# The Membership Problem for Hypergeometric Sequences with Rational Parameters

Séminaire différentiel — 29 november 2022

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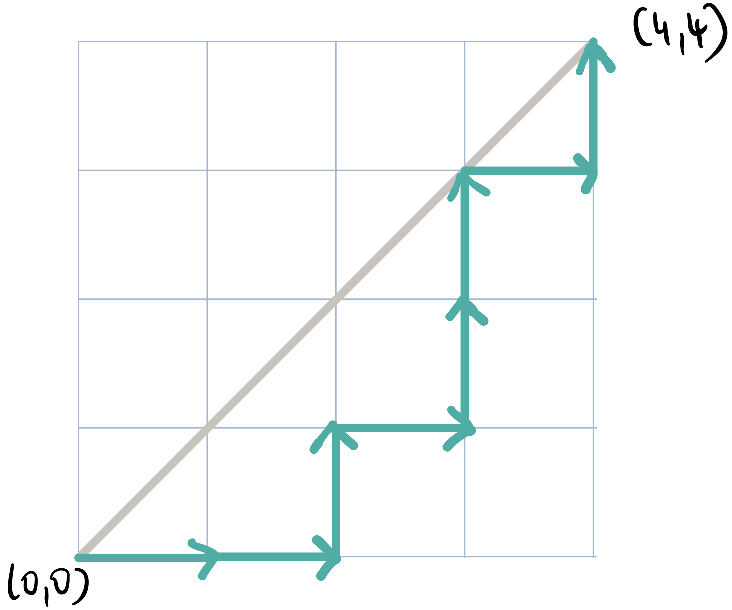
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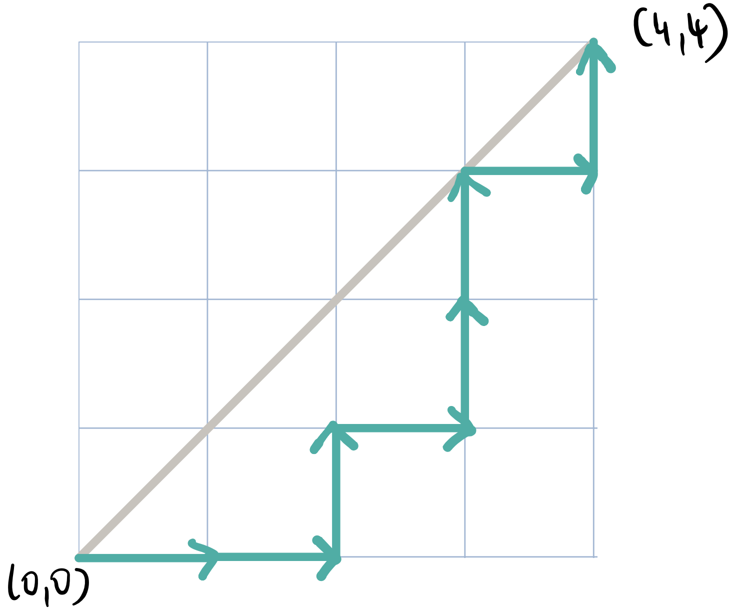


Catalan numbers:

1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, ...

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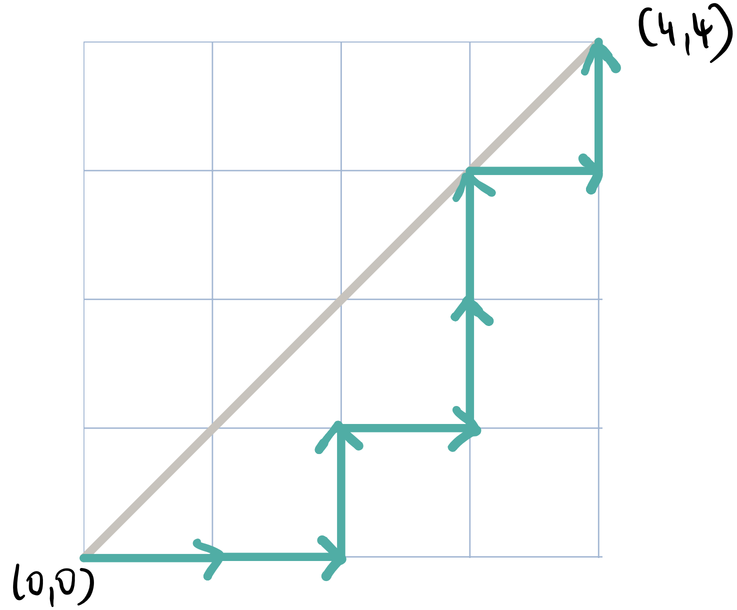


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$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

$$C_0 = 1 \quad \text{and} \quad C_{n+1} = \frac{2(2n+1)}{n+2} C_n$$

# Hypergeometric sequences

* A HYPERGEOMETRIC SEQUENCE is a sequence $\langle u_0, u_1, u_2, \ldots \rangle$ of rational numbers satisfying a recurrence of the form

$$\boxed{p(n)u_n - q(n)u_{n-1} = 0}$$

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$$\boxed{u_n = r(n)\,u_{n-1}}$$

shift quotient $\quad r(x) = \dfrac{q(x)}{p(x)}$

$$\langle u_n \rangle_{n=0}^{\infty} \qquad u_0 = 1 \qquad r(x) = \frac{\left(x+\frac{9}{2}\right)\left(x+\frac{7}{2}\right)\left(x+\frac{5}{2}\right)}{\left(x+\frac{11}{2}\right)(x+4)(x+1)}$$

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Does the value $\frac{13}{6}$ $\approx 2.167$ appear in the sequence?

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Prob. (MP)

Given $u_0 \in \mathbb{Q}$, $r(x) \in \mathbb{Q}(x)$ and $t \in \mathbb{Q}$:

$\exists n$ s.t. $u_n = t$ ?

# A step back

( Hypergeometric sequences
$p(n)u_n - q(n)u_{n-1} = 0$ )

C-recursive sequences $\quad u_n = \sum_{k=1}^{d} c_k \cdot u_{n-k}$

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The set of zeros of $\langle u_n \rangle_{n=0}^{\infty}$ is a union of finitely many arithmetic progressions and a finite set.

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- order $\leq 4$: decidable
(Mignotte, Shorey, Tijdeman '84 & Vereschagin '85)
- order 5: open!

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- order 1: trivial
  $u_n = p_1(n) u_{n-1}$

- order 2: open!
  $u_n = p_1(n) u_{n-1} + p_2(n) u_{n-2}$

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- Order 1.5: MP
  Sum of 2 HS
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How?

Divisibility by primes!

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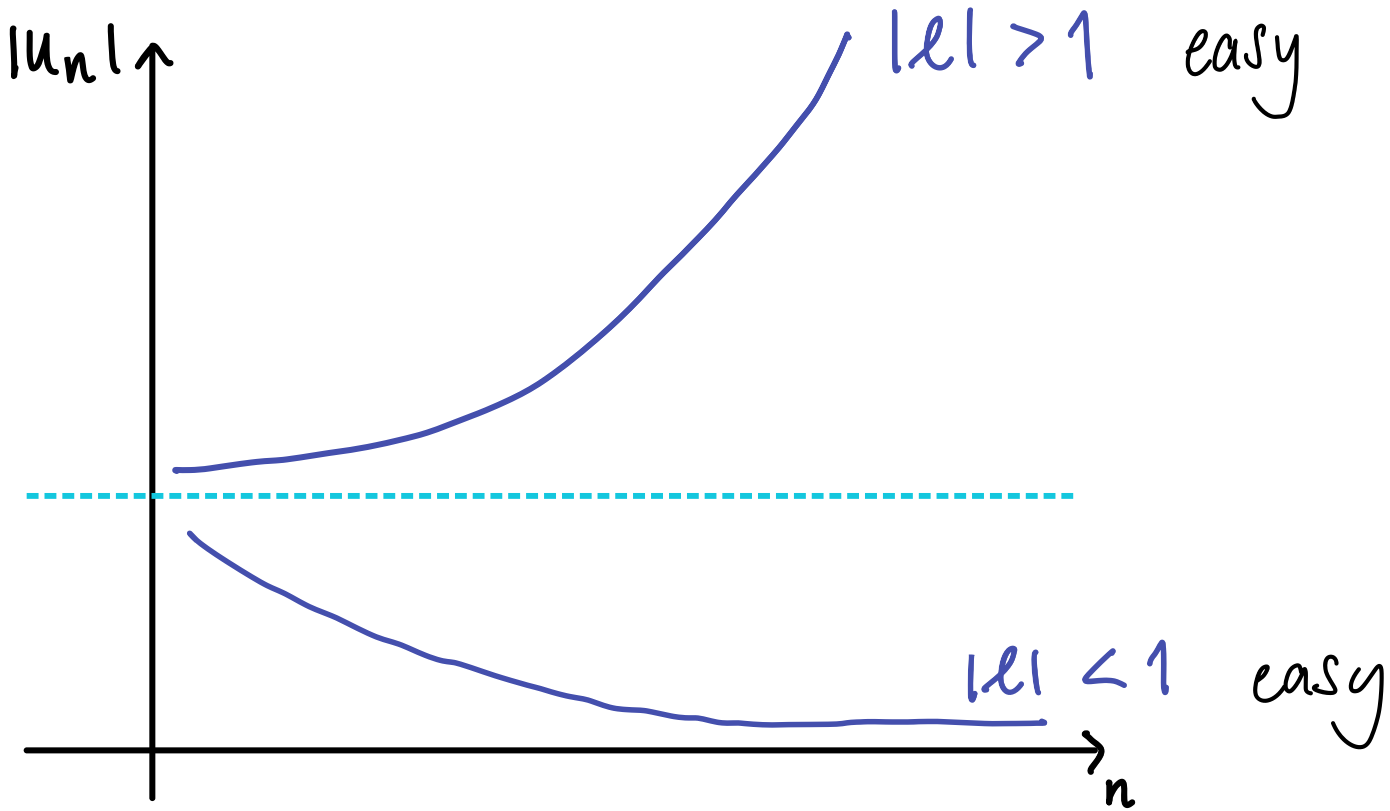

