Classification of q-difference equations using holomorphic vector bundles

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Abstract

Let q a complex number such that $0 < |q| < 1$. With every analytic q-difference module is associated a holomorphic vector bundle over the corresponding elliptic curve $\mathbf{E}_q := \mathbf{C}^*/q^Z$.
For fuchsian q-difference modules, this allows for analytic classification, including the calculation of the Galois group (Baranovsky-Ginzburg-Kontsevich).
For arbitrary analytic q-difference modules, one can also obtain in this way a formal classification closely related to Atiyah's classification of holomorphic vector bundles over $\mathbf{E}_q$ (van der Put-Reversat).
In order to get the analytic classification of arbitrary modules (irregular equations), it is necessary (and sufficient) to take in account the canonical slope filtration on the side of the modules; and something called anti-HN filtration (HN stands for Harder-Narasimhan) on the side of the bundles. This is a variant of a statement attributed by Kontsevich and Soibelman to Ramis-Sauloy-Zhang.

Plan

How HVBs (= holomorphic vector bundles) come in

The q-world (local analytic view)

The functor $M \rightsquigarrow \mathcal{F}_M$

Slopes and some positive results

The canonical slope filtration

"Anti-HN" filtrations

Let Σ a compact Riemann surface and $p : \tilde{\Sigma} \to \Sigma$ its universal covering. Then every vector bundle $\mathcal{F}$ over Σ can be pulled back to a vector bundle $p^*\mathcal{F}$ over $\tilde{\Sigma}$.
Except if Σ is the Riemann sphere, $\tilde{\Sigma}$ is contractile and therefore $p^*\mathcal{F}$ is trivial: $p^*\mathcal{F} = \tilde{\Sigma} \times \mathbf{C}^n$.

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Moreover, $p^*\mathcal{F}$ is *equivariant* under $\mathrm{Aut}(p) = \pi_1(\Sigma)$, *i.e.* every $g \in \pi_1(\Sigma)$ gives rise to a map:

$$(x, X) \mapsto (gx, A(g,x)X) \text{ where } \forall g \in \mathrm{Aut}(p) \ , \ A(g,-) \in \mathcal{O}(\tilde{\Sigma}).$$

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This can be used to describe the vector bundle $\mathcal{F}$ as a quotient of $p^*\mathcal{F} = \tilde{\Sigma} \times \mathbf{C}^n$ under the action of $\pi_1(\Sigma)$:

$$\mathcal{F} = \frac{\tilde{\Sigma} \times \mathbf{C}^n}{(x, X) \sim \ \text{all } (gx, A(g,x)X)}.$$

One also gets a description of the sheaf of sections of $\mathcal{F}$ as a sheaf on Σ of functions on $\tilde{\Sigma}$ satisfying some "automorphy" conditions under the action of $\mathrm{Aut}(p)$.

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$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{\quad \mathbf{e} \quad} & \mathbf{C}^* \\ {\scriptstyle \text{universal covering } p} \big\downarrow & & \big\downarrow {\scriptstyle \pi \text{ (also an etale covering)}} \\ \mathbf{C}/(\mathbf{Z} + \mathbf{Z}\tau) & \xrightarrow{\simeq} & \mathbf{C}^*/q^{\mathbf{Z}} =: \mathbf{E}_q \end{array}$$

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If we lift a *holomorphic* vector bundle (abreviated *HVB*):

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Of course, $\pi^*\mathcal{F}$ is also equivariant under $\mathrm{Aut}(\pi) = q^{\mathbf{Z}}$.

The equivariant action of the cyclic group $\mathrm{Aut}(\pi) = q^{\mathbf{Z}}$ on the trivial HVB $\pi^* \mathcal{F} = \mathbf{C}^* \times \mathbf{C}^n$ is completely determined by the action of its generator q, which has the form:

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The sheaf on $\mathbf{E}_q$ of sections of the HVB $\mathcal{F}$ can then be described as:

$$\mathcal{F}_A(U) := \{X \in \mathcal{O}(\pi^{-1}(U))^n \mid X(qx) = A(x)X(x)\}$$

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However this $A(x) \in \mathrm{GL}_n(\mathcal{O}(\mathbf{C}^*))$ is "wild" at 0, while in the theory of functional equations the coefficients are usually taken in $\mathbf{C}(x)$ (global study) or $\mathbf{C}(\{x\})$ (local analytic study) or $\mathbf{C}((x))$ (formal study).

Plan

Recall notations: $\Im \tau > 0$, $q := e^{2\imath\pi\tau}$, and $\mathbf{E}_q := \mathbf{C}^*/q^{\mathbf{Z}}$. Also we denote $K := \mathbf{C}(\{x\}) \subset \hat{K} := \mathbf{C}((x))$ and σ_q their q-dilatation automorphism $f(x) \mapsto f(qx)$.

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The Ore-Laurent ring of q-difference operators over K is:

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We shall rather see it as a pair (V, Φ) of

- a finite dimensional K-vector space V and
- a σ_q-linear automorphism Φ of V (*i.e.* a group automorphism such that $\Phi(av) = \sigma_q(a)\Phi(v)$).

Up to the choice of a basis, any qdm can be put in the form:

$$M_A := (K^n, \Phi_A), \text{ where } \Phi_A(X) := A^{-1}\sigma_q X, A \in \mathrm{GL}_n(K).$$

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We call $\mathcal{E}_q$ the category of qdms over K. It is abelian, **C**-linear and has a tensor structure (it is tannakian).

The sheaf (on $\mathbf{E}_q$) of solutions (in $\mathbf{C}^*$) of $\sigma_q X = AX$ is:

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The sheaf $\mathcal{F}_A$ is locally free of rank n and corresponds to the HVB (denoted, as is customary, by the same letter):

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There is an intrinsic definition of the sheaf and HVB $\mathcal{F}_M$ from an arbitrary qdm M, such that $\mathcal{F}_A = \mathcal{F}_{M_A}$.

Plan

Recall that $K = \mathbf{C}(\{x\})$. Let $A \in \mathrm{GL}_n(K)$, $B \in \mathrm{GL}_p(K)$ and $F \in \mathrm{Mat}_{p,n}(K)$ such that $(\sigma_q F)A = BF$, *i.e.* a morphism of qdms:

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Now let $R := \mathcal{O}(\mathbf{C}^*, 0) \supset K$. Then $A \in \mathrm{GL}_n(R)$, $B \in \mathrm{GL}_p(R)$ and $F \in \mathrm{Mat}_{p,n}(R)$ induces a morphism of HVBs:

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This can be made intrinsic: every $F : M \to N$ in the category $\mathcal{E}_q$ of qdms over K defines a morphism $\mathcal{F}_M \to \mathcal{F}_N$ in the category $\mathcal{V}_q$ of HVBs over $\mathbf{E}_q$.

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We thereby get a functor $M \rightsquigarrow \mathcal{F}_M$ from $\mathcal{E}_q$ to $\mathcal{V}_q$.

The category $\mathcal{E}_q$ of qdms over K is abelian $\mathbf{C}$-linear (and tannakian).
However, the category $\mathcal{V}_q$ of HVBs over $\mathbf{E}_q$ is not abelian (it is only additive).

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It is exact, $\mathbf{C}$-linear, faithful (and $\otimes$-compatible).
Its essential image is $\mathcal{V}_q$.

However, it is *not* fully faithful.

Example 1

Let $n = p = 1$, $A := (x)$ and $B := (1)$.

Morphisms $M_A \to M_B$, resp. $\mathcal{F}_A \to \mathcal{F}_B$, are (scalar) functions $f \in K = \