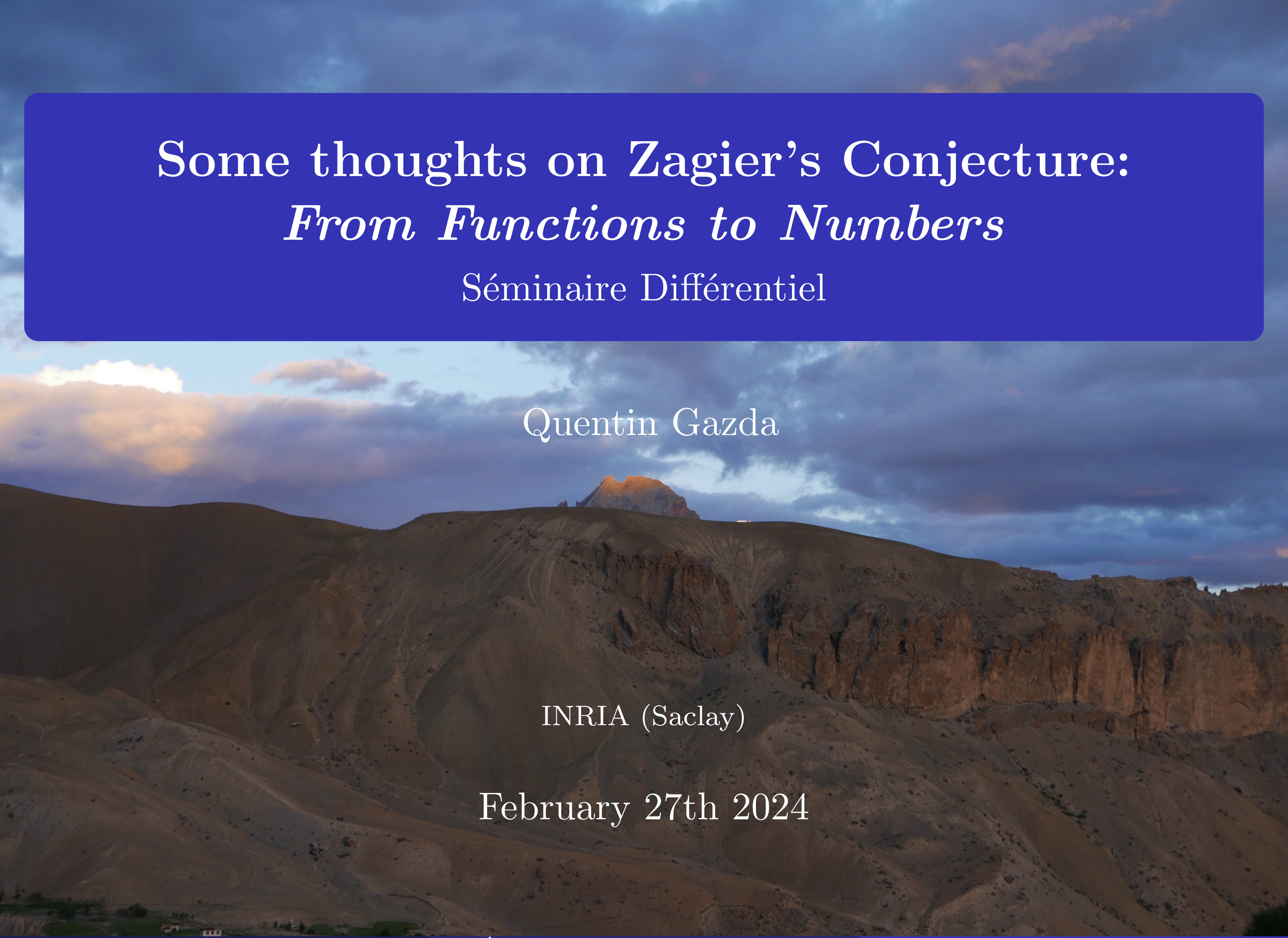


Some thoughts on Zagier's Conjecture: *From Functions to Numbers*

Séminaire Différentiel

Quentin Gazda

INRIA (Saclay)