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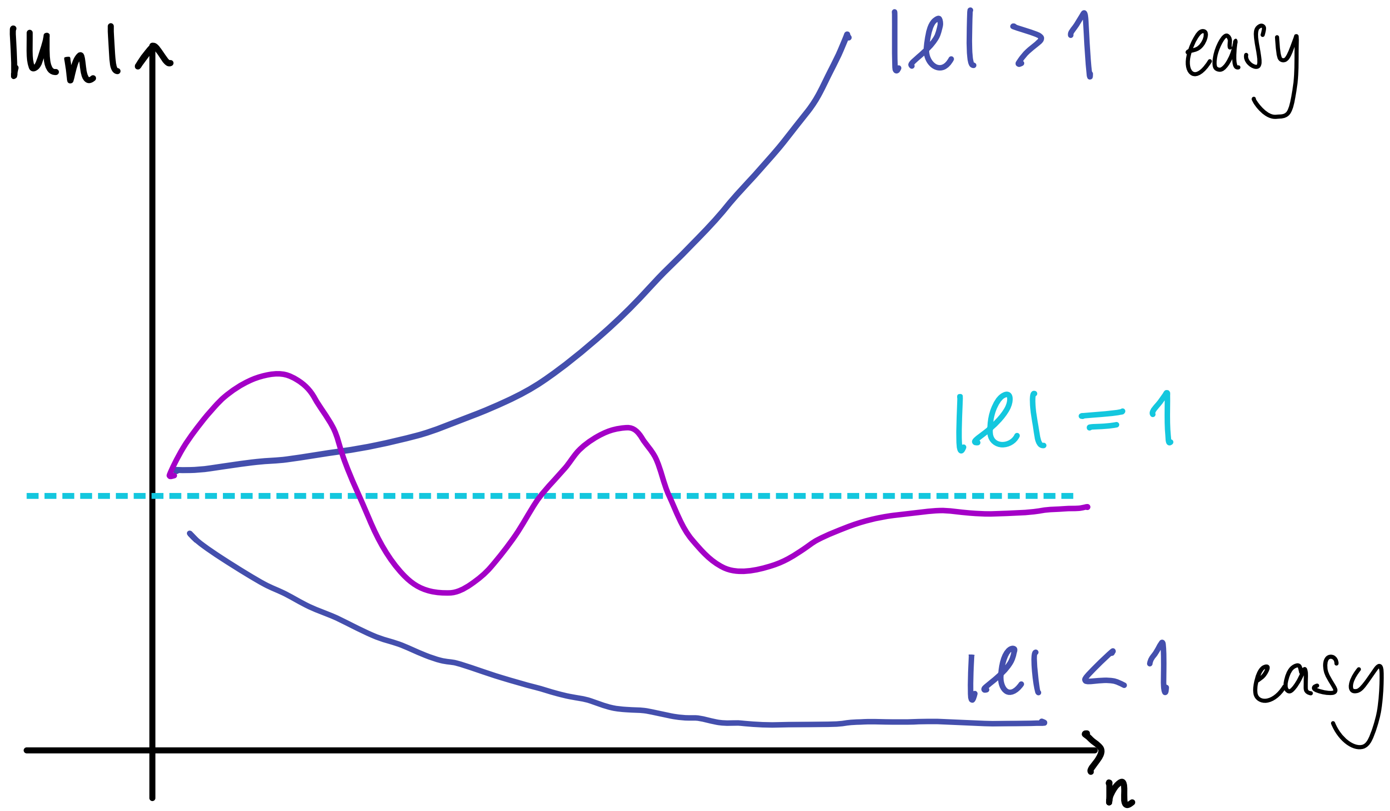

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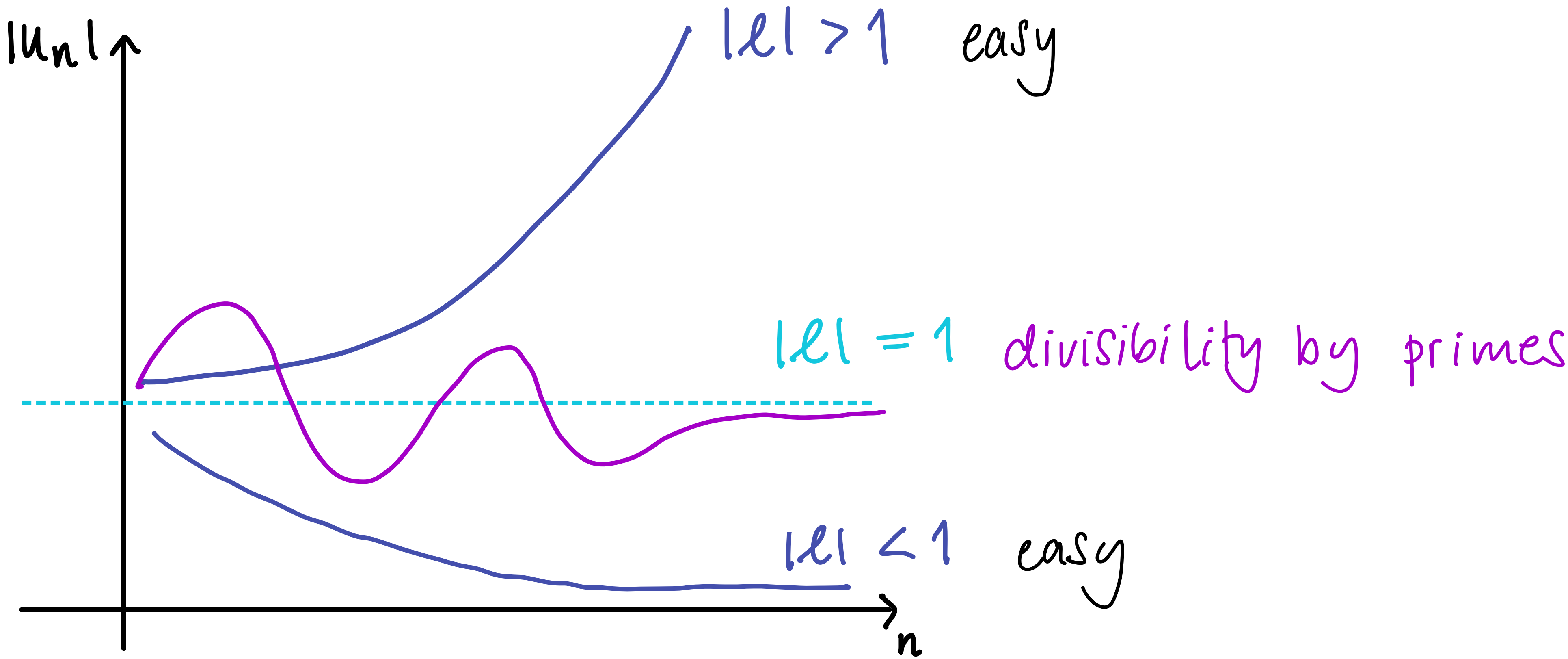

$\langle u_n \rangle_{n=0}^{\infty}$ $u_0 = 1$ $r(x) = \dfrac{(x+\frac{9}{2})(x+\frac{7}{2})(x+\frac{5}{2})}{(x+\frac{11}{2})(x+4)(x+1)}$

$\boxed{\exists n \in \mathbb{N} \text{ s.t. } u_n = \frac{13}{6} \, ?}$

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$$\lim_{n \to \infty} u_n = \frac{3 \cdot 2^5}{5\pi} \neq \frac{13}{6}$$

# A number theoretical diversion

$$u_n = u_0 \prod_{k=1}^{n} r(k) = u_0 \prod_{k=1}^{n} \frac{(k+\alpha_1)\cdots(k+\alpha_d)}{(k+\beta_1)\cdots(k+\beta_d)}$$

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for $m \in \mathbb{N}$ $\quad \Gamma(m) = (m-1)!$

for $c \in \mathbb{C}$ $\quad \Gamma(c+1) = c \cdot \Gamma(c)$

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What happens as $n \to \infty$ ?

$$\prod_{k=1}^{n} (k+\alpha) = \,?$$

$$\begin{aligned}
&\Gamma(n+\alpha+1) \\
&= (n+\alpha)\,\Gamma(n+\alpha) \\
&= (n+\alpha)(n-1+\alpha)\,\Gamma(n-1+\alpha) \\
&\;\;\vdots \\
&= \boxed{\prod_{k=1}^{n}(k+\alpha)} \cdot \alpha\,\Gamma(\alpha)
\end{aligned}$$

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**Prop.** Let $\alpha_1, \dots, \alpha_d, \beta_1, \dots, \beta_d \in \mathbb{C} \setminus \mathbb{Z}_{\le 0}$.

If $\alpha_1 + \dots + \alpha_d = \beta_1 + \dots + \beta_d$ then

$$\prod_{k=1}^{\infty} \frac{(k+\alpha_1) \cdots (k+\alpha_d)}{(k+\beta_1) \cdots (k+\beta_d)} = \frac{\beta_1 \cdots \beta_d}{\alpha_1 \cdots \alpha_d} \frac{\Gamma(\alpha_1) \cdots \Gamma(\alpha_d)}{\Gamma(\beta_1) \cdots \Gamma(\beta_d)}$$

otherwise the infinite product diverges.

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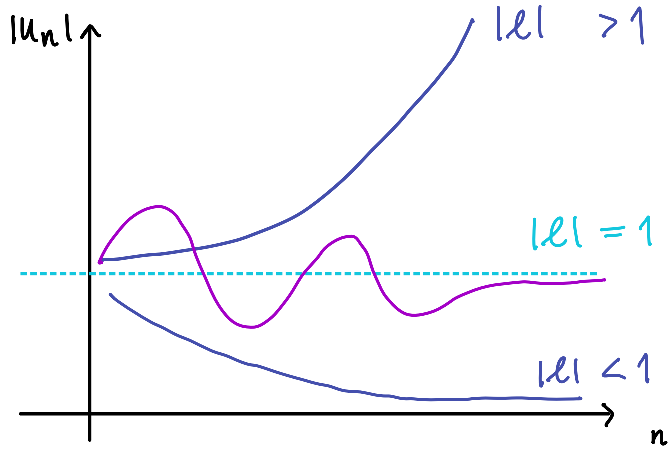
otherwise the infinite product diverges.

$$\lim_{n\to\infty} u_n =$$

$$= \lim_{n\to\infty} u_0 \prod_{k=1}^{\infty} \frac{(k+\alpha_1)\cdots(k+\alpha_d)}{(k+\beta_1)\cdots(k+\beta_d)}$$

$$= \text{const.} \frac{\Gamma(\alpha_1)\cdots\Gamma(\alpha_d)}{\Gamma(\beta_1)\cdots\Gamma(\beta_d)}$$



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**Prop.** The Membership Problem for hypergeometric sequences reduces to deciding, given $\alpha_1, \dots, \alpha_d$ and $\beta_1, \dots, \beta_d \in \mathbb{R} \setminus \mathbb{Z}_{<0}$, whether

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Values of the Gamma function?

$$\frac{\Gamma(\frac{1}{14})\,\Gamma(\frac{9}{14})\,\Gamma(\frac{11}{14})}{\Gamma(\frac{3}{14})\,\Gamma(\frac{5}{14})\,\Gamma(\frac{13}{14})} = 2$$

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**Conj.** (Rohrlich-Lang '78) Any multiplicative relation of Gamma values on rational points that is algebraic is a consequence of the standard relations

translation, reflection, multiplication.

Can we prove decidability unconditionally?

# Divisibility by p

p-adic valuation

$\nu_p : \mathbb{Q} \to \mathbb{Z} \cup \{\infty\}$

$\nu_p(x) := k$ if $x = p^k \frac{a}{b}$ and $p \nmid ab$

$\nu_p(0) := \infty$

$\nu_3\left(\frac{36}{5}\right) = \nu_3\left(9 \cdot \frac{4}{5}\right) = 2$

$\nu_p(a \cdot b) = \nu_p(a) + \nu_p(b)$

# Divisibility by p

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## ring $\mathbb{Z}_{(p)}$

$\mathbb{Z}_{(p)} = \{x \in \mathbb{Q} : v_p(x) \geq 0\}$

$\mathbb{Z}_{(p)} = \{\frac{a}{b} \in \mathbb{Q} : p \nmid b\}$

# Divisibility by p

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$\mathbb{Z}_{(p)} = \{x \in \mathbb{Q} : \nu_p(x) \geq 0\}$

$\mathbb{Z}_{(p)} = \{\frac{a}{b} \in \mathbb{Q} : p \nmid b\}$

- remainder with respect to p

$rem_p : \mathbb{Z}_{(p)} \to \mathbb{F}_p$

$rem_p\left(\frac{a}{b}\right) := ab^{-1} \pmod{p}$

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$$\boxed{u_n = u_0 \prod_{k=1}^{n} r(k)} \qquad \boxed{t}$$

$$v_p(u_n) = v_p(u_0) + v_p\left(\prod_{k=1}^{n} r(k)\right) \qquad v_p(t)$$

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$\exists M \in \mathbb{N}$ s.t. for all $n > M$ $\exists$ prime $p$ s.t.

$$v_p\left(\prod_{k=1}^{n} r(k)\right) \neq 0 \quad \text{and} \quad v_p(t) = v_p(u_0) = 0$$

# The nontrivial instances of MP

Shift quotient $r(x) \to \pm 1$ as $x \to \infty$

$$r(x) = \frac{(x-\alpha_1) \cdots (x-\alpha_d)}{(x-\beta_1) \cdots (x-\beta_d)} \qquad A := \{\alpha_1, \dots, \alpha_d\}$$
$$B := \{\beta_1, \dots, \beta_d\}$$

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$$\nu_p\left(\prod_{k=1}^{n} r(k)\right)$$

$$S_p(n) := \sum_{k=1}^{n}\left(\sum_{\alpha\in A}\nu_p(k-\alpha) - \sum_{\beta\in B}\nu_p(k-\beta)\right)$$

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Certificate that $u_n \neq t$: prime $p$ s.t. $S_p(n) \neq 0$

$\langle u_n \rangle_{n=0}^{\infty}$ $\qquad u_0 = 1$ $\qquad r(x) = \dfrac{(x+\frac{9}{2})(x+\frac{7}{2})(x+\frac{5}{2})}{(x+\frac{11}{2})(x+4)(x+1)}$

$$\exists n \text{ s.t. } u_n = \frac{13}{6} \, ?$$

$\alpha_1 = -\frac{9}{2}$ $\qquad \beta_1 = -\frac{11}{2}$

$\alpha_2 = -\frac{7}{2}$ $\qquad \beta_2 = -4$

$\alpha_3 = -\frac{5}{2}$ $\qquad \beta_3 = -1$

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$$\alpha_2 = -\frac{7}{2} = -3 - \frac{1}{2} \qquad \beta_2 = -4 = -3 - \frac{2}{2}$$

$$\alpha_3 = -\frac{5}{2} = -2 - \frac{1}{2} \qquad \beta_3 = -1 = 0 - \frac{2}{2}$$

**Def.** (canonical representation)
$\gamma = c - \frac{a}{b}$ where $c \in \mathbb{Z}$ and $a \in \{1, \dots, b\}$.

# Choosing the prime

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# Choosing the prime

Prime $p$ in $b\mathbb{N}+1$:

$$\alpha_1, \dots, \alpha_d, \beta_1, \dots, \beta_d \in \mathbb{Z}_{(p)}$$

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3, 5, 7, 9, 11, 13, 17, 21, 23, 25, 27, 29, . . .

# Choosing the prime

Prime $p$ in $b\mathbb{N}+1$:

$$\alpha_1, \dots, \alpha_d, \beta_1, \dots, \beta_d \in \mathbb{Z}_{(p)}$$

$\alpha_1 = -\frac{9}{2} = -4 - \frac{1}{2}$

$\beta_1 = -\frac{11}{2} = -5 - \frac{1}{2}$

$\alpha_2 = -\frac{7}{2} = -3 - \frac{1}{2}$

$\beta_2 = -4 = -3 - \frac{2}{2}$

$\alpha_3 = -\frac{5}{2} = -2 - \frac{1}{2}$

$\beta_3 = -1 = 0 - \frac{2}{2}$

3, 5, 7, 9, 11, 13, 17, 21, 23, 25, 27, 29, ...

$\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3 \in \mathbb{Z}_{(17)}, \mathbb{Z}_{(23)}, \mathbb{Z}_{(29)}$

# The remainder

**Prop.** Let $\gamma = c - \frac{a}{b}$. For all primes $p > M$ in $b\mathbb{N}+1$:

$$\mathrm{rem}_p(\gamma) = c + \frac{(p-1)a}{b}$$

$\alpha_1 = -4 - \frac{1}{2}$

$\alpha_2 = -3 - \frac{1}{2}$

$\alpha_3 = -2 - \frac{1}{2}$

$\beta_1 = -5 - \frac{1}{2}$

$\beta_2 = -3 - \frac{2}{2}$

$\beta_3 = 0 - \frac{2}{2}$

# The remainder

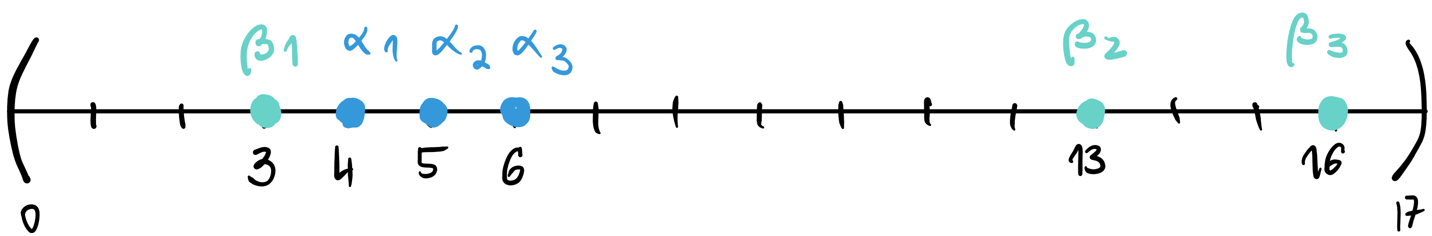
**Prop.** Let $\gamma = c - \frac{a}{b}$. For all primes $p > M$ in $b\mathbb{N} + 1$:

$$\mathrm{rem}_p(\gamma) = c + \frac{(p-1)a}{b}$$

$\alpha_1 = -4 - \frac{1}{2}$ $\quad \mathrm{rem}_p(\alpha_1) = 4$ $\quad \beta_1 = -5 - \frac{1}{2}$ $\quad \mathrm{rem}_p(\beta_1) = 3$

$\alpha_2 = -3 - \frac{1}{2}$ $\quad \mathrm{rem}_p(\alpha_2) = 5$ $\quad \beta_2 = -3 - \frac{2}{2}$ $\quad \mathrm{rem}_p(\beta_2) = 13$

$\alpha_3 = -2 - \frac{1}{2}$ $\quad \mathrm{rem}_p(\alpha_3) = 6$ $\quad \beta_3 = 0 - \frac{2}{2}$ $\quad \mathrm{rem}_p(\beta_3) = 16$



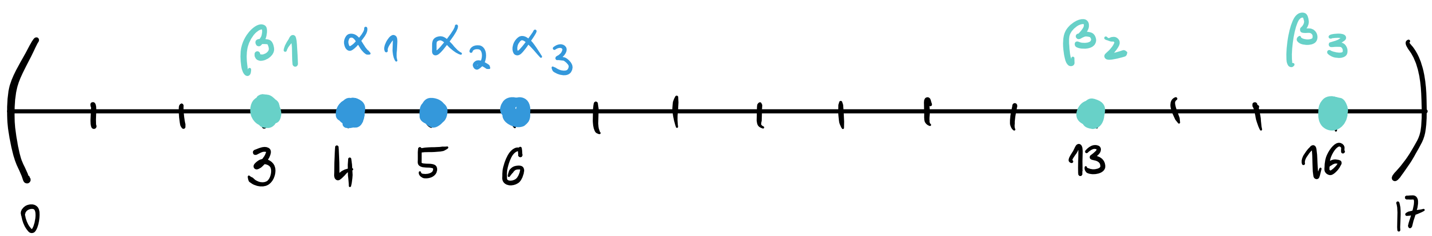
# An ordering on $\mathbb{Z}_{(p)}$

**Def.**

$\gamma, \gamma' \in \mathbb{Z}_{(p)}$:

$$\gamma \preceq_p \gamma' \quad \text{iff} \quad \mathrm{rem}_p(\gamma) \leq \mathrm{rem}_p(\gamma')$$

$$\beta_1 \preceq_{17} \alpha_1 \preceq_{17} \alpha_2 \preceq_{17} \alpha_3 \preceq_{17} \beta_2 \preceq_{17} \beta_3$$



# The preorder $\preceq$

**Prop.** Let $\gamma = c - \frac{a}{b}$ and $\gamma' = c' - \frac{a'}{b'}$. For all primes $p > M$ in $b\mathbb{N}+1$:

$$\gamma \prec_p \gamma' \quad \text{iff} \quad ((a < a') \text{ or } (a = a' \text{ and } c < c'))$$

$$\beta_1 \preceq_{17} \alpha_1 \preceq_{17} \alpha_2 \preceq_{17} \alpha_3 \preceq_{17} \beta_2 \preceq_{17} \beta_3$$

# The preorder $\preceq$

**Prop.** Let $\gamma = c - \frac{a}{b}$ and $\gamma' = c' - \frac{a'}{b'}$. For all primes $p > M$ in $b\mathbb{N}+1$:

$$\gamma \prec_p \gamma' \quad \text{iff} \quad ((a < a') \text{ or } (a = a' \text{ and } c < c'))$$

$$\beta_1 \preceq_{17} \alpha_1 \preceq_{17} \alpha_2 \preceq_{17} \alpha_3 \preceq_{17} \beta_2 \preceq_{17} \beta_3$$

$$\beta_1 \preceq_{23} \alpha_1 \preceq_{23} \alpha_2 \preceq_{23} \alpha_3 \preceq_{23} \beta_2 \preceq_{23} \beta_3$$