February 27th 2024

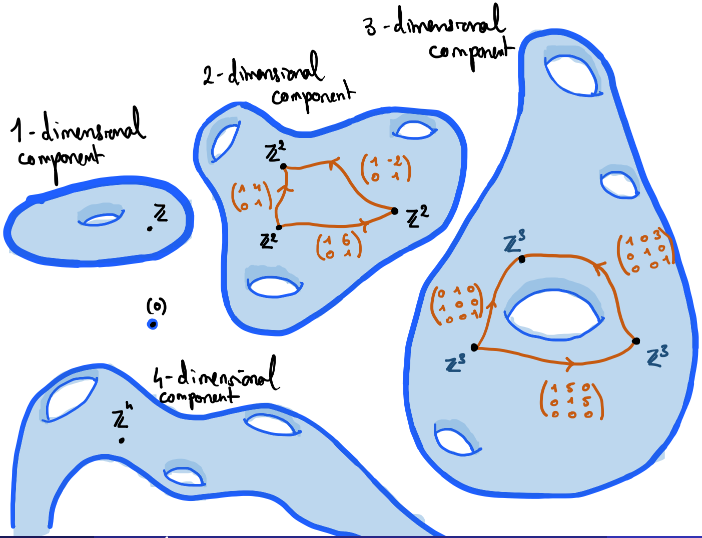
Zagier's Conjecture

"Linear relations among ***polylogarithms*** *evaluated at algebraic numbers come from relations among symbols in* K***-theory****"*

K-theory

R : commutative ring with unit,
$S(R)$: *spectrum* built out of the commutative algebra of R.

Points of $S(R)$	:	finite projective modules over R,
Paths on $S(R)$	:	isomorphisms of R-modules,
Filled triangles on $S(R)$	:	commutative diagrams,
		etc.



K-theory

Operations $\oplus$ and $\otimes$ extend to $S(R)$ into operations that are associative and commutative only *up to homotopy*.
The K-theory spectrum is obtained by *groupifying* $S(R)$ for $\oplus$:

$$K(R) := S(R)^{\text{grp}}$$

For $i \geq 0$, we set $K_i(R) := \pi_i K(R)$.

Examples

$K_0(R)$ is the Grothendieck ring of R,
$K_1(R)$ is the abelianization of the infinite linear group $\mathrm{GL}(R)$,
$K_2(R) \ldots =?$

Polylogarithms

Let $n \geq 1$ be an integer.

$$\text{For } |x| < 1: \quad \mathrm{Li}_n(x) = \sum_{m>0} \frac{x^m}{m^n}$$

Bloch–Wigner–Ramakrishnan polylogarithms

$$\mathcal{D}_n(x) = (-1)^n (n-1)! \sum_{n>k\geq 0} \frac{(-1)^k}{k!} \log(x)^k \,\mathrm{Li}_{n-k}(x).$$

The monodromy of $\mathcal{D}_n$ is simpler than that of Li_n. We get a univaluated function

$$\mathcal{D}_n : \mathbb{C} \setminus \{0,1\} \longrightarrow \mathbb{C}/(2\pi i)^n \mathbb{Z}.$$

Examples

$\mathcal{D}_1(x) = -\mathrm{Li}_1(x) = \log(1-x)$, $\mathcal{D}_2(x) = \mathrm{Li}_2(x) - \log(x)\log(1-x)$, etc.

Statement

Let $n \geq 1$ be an integer.
Let F be a number field (which we assume included in $\mathbb{C}$).

Conjecture (Zagier)

There exist an (explicit) subgroup $\mathrm{Symb}_n(F) \subseteq \mathbb{Z}[F \setminus \{0,1\}]$ together with a map $\mathrm{Symb}_n(F) \longrightarrow K_{2n-1}(F)$ surjective up to torsion such that, for all $\sum_\alpha n_\alpha[\alpha] \in \mathrm{Symb}_n(F)$,

$$\sum_\alpha n_\alpha[\alpha] = 0 \text{ dans } K_{2n-1}(F) \quad \Longleftrightarrow \quad \sum_\alpha n_\alpha \mathcal{D}_n(\alpha) \in (2\pi i)^n \mathbb{Z}$$

For $n = 1$. $\mathrm{Symb}_1(F) = \mathbb{Z}[F \setminus \{0,1\}] \longrightarrow F^\times$, $[\alpha] \longmapsto 1 - \alpha$.
The conjecture is equivalent to the injectivity of $\log : F^\times \to \mathbb{C}/(2\pi i)\mathbb{Z}$.