# The preorder $\preceq$

**Prop.** Let $\gamma = c - \frac{a}{b}$ and $\gamma' = c' - \frac{a'}{b'}$. For all primes $p > M$ in $b\mathbb{N} + 1$:

$$\gamma \prec_p \gamma' \quad \text{iff} \quad ((a < a') \text{ or } (a = a' \text{ and } c < c'))$$

$$\beta_1 \preceq_{17} \alpha_1 \preceq_{17} \alpha_2 \preceq_{17} \alpha_3 \preceq_{17} \beta_2 \preceq_{17} \beta_3$$

$$\beta_1 \preceq_{23} \alpha_1 \preceq_{23} \alpha_2 \preceq_{23} \alpha_3 \preceq_{23} \beta_2 \preceq_{23} \beta_3$$

$$\beta_1 \preceq_{29} \alpha_1 \preceq_{29} \alpha_2 \preceq_{29} \alpha_3 \preceq_{29} \beta_2 \preceq_{29} \beta_3$$

$$\preceq_{17} = \preceq_{23} = \preceq_{29} = \preceq$$

# Back to the problem

Certificate that $u_n \neq t$: prime $p$ s.t.

$$S_p(n) = \sum_{k=1}^{n} \left( \sum_{\alpha \in A} \nu_p(k-\alpha) - \sum_{\beta \in B} \nu_p(k-\beta) \right) \neq 0$$

# Back to the problem

Certificate that $u_n \neq t$: prime $p$ s.t.

$$S_p(n) = \sum_{k=1}^{n} \left( \sum_{\alpha \in A} \nu_p(k-\alpha) - \sum_{\beta \in B} \nu_p(k-\beta) \right) \neq 0$$

**Prop.**
$\exists P \in \mathbb{N}$ s.t. for primes $p > P$, for all $k < p$ and $\gamma \in A \uplus B$

$$\nu_p(k-\gamma) = \begin{cases} 1 & \text{if } \mathrm{rem}_p(\gamma) = \mathrm{rem}_p(k), \\ 0 & \text{otherwise} \end{cases}$$

# Back to the problem

Certificate that $u_n \neq t$: prime $p$ s.t.

$$S_p(n) = \sum_{k=1}^{n} \left( \sum_{\alpha \in A} \nu_p(k-\alpha) - \sum_{\beta \in B} \nu_p(k-\beta) \right) \neq 0$$

**Prop.**
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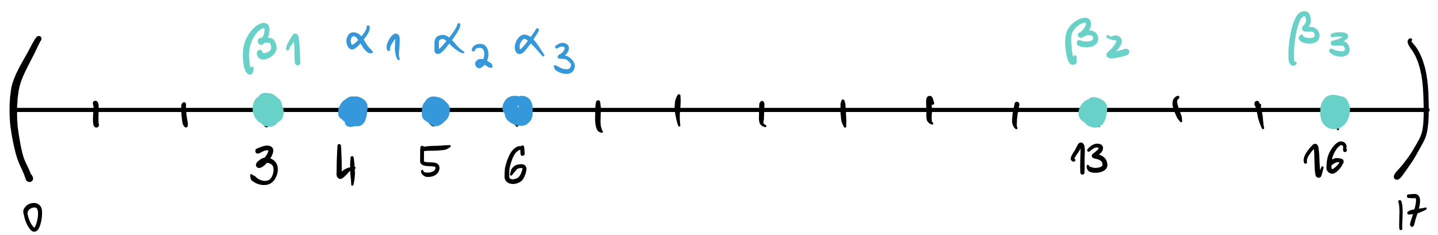
$$\nu_p(k-\gamma) = \begin{cases} 1 & \text{if } \mathrm{rem}_p(\gamma) = \mathrm{rem}_p(k), \\ 0 & \text{otherwise} \end{cases}$$




| | |
|---|---|
| $k$ | 1 |
| contribution | 0 |
| $S_p(k)$ | 0 |

# Back to the problem

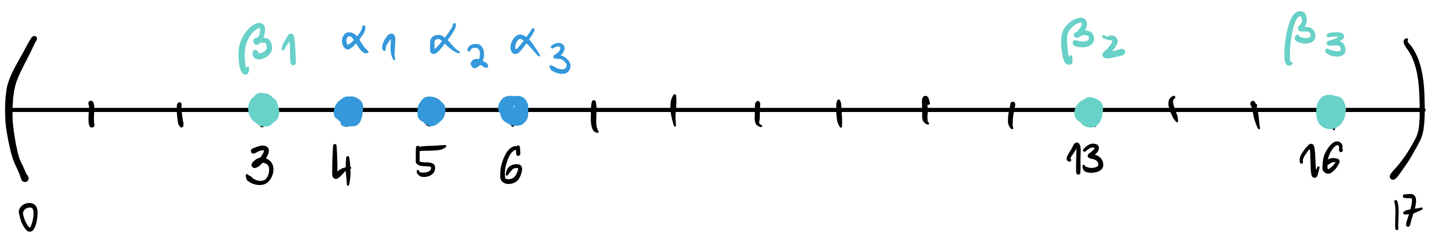
Certificate that $u_n \neq t$: prime $p$ s.t.

$$S_p(n) = \sum_{k=1}^{n} \Big( \sum_{\alpha \in A} \nu_p(k-\alpha) - \sum_{\beta \in B} \nu_p(k-\beta) \Big) \neq 0$$

**Prop.**

$\exists P \in \mathbb{N}$ s.t for primes $p > P$, for all $k < p$ and $\gamma \in A \uplus B$,

$$\nu_p(k-\gamma) = \begin{cases} 1 & \text{if } \mathrm{rem}_p(\gamma) = \mathrm{rem}_p(k), \\ 0 & \text{otherwise} \end{cases}$$




| | | |
|---|---|---|
| $k$ | 1 | 2 |
| contribution | 0 | 0 |
| $S_p(k)$ | 0 | 0 |

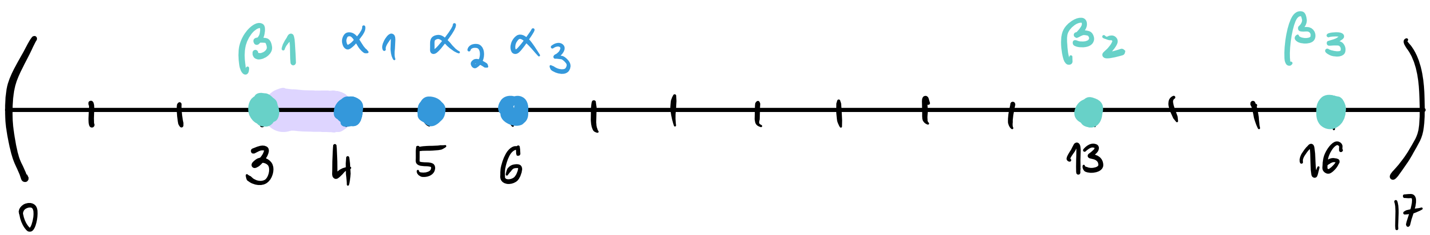
# Back to the problem

Certificate that $u_n \neq t$: prime $p$ s.t.

$$S_p(n) = \sum_{k=1}^{n} \Big( \sum_{\alpha \in A} \nu_p(k-\alpha) - \sum_{\beta \in B} \nu_p(k-\beta) \Big) \neq 0$$

**Prop.**
$\exists P \in \mathbb{N}$ s.t for primes $p > P$, for all $k < p$ and $\gamma \in A \uplus B$

$$\nu_p(k-\gamma) = \begin{cases} 1 & \text{if } \mathrm{rem}_p(\gamma) = \mathrm{rem}_p(k), \\ 0 & \text{otherwise} \end{cases}$$




| | | | |
|---|---|---|---|
| $k$ | 1 | 2 | 3 |
| contribution | 0 | 0 | -1 |
| $S_p(k)$ | 0 | 0 | -1 |

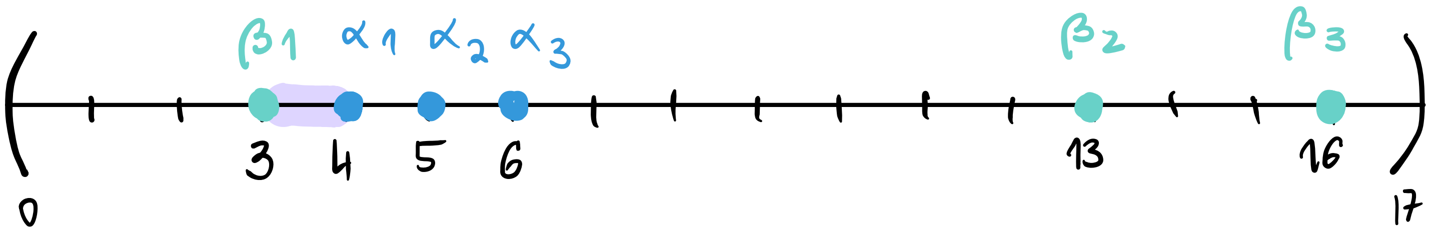
# Back to the problem

Certificate that $u_n \neq t$: prime $p$ s.t.

$$S_p(n) = \sum_{k=1}^{n} \left( \sum_{\alpha \in A} \eta_p(k-\alpha) - \sum_{\beta \in B} \eta_p(k-\beta) \right) \neq 0$$

**Prop.**
$\exists P \in \mathbb{N}$ s.t for primes $p > P$, for all $k < p$ and $\gamma \in A \uplus B$

$$\eta_p(k-\gamma) = \begin{cases} 1 & \text{if } \mathrm{rem}_p(\gamma) = \mathrm{rem}_p(k), \\ 0 & \text{otherwise} \end{cases}$$




| | | | | |
|---|---|---|---|---|
| $k$ | 1 | 2 | 3 | 4 |
| contribution | 0 | 0 | -1 | 1 |
| $S_p(k)$ | 0 | 0 | -1 | 0 |

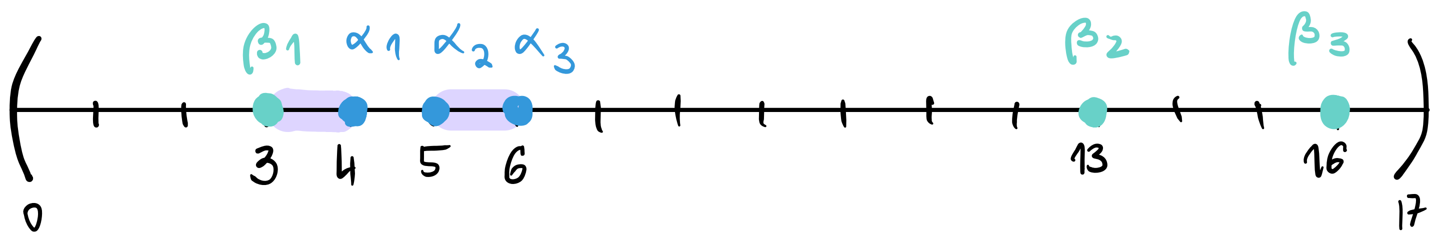
# Back to the problem

Certificate that $u_n \neq t$: prime $p$ s.t.

$$S_p(n) = \sum_{k=1}^{n} \left( \sum_{\alpha \in A} r_p(k-\alpha) - \sum_{\beta \in B} r_p(k-\beta) \right) \neq 0$$

**Prop.**
$\exists P \in \mathbb{N}$ s.t for primes $p > P$, for all $k < p$ and $\gamma \in A \uplus B$,

$$r_p(k-\gamma) = \begin{cases} 1 & \text{if } \mathrm{rem}_p(\gamma) = \mathrm{rem}_p(k), \\ 0 & \text{otherwise} \end{cases}$$




| | | | | | |
|---|---|---|---|---|---|
| $k$ | 1 | 2 | 3 | 4 | 5 |
| contribution | 0 | 0 | -1 | 1 | 1 |
| $S_p(k)$ | 0 | 0 | -1 | 0 | 1 |

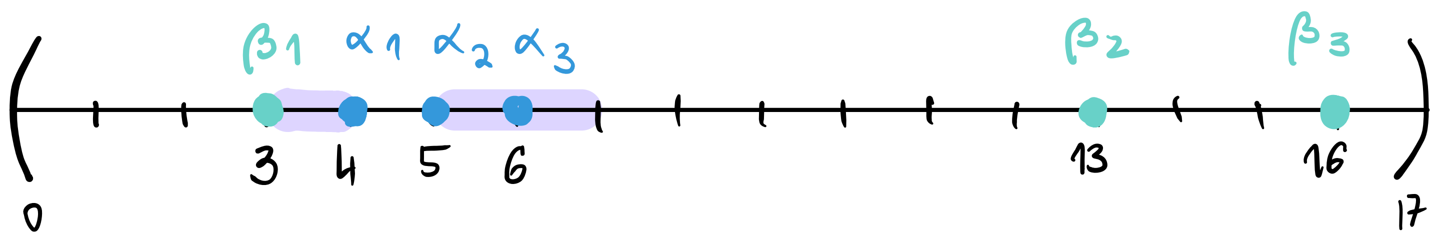
# Back to the problem

Certificate that $u_n \neq t$: prime $p$ s.t.

$$S_p(n) = \sum_{k=1}^{n} \left( \sum_{\alpha \in A} \nu_p(k-\alpha) - \sum_{\beta \in B} \nu_p(k-\beta) \right) \neq 0$$

**Prop.**
$\exists P \in \mathbb{N}$ s.t for primes $p > P$, for all $k < p$ and $\gamma \in A \uplus B$

$$\nu_p(k-\gamma) = \begin{cases} 1 & \text{if } \mathrm{rem}_p(\gamma) = \mathrm{rem}_p(k), \\ 0 & \text{otherwise} \end{cases}$$




| | | | | | | |
|---|---|---|---|---|---|---|
| $k$ | 1 | 2 | 3 | 4 | 5 | 6 |
| contribution | 0 | 0 | -1 | 1 | 1 | 1 |
| $S_p(k)$ | 0 | 0 | -1 | 0 | 1 | 2 |

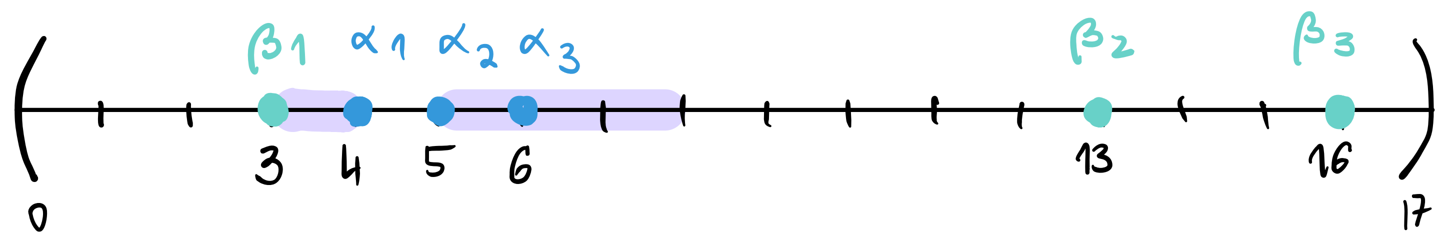
# Back to the problem

Certificate that $u_n \neq t$: prime $p$ s.t.

$$S_p(n) = \sum_{k=1}^{n} \left( \sum_{\alpha \in A} \nu_p(k-\alpha) - \sum_{\beta \in B} \nu_p(k-\beta) \right) \neq 0$$

**Prop.**
$\exists P \in \mathbb{N}$ s.t for primes $p > P$, for all $k < p$ and $\gamma \in A \uplus B$

$$\nu_p(k-\gamma) = \begin{cases} 1 & \text{if } rem_p(\gamma) = rem_p(k), \\ 0 & \text{otherwise} \end{cases}$$




| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| contribution | 0 | 0 | -1 | 1 | 1 | 1 | 0 |
| $S_p(k)$ | 0 | 0 | -1 | 0 | 1 | 2 | 2 |

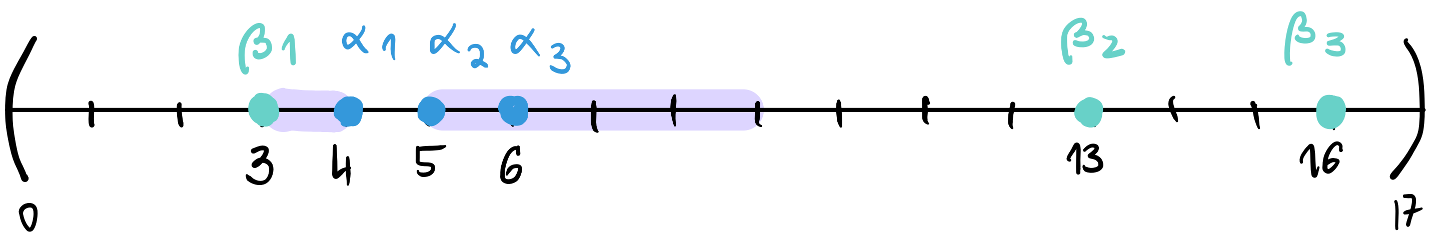
# Back to the problem

Certificate that $u_n \neq t$: prime $p$ s.t.

$$S_p(n) = \sum_{k=1}^{n} \left( \sum_{\alpha \in A} \nu_p(k-\alpha) - \sum_{\beta \in B} \nu_p(k-\beta) \right) \neq 0$$

**Prop.**
$\exists P \in \mathbb{N}$ s.t for primes $p > P$, for all $k < p$ and $\gamma \in A \uplus B$

$$\nu_p(k-\gamma) = \begin{cases} 1 & \text{if } \mathrm{rem}_p(\gamma) = \mathrm{rem}_p(k), \\ 0 & \text{otherwise} \end{cases}$$




| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| contribution | 0 | 0 | -1 | 1 | 1 | 1 | 0 | 0 |
| $S_p(k)$ | 0 | 0 | -1 | 0 | 1 | 2 | 2 | 2 |

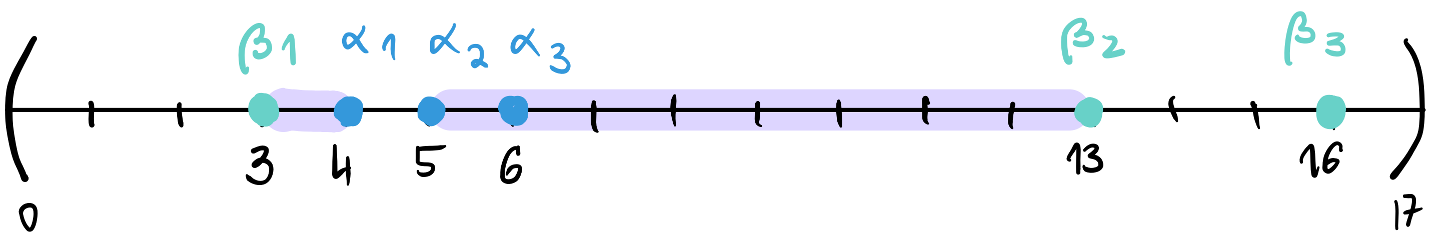
# Back to the problem

Certificate that $u_n \neq t$ : prime $p$ s.t.

$$S_p(n) = \sum_{k=1}^{n} \left( \sum_{\alpha \in A} \nu_p(k-\alpha) - \sum_{\beta \in B} \nu_p(k-\beta) \right) \neq 0$$

**Prop.**
$\exists P \in \mathbb{N}$ s.t for primes $p > P$, for all $k < p$ and $\gamma \in A \uplus B$

$$\nu_p(k-\gamma) = \begin{cases} 1 & \text{if } \mathrm{rem}_p(\gamma) = \mathrm{rem}_p(k), \\ 0 & \text{otherwise} \end{cases}$$




| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | |
| contribution | 0 | 0 | -1 | 1 | 1 | 1 | 0 | 0 | . . . |
| $S_p(k)$ | 0 | 0 | -1 | 0 | 1 | 2 | 2 | 2 | |

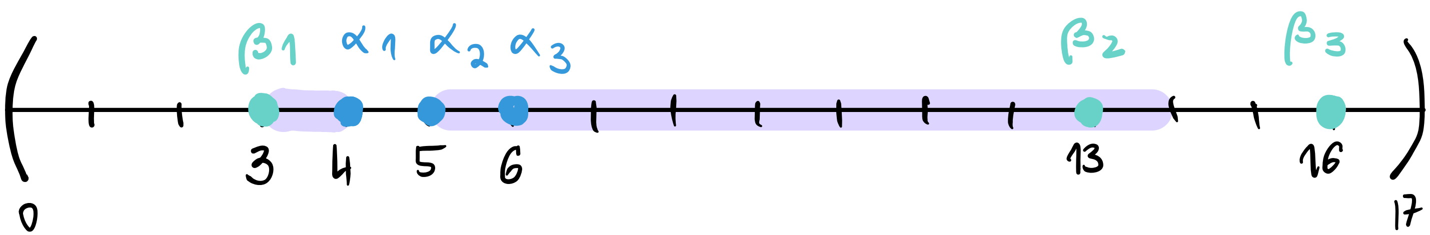
# Back to the problem

Certificate that $u_n \neq t$: prime $p$ s.t.

$$S_p(n) = \sum_{k=1}^{n} \left( \sum_{\alpha \in A} \nu_p(k-\alpha) - \sum_{\beta \in B} \nu_p(k-\beta) \right) \neq 0$$

**Prop.**
$\exists P \in \mathbb{N}$ s.t for primes $p > P$, for all $k < p$ and $\gamma \in A \uplus B$

$$\nu_p(k-\gamma) = \begin{cases} 1 & \text{if } \mathrm{rem}_p(\gamma) = \mathrm{rem}_p(k), \\ 0 & \text{otherwise} \end{cases}$$




| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | | 13 |
| contribution | 0 | 0 | -1 | 1 | 1 | 1 | 0 | 0 | ... | -1 |
| $S_p(k)$ | 0 | 0 | -1 | 0 | 1 | 2 | 2 | 2 | | 1 |