Examples

For $n = 2$. $\mathrm{Symb}_2(F)$ consists in elements $\sum_\alpha n_\alpha[\alpha]$ such that

$$\sum_\alpha n_\alpha \cdot \alpha \wedge (1-\alpha) = 0 \quad \text{dans} \quad F^\times \bigwedge F^\times.$$

We have a map $\mathrm{Symb}_2(F) \longrightarrow K_3(F)$ constructed by Bloch–Suslin. The elements

$$2\left([x] + \left[\frac{1}{x}\right]\right), \quad [x]+[1-x], \quad [x]+[y]+\left[\frac{1-x}{1-xy}\right]+[1-xy]+\left[\frac{1-y}{1-xy}\right],$$

are in its kernel, *e.g.* testifying of the celebrated *five-term relation*:

$$\mathcal{D}_2(x) + \mathcal{D}_2(y) + \mathcal{D}_2\left(\frac{1-x}{1-xy}\right) + \mathcal{D}_2(1-xy) + \mathcal{D}_2\left(\frac{1-y}{1-xy}\right) \in (2\pi i)^2\mathbb{Z}.$$

Motivic interpretation (after Beilinson–Deligne '94)

$\mathcal{MM}_F$: hypothetical category of *mixed motives* over F.
It is supposed to be abelian, $\mathbb{Q}$-linear, symmetric monoidal $\otimes$
It possesses objects $\mathbb{Q}(n)$ called *Tate twists*.

$$\mathrm{Ext}^1_{\mathcal{MM}_F}(\mathbb{1}, \mathbb{Q}(n))$$

Recall: In any abelian category $\mathcal{A}$, the group of extensions of two objects X and Y of $\mathcal{A}$ is given by

$$\mathrm{Ext}^1_{\mathcal{A}}(X, Y) \stackrel{\mathrm{def}}{=} \{[Z] : 0 \to Y \to Z \to X \to 0\} / \{[Z] \sim [Z']\}$$

where $[Z] \sim [Z']$ means that there exists a diagram of the form:

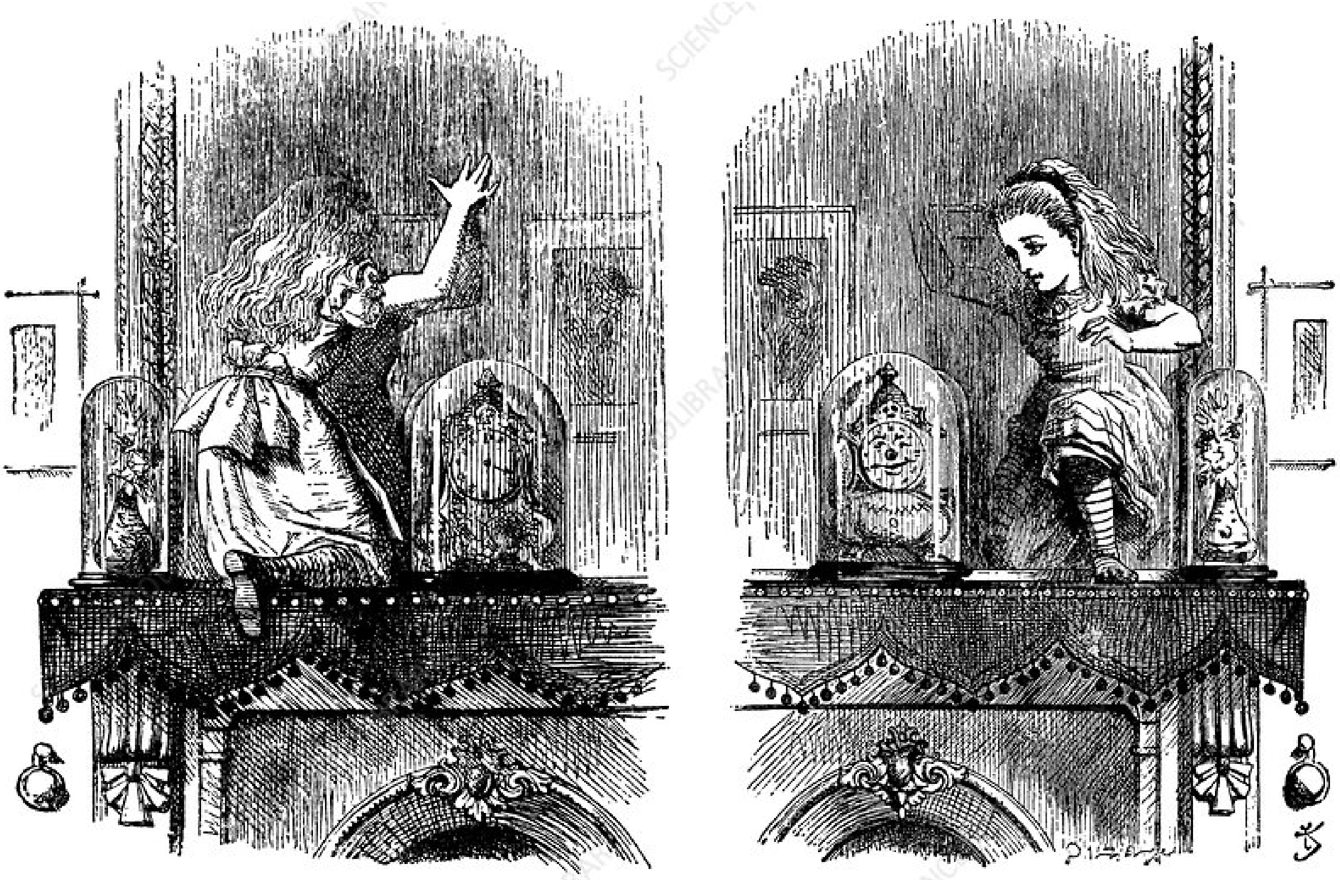
$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X & \longrightarrow & 0 \\ & & \downarrow \mathrm{id} & & \downarrow & & \downarrow \mathrm{id} & & \\ 0 & \longrightarrow & Y & \longrightarrow & Z' & \longrightarrow & X & \longrightarrow & 0 \end{array}$$

Motivic interpretation (after Beilinson–Deligne '94)

$$\begin{array}{ccccc}
\mathrm{Ext}^1_{\mathcal{MM}_F}(\mathbb{1},\mathbb{Q}(n)) & \xleftrightarrow{\sim} & K_{2n-1}(F)_{\mathbb{Q}} & \twoheadleftarrow & \mathrm{Symb}_n(F)_{\mathbb{Q}} \\
\text{réal. Hodge}\downarrow & & \downarrow & & \downarrow \\
\mathrm{Ext}^1_{\mathrm{Hdg}_{\mathbb{Q}}}(\mathbb{1}_{\mathrm{Hdg}},\mathbb{Q}(n)_{\mathrm{Hdg}}) & \xleftrightarrow{\sim} & \mathbb{C}/(2\pi i)^n\mathbb{Q} & \xleftarrow{\mathcal{D}_n} & \mathbb{Z}[F\setminus\{0,1\}]
\end{array}$$

The vertical map, called *regulator*, is expected to be injective: this is *Beilinson's conjecture*. It implies Zagier's conjecture:

$$\ker\left(\mathrm{Symb}_n(F)_{\mathbb{Q}} \to K_{2n-1}(F)_{\mathbb{Q}}\right) = \ker\left(\mathcal{D}_n \mid \mathrm{Symb}_n(F)\right).$$



Function fields arithmetic

Let $\mathbb{F}$ be a finite field with q elements.