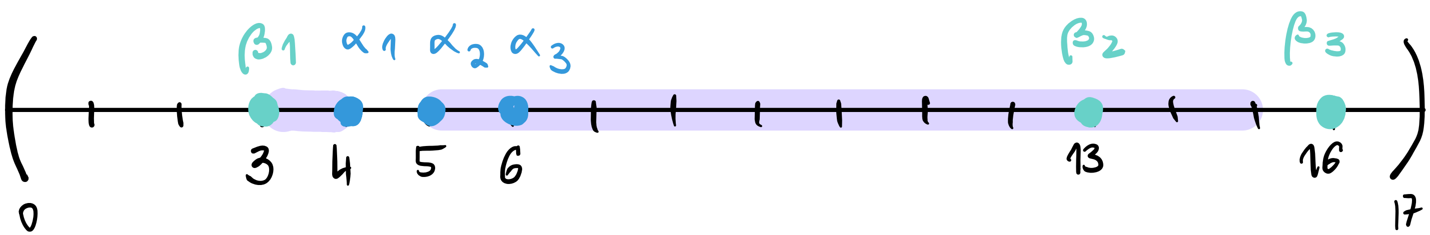
# Back to the problem

Certificate that $u_n \neq t$: prime $p$ s.t.

$$S_p(n) = \sum_{k=1}^{n} \left( \sum_{\alpha \in A} \nu_p(k-\alpha) - \sum_{\beta \in B} \nu_p(k-\beta) \right) \neq 0$$

**Prop.**
$\exists P \in \mathbb{N}$ s.t. for primes $p > P$, for all $k < p$ and $\gamma \in A \uplus B$

$$\nu_p(k-\gamma) = \begin{cases} 1 & \text{if } \mathrm{rem}_p(\gamma) = \mathrm{rem}_p(k), \\ 0 & \text{otherwise} \end{cases}$$




| $k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | ... | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| contribution | 0 | 0 | -1 | 1 | 1 | 1 | 0 | 0 | ... | -1 | 0 |
| $S_p(k)$ | 0 | 0 | -1 | 0 | 1 | 2 | 2 | 2 | | 1 | 1 |

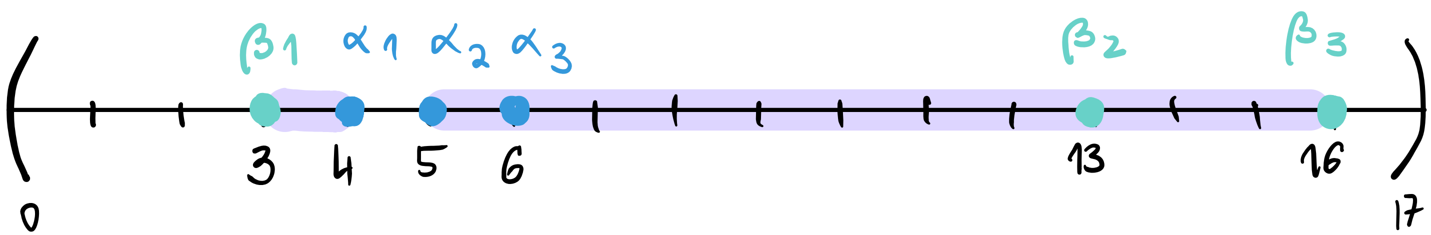
# Back to the problem

Certificate that $u_n \neq t$: prime $p$ s.t.

$$S_p(n) = \sum_{k=1}^{n} \left( \sum_{\alpha \in A} \nu_p(k-\alpha) - \sum_{\beta \in B} \nu_p(k-\beta) \right) \neq 0$$

**Prop.**
$\exists P \in \mathbb{N}$ s.t. for primes $p > P$, for all $k < p$ and $\gamma \in A \uplus B$

$$\nu_p(k-\gamma) = \begin{cases} 1 & \text{if } \mathrm{rem}_p(\gamma) = \mathrm{rem}_p(k), \\ 0 & \text{otherwise} \end{cases}$$




| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | | 13 | 14 | 15 |
| contribution | 0 | 0 | -1 | 1 | 1 | 1 | 0 | 0 | ... | -1 | 0 | 0 |
| $S_p(k)$ | 0 | 0 | -1 | 0 | 1 | 2 | 2 | 2 | | 1 | 1 | 1 |

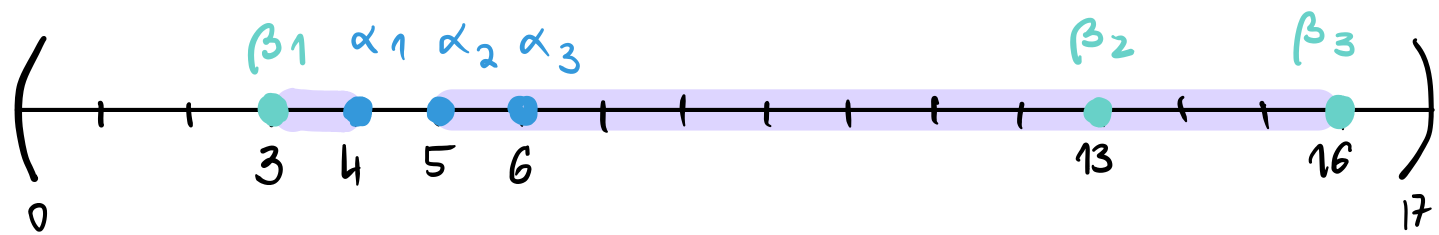
# Back to the problem

Certificate that $u_n \neq t$: prime $p$ s.t.

$$S_p(n) = \sum_{k=1}^{n} \Big( \sum_{\alpha \in A} \nu_p(k-\alpha) - \sum_{\beta \in B} \nu_p(k-\beta) \Big) \neq 0$$

**Prop.**
$\exists P \in \mathbb{N}$ s.t for primes $p > P$, for all $k < p$ and $\gamma \in A \uplus B$,

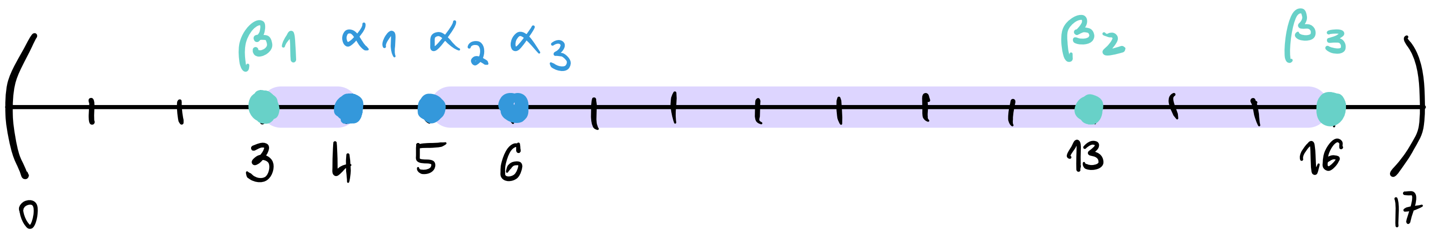
$$\nu_p(k-\gamma) = \begin{cases} 1 & \text{if } \mathrm{rem}_p(\gamma) = \mathrm{rem}_p(k), \\ 0 & \text{otherwise} \end{cases}$$




| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | ... | 13 | 14 | 15 | 16 |
| contribution | 0 | 0 | -1 | 1 | 1 | 1 | 0 | 0 | ... | -1 | 0 | 0 | -1 |
| $S_p(k)$ | 0 | 0 | -1 | 0 | 1 | 2 | 2 | 2 | | 1 | 1 | 1 | 0 |

# Unbalanced intervals and families

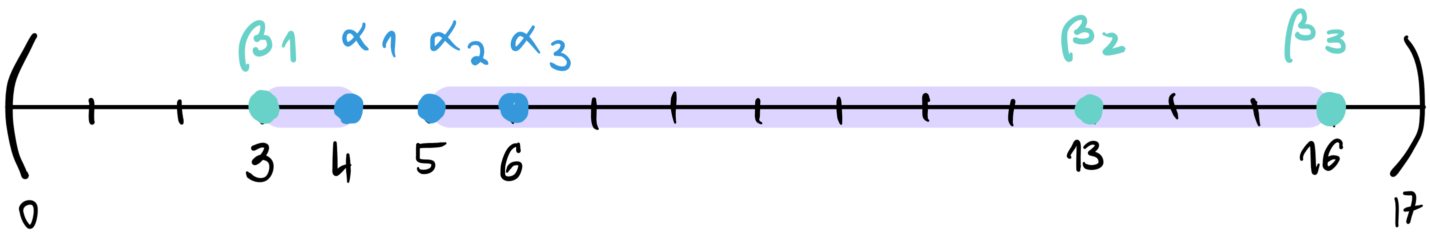
$\boxed{S_p(n) \neq 0}$ iff $|\{\alpha \in A : \alpha \trianglelefteq n\}| \neq |\{\beta \in B : \beta \trianglelefteq n\}|$



# Unbalanced intervals and families

$\boxed{S_p(n) \neq 0}$ iff $|\{\alpha \in A : \alpha \preceq n\}| \neq |\{\beta \in B : \beta \preceq n\}|$

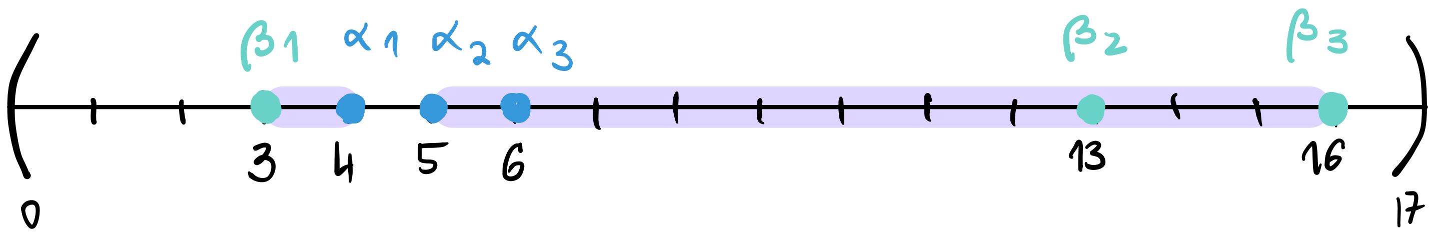
$n$ is unbalanced



# Unbalanced intervals and families

$\boxed{S_p(n) \neq 0}$ iff $|\{\alpha \in A : \alpha \trianglelefteq n\}| \neq |\{\beta \in B : \beta \trianglelefteq n\}|$

$\rightarrow$ $n$ is unbalanced

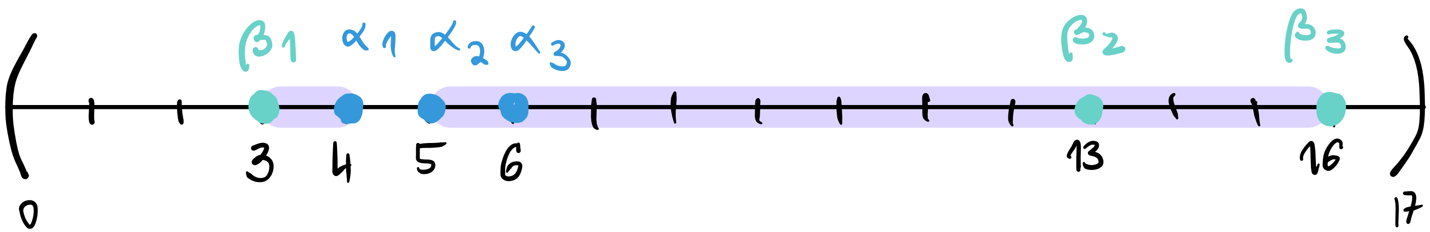



Unbalanced intervals: $\{3\}$ and $\{5, \dots, 15\}$

# Unbalanced intervals and families

$S_p(n) \neq 0$ iff $|\{\alpha \in A : \alpha \preceq n\}| \neq |\{\beta \in B : \beta \preceq n\}|$

$n$ is unbalanced

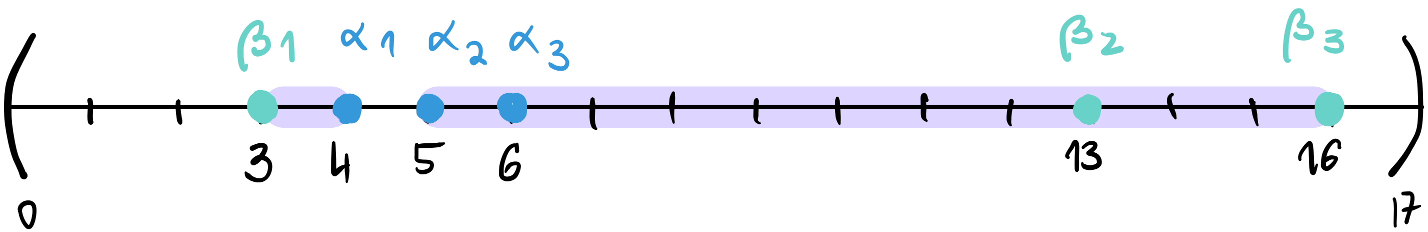


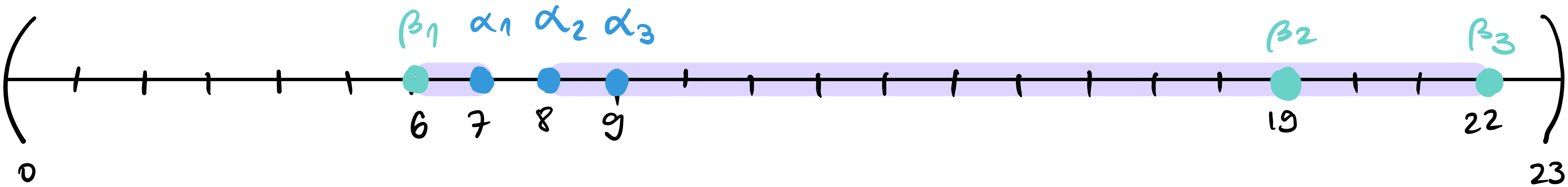

Unbalanced intervals: $\{3\}$ and $\{5, \ldots, 15\}$

$$\beta_1 \preceq \alpha_1 \preceq \alpha_2 \preceq \alpha_3 \preceq \beta_2 \preceq \beta_3$$

Unbalanced families: $\overline{(\beta_1, \alpha_1)}$ and $\overline{(\alpha_2, \beta_2)}$

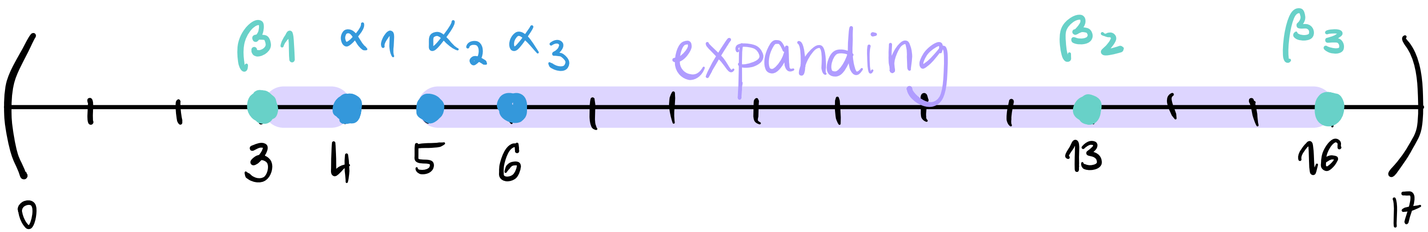
# Expanding families

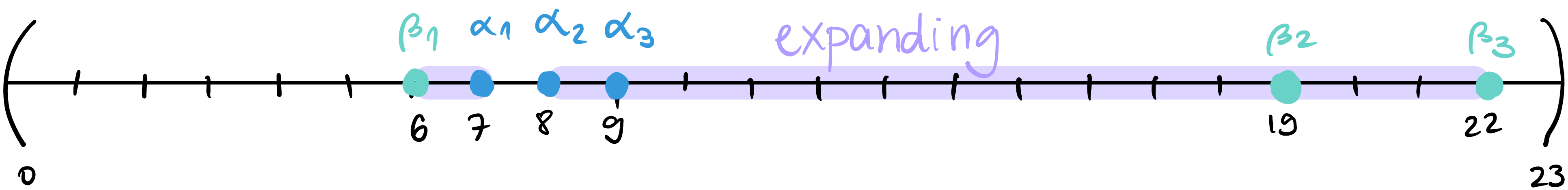



$\alpha_1 = -\frac{9}{2}$ $\alpha_2 = -\frac{7}{2}$ $\alpha_3 = -\frac{5}{2}$ $\beta_1 = -\frac{11}{2}$ $\beta_2 = -4$ $\beta_3 = -1$

# Expanding families




**Prop.** Let $\gamma, \gamma' \in A \uplus B$ be s.t. $\gamma \pitchfork \gamma'$

$$\gamma - \gamma' \notin \mathbb{Z} \quad \text{iff} \quad \overline{(\gamma, \gamma')} \text{ is expanding}$$

$\alpha_1 = -\frac{9}{2}$ $\quad$ $\boxed{\alpha_2 = -\frac{7}{2}}$ $\quad$ $\alpha_3 = -\frac{5}{2}$ $\quad$ $\beta_1 = -\frac{11}{2}$ $\quad$ $\beta_2 = -4$ $\quad$ $\boxed{\beta_3 = -1}$

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No, but ...

**Prop.**

Given $r(x) \in \mathbb{Q}(x)$ converging to $\pm 1$ as $x \to \infty$, either

1. $\exists$ an expanding unbalanced family of intervals, or
2. every hypergeometric sequence $\langle u_n \rangle_{n=0}^{\infty}$ with shift quotient $r(x)$ is a rational function of $n$.

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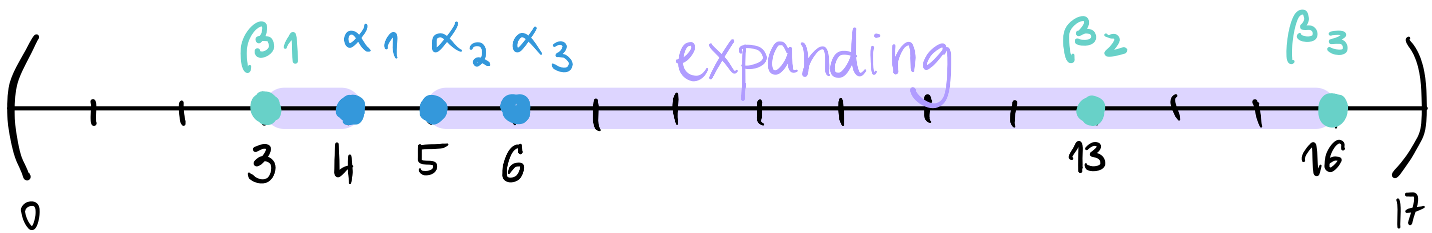
Item 2: $u_n = \frac{f(n)}{g(n)}$ $\quad f, g \in \mathbb{Q}[x]$

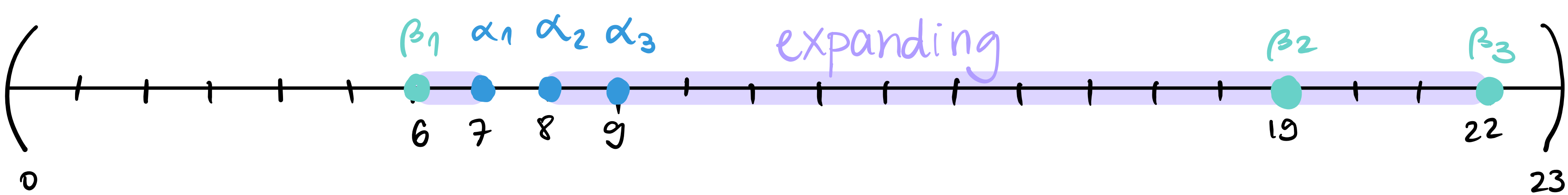
$\exists n \in \mathbb{N}$ s.t. $u_n = t$?

$\exists n \in \mathbb{N}$ s.t. $f(n) - t \cdot g(n) = 0$?

# Where are we now?

Expanding unbalanced family $\overline{(\alpha_2, \beta_3)}$




Prime 17: $\forall n \in \{5, \dots, 15\}: \ u_n \neq t$

Prime 23: $\forall n \in \{8, \dots, 21\}: \ u_n \neq t$

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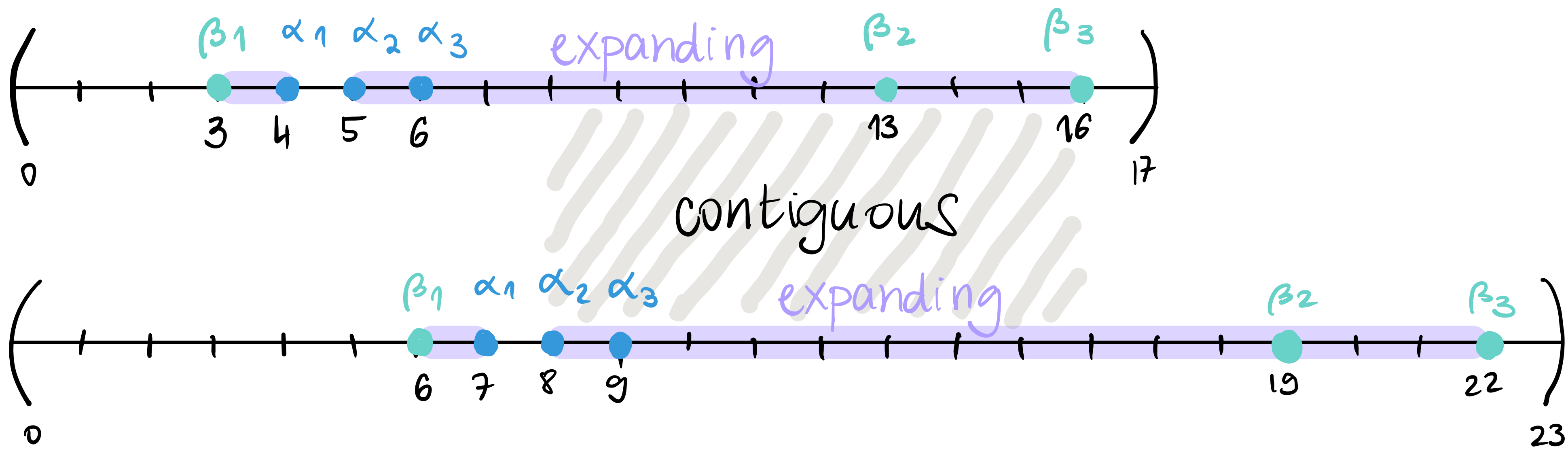