$$\begin{array}{ccccccc} \mathbb{F}[\theta] & \subset & \mathbb{F}(\theta) & \subset & K_\infty := \mathbb{F}(\!(\frac{1}{\theta})\!) & \subset & \mathbb{C}_\infty := (K_\infty^{\mathrm{alg}})^\wedge \\ \wr & & \wr & & \wr & & \wr \\ \mathbb{Z} & \subset & \mathbb{Q} & \subset & \mathbb{R} & \subset & \mathbb{C} \end{array}$$

Carlitz exponential

For $z \in \mathbb{C}_\infty$, $\quad \exp_C(z) := \sum_{i \geq 0} \frac{z^{q^i}}{(\theta^{q^i} - \theta)(\theta^{q^i} - \theta^q) \cdots (\theta^{q^i} - \theta^{q^{i-1}})}.$

The *fundamental exact sequence*

$$0 \longrightarrow \tilde{\pi} \cdot \mathbb{F}[\theta] \longrightarrow \mathbb{C}_\infty \xrightarrow{\exp_C} \mathbb{C}_\infty \longrightarrow 0.$$

where $\tilde{\pi}$ is *Carlitz's period*, analogue to $2\pi i$,

$$\tilde{\pi} := (-\theta)^{\frac{q}{q-1}} \prod_{i \geq 1} \left(1 - \theta^{1-q^i}\right)^{-1}.$$

Carlitz's polylogarithms

$$0 \longrightarrow \tilde{\pi} \cdot \mathbb{F}[\theta] \longrightarrow \mathbb{C}_\infty \underset{\log_C}{\overset{\exp_C}{\longrightarrow}} \mathbb{C}_\infty \longrightarrow 0.$$

Carlitz's logarithm:

$$\text{For } |z| < |\theta|^{\frac{q}{q-1}}, \quad \log_C(z) = \sum_{i \geq 0} \frac{z^{q^i}}{(\theta - \theta^q)(\theta - \theta^{q^2}) \cdots (\theta - \theta^{q^i})}.$$

For $n \geq 1$, *the nth Carlitz's polylogarithm*:

$$\text{For } |z| < |\theta|^{\frac{qn}{q-1}}, \quad \mathrm{Li}_n(z) = \sum_{i \geq 0} \frac{z^{q^i}}{(\theta - \theta^q)^n(\theta - \theta^{q^2})^n \cdots (\theta - \theta^{q^i})^n}.$$

t-motives (after Anderson, 1986)

Let R be an $\mathbb{F}[\theta]$-algebra.

Definition (t-motive)

A *t-motive over R* is the data of $\underline{M} = (M, \phi_M)$

1. a finite projective $R[t]$-module M of constant rank,
2. an isomorphism ϕ_M

$$\phi_M : (\mathrm{Frob}_R^* M)\left[\frac{1}{t-\theta}\right] \xrightarrow{\sim} M\left[\frac{1}{t-\theta}\right].$$

The neutral t-motive : $\mathbb{1} = (R[t], 1)$,
The nth Carlitz's twist : $\underline{A}(n) = (R[t], (t-\theta)^{-n})$,
Let $\alpha \in R$.

$$\mathcal{L}_n(\alpha) = \left(R[t]^{\oplus 2}, \begin{pmatrix} (t-\theta)^{-n} & (t-\theta)^{-n}\alpha \\ 0 & 1 \end{pmatrix}\right).$$

Zagier's Conjecture in equal characteristic

This t-motive inserts in a short exact sequence $0 \to \underline{A}(n) \to \mathcal{L}_n(\alpha) \to \mathbb{1} \to 0$. We denote its class by

$$[\alpha] \in \mathrm{Ext}^1_{t\mathbf{Mot}_R}(\mathbb{1}, \underline{A}(n)).$$

Let $F \subset \mathbb{C}_\infty$ be a finite field extension of $\mathbb{F}(\theta)$ (not necessarily separated).

Theorem (G–Maurischat)

Let $\alpha_1, \ldots, \alpha_s \in F$ of norm $< |\theta|^{\frac{qn}{q-1}}$, $a_1(t), \ldots, a_s(t) \in \mathbb{F}[t]$. Then,

$$\sum_{i=1}^{s} a_i(t)[\alpha_i] = 0 \text{ in } \mathrm{Ext}^1_{t\mathbf{Mot}_F}(\mathbb{1}, \underline{A}(n)) \iff \sum_{i=1}^{s} a_i(\theta)\,\mathrm{Li}_n(\alpha_i) \in \tilde{\pi}^n \mathbb{F}[\theta]$$

Sketch of proof

We consider the operator:

$$\xi(t) \in \mathbb{C}_\infty\langle t\rangle, \quad \Delta_n(\xi) = (t-\theta)^n \xi - \mathrm{Frob}_{\mathbb{C}_\infty}(\xi).$$

Solutions of $\Delta_n(\xi) = 0$ are the $\mathbb{F}[t]$-multiples of the nth power of the *Anderson–Thakur function*

$$\omega(t) := (-\theta)^{\frac{1}{q-1}} \prod_{i\geq 0} \left(1 - \frac{t}{\theta^{q^i}}\right)^{-1}.$$

For $\alpha \in F$ of norm $< |\theta|^{\frac{qn}{q-1}}$, a solution of $\Delta_n(\xi) = \alpha$ is given by

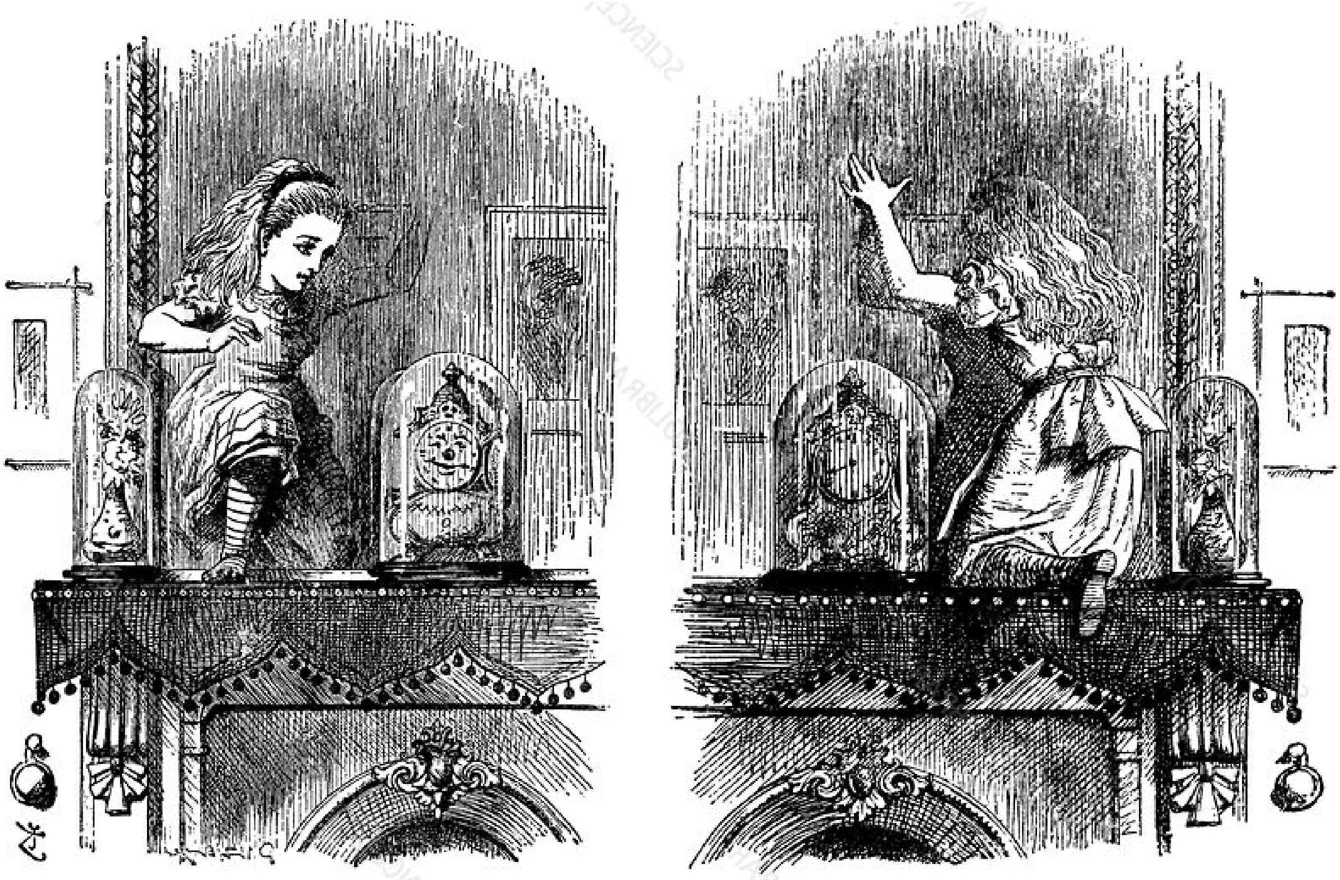
$$\mathrm{Li}_n(\alpha)_t := \sum_{i\geq 0} \frac{\alpha^{q^i}}{(t-\theta)^n (t-\theta^q)^n \cdots (t-\theta^{q^i})^n}.$$

Sketch of proof

$$\xi \longmapsto \mathcal{L}_n\left(\frac{\Delta_n(\xi)}{(t-\theta)^n}\right)$$

$$\begin{array}{ccc}
\xi & & \dfrac{\{\xi \in \mathbb{C}_\infty\langle t\rangle \mid \Delta_n(\xi) \in F[t]\}}{\mathbb{F}[t]\omega(t)^n + F[t]} \to \operatorname{Ext}^1_{t\mathbf{Mot}_F}(\mathbb{1}, \underline{A}(n)) \\
\downarrow & & \downarrow \\
(t-\theta)^n\xi(t)|_{t=\theta} & & \mathbb{C}_\infty/\tilde{\pi}^n\mathbb{F}[\theta]
\end{array}$$

The class of $\mathrm{Li}_n(\alpha)_t$ is mapped horizontally to $[\alpha]$, vertically to $\mathrm{Li}_n(\alpha)$. The two arrows are injective. For the horizontal, this is a formality. For the vertical one, this follows from *Anderson–Brownawell–Papanikolas criterion.*



q-Delirium

Hope

$$\mathrm{Li}_n(\alpha)_t = \sum_{i\geq 0} \frac{\alpha^{q^i}}{(t-\theta)^n(t-\theta^q)^n\cdots(t-\theta^{q^i})^n} \quad \textit{is to} \quad \mathrm{Li}_n(\alpha)$$

$$\textit{what} \qquad \mathrm{Li}_n(T)_q = \sum_{m>0} \frac{T^m}{[m]_q^n} \qquad \textit{is to} \quad \mathrm{Li}_n(T)$$

where $[m]_q = 1 + q + \ldots + q^{m-1}$ is the q-analogue of m.

Today, we do not possess the technology to redo this proof in number theory. We find ourselves powerless at the very definition of a *t-motive* as we do not have a counterpart for

$$\mathbb{F}[t,\theta] \quad \sim \quad \text{“}\mathbb{Z} \otimes_{\mathbb{F}_1} \mathbb{Z}\text{”}$$

For p a prime, however, recent breakthroughs allow to realize some parts of this paradigm *after p-completion*:

$$\left(\mathbb{F}[t,\theta]^{\wedge}_{(\mathfrak{p}(t),\mathfrak{p}(\theta))},\ \mathrm{Frob}_\theta^{\deg \mathfrak{p}}\right) \quad \sim \quad (\Delta_{\mathbb{Z}_p}, \phi)$$

where $\mathfrak{p}(t) \in \mathbb{F}[t]$ is a prime element; i.e. an irreducible polynomial. Δ_R denotes the *prismatic cohomology*[1] of a $\mathbb{Z}_p$-algebra R and ϕ its Frobenius.
If $K/\mathbb{Q}_p$ is an unramified finite field extension, then

$$(\Delta_{\mathcal{O}_K}, \phi) = (\mathcal{O}_K[\![q-1]\!],\ \phi_K \ + \ q \mapsto q^p)$$

[1] ↑ We rather consider q-crystalline cohomology.

prismatic q-crystals (after Bhatt–Scholze)

Definition (q-crystal)

A *q-crystal $\underline{M}$ over $\mathcal{O}_K$* is the data of a couple (M, ϕ_M) where

1. M is a finite projective module over $\Delta_{\mathcal{O}_K}$
2. ϕ_M is an isomorphism

$$\phi_M : (\phi^* M)\left[\frac{1}{[p]_q}\right] \xrightarrow{\sim} M\left[\frac{1}{[p]_q}\right].$$

Similarly, we have a neutral q-crystal $\mathbb{1} = (\Delta_{\mathcal{O}_K}, 1)$ and Tate twists $\mathbb{Z}_p(n) = (\Delta_{\mathcal{O}_K}, [p]_q^{-n})$. A subgroup of the group of extensions of q-crystals $\mathrm{Ext}^1_q(\mathbb{1}, \mathbb{Z}_p(n))$ is computed as the H^1 of the *q-syntomic complex*

$$R\Gamma_{q\mathrm{syn}}(\mathcal{O}_K, \mathbb{Z}_p(n)) = \left[(q-1)^n \Delta_{\mathcal{O}_K} \xrightarrow{\mathrm{id} - \frac{\phi}{[p]_q^n}} \Delta_{\mathcal{O}_K}\right].$$

Case $n = 1$: q-syntomic Chern classes

We are looking for a group homomorphism $\mathcal{O}_K^\times \to \mathrm{Ext}^1_q(\mathbb{1}, \mathbb{Z}_p(1))$. Such a morphism was recently constructed by Bhatt–Lurie! This is the H^1 of the *q-syntomic Chern class*

$$c_1^{q\mathrm{syn}} : R\Gamma_{\text{ét}}(R, \mathbb{G}_m)[-1] \longrightarrow R\Gamma_{q\mathrm{syn}}(R, \mathbb{Z}_p(1))$$

Conjecture (joint work with T. Bouis)

Let $r \in 1 + \mathcal{O}_K^\times$. The q-syntomic Chern class $c_1^{q\,\mathrm{syn}}$ of Bhatt–Lurie in cohomological degree one $\mathcal{O}_K^\times \to \mathrm{H}^1_{q\,\mathrm{syn}}(\mathcal{O}_K, \mathbf{Z}_p(1))$, maps the element $1 - r$ towards the class of $-\,\mathrm{Li}_1^{(p)}(r)_q$. The underlying extension of q-crystals is represented by

$$\mathcal{L}_1(1-r) := \left[\Delta_{\mathcal{O}_K}^{\oplus 2}, \begin{pmatrix} [p]_q^{-1} & -\,\mathrm{Li}_1^{(p)}(r)_q \\ 0 & 1 \end{pmatrix} \right].$$

p-adic q-polylogarithms

For $n \geq 1$ an integer, it is formally given by

$$\mathrm{Li}_n^{(p)}(T)_q = \sum_{\substack{m>0 \\ (m,p)=1}} \frac{T^m}{[m]_q^n}.$$

Proposition

Consider the ring $B = \varprojlim_r \mathbb{Z}[q, T^{\pm 1}, (1 - T^{p^r})^{-1}]/[p^r]_q$.

1. The sequence
$$\left(\frac{1}{1 - T^{p^r}} \sum_{\substack{p^r > m > 0 \\ (p,m)=1}} \frac{T^m}{[m]_q^n} \right)_{r>0}$$
forms a projective system in B. We denote by $\mathrm{Li}_n^{(p)}(T)_q$ its limit.
2. We have the relation: $\mathrm{Li}_n^{(p)}(T)_q + (-1)^n \mathrm{Li}_n^{(p)}(1/T)_q = 0$.

q-motives?

Two observations:

1. No need of the enormous ring $\mathcal{O}_K[\![q-1]\!]$. The ring

$$\varprojlim_r \mathcal{O}_K[q]/[p^r]_q$$

   suffices. It is a q-deformation of the p-completion.

2. Let p and ℓ be two prime numbers. The p-Frobenius commutes to the ℓ-Frobenius!

Many thanks for your kind attention!