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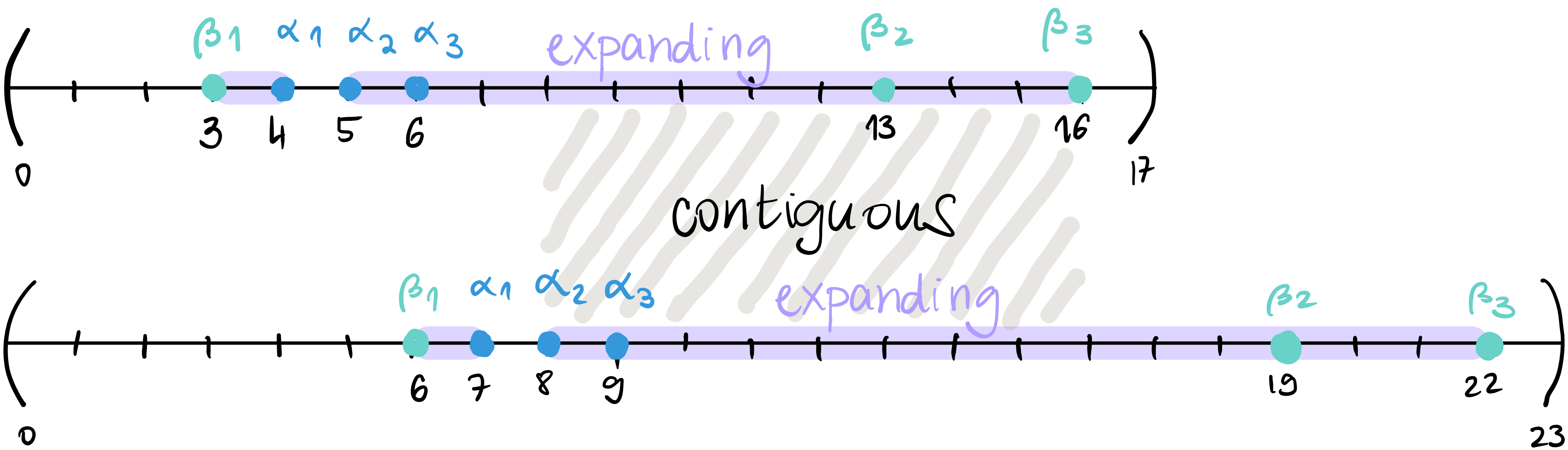


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Can we always ensure contiguity?

# Contiguous intervals

Prop.

Let $p, q \in b\mathbb{N}+1$ be primes. Let $\gamma, \gamma' \in A \uplus B$ s.t. $\gamma - \gamma' \notin \mathbb{Z}$ and $\gamma \prec \gamma'$. If $p < q$ and $q < p + \frac{p-1}{b} + C$

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Our example: $17 < 23 < 17 + \frac{16}{2} + C$ ✓

# Constructing the prime sequence

$\pi_{n,a}(x)$ = #primes in $n\mathbb{N}+a$ that are $\leq x$

$$\pi_{b,1}\left(p+\frac{p-1}{b}+C\right)-\pi_{b,1}(p)>1$$

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**Thm.** (effective Dirichlet)

Given $n \geq 3$ and $a \in \mathbb{N}$ coprime to $n$, there exist explicit positive constants $c$ and $x_0$ depending on $n$ such that

$$\left| \boxed{\pi_{n,a}(x)} - \frac{1}{\varphi(n)} \cdot \frac{x}{\log x} \right| < c \frac{x}{\log^2 x} \quad \text{for all } x > x_0$$

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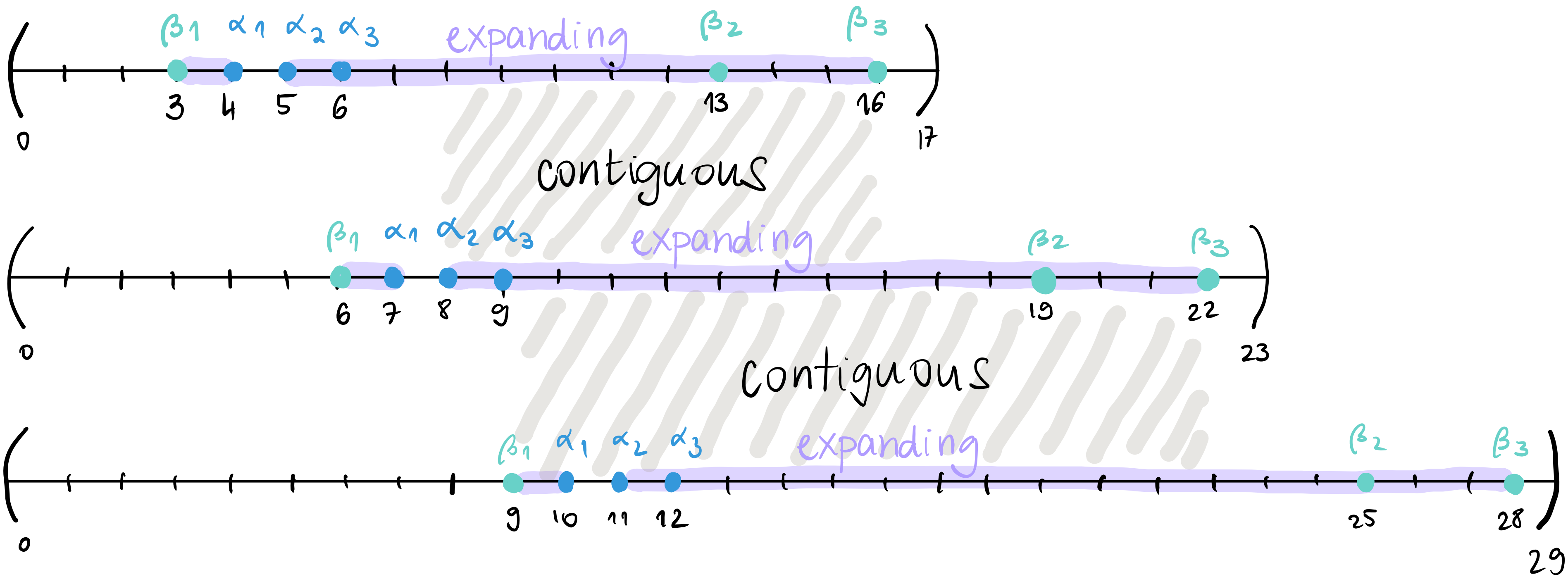
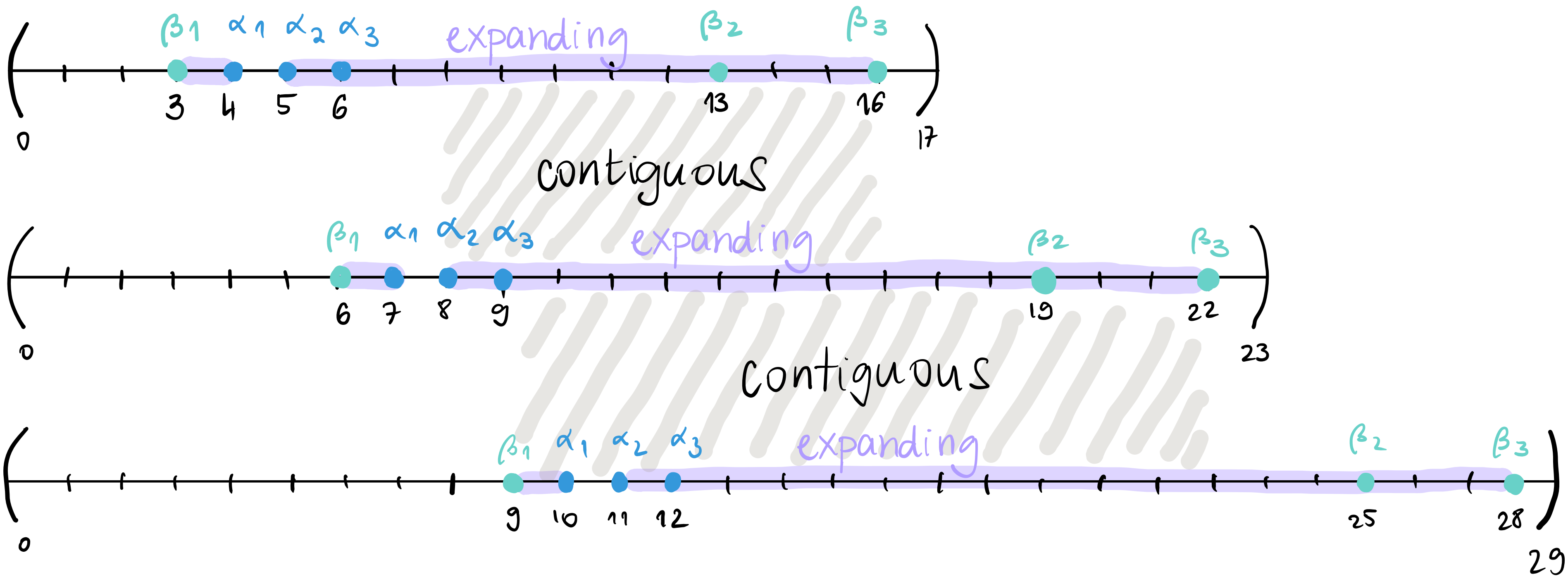
**Prop.**

There exists an effectively computable bound $P \in \mathbb{N}$ s.t. for all primes $p \in b\mathbb{N}+1$ greater than $P$, there exists a prime $Q \in b\mathbb{N}+1$ with $p < Q < p + \frac{p-1}{b} + C$.




⋮ sequence of primes in $2\mathbb{N}+1$




$\vdots$ sequence of primes in $2\mathbb{N}+1$

$\exists n \in \mathbb{N}$ s.t. $u_n = \frac{13}{6}$ ? $\longrightarrow$ check only $\{u_0, \ldots, u_4\}$

# The result

**Thm.**

The Membership Problem for hypergeometric sequences with rational parameters is decidable.

Proof:

$p$-adic valuations + results on the density of primes

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Follow-up work (in preparation):

algebraic parameters of degree 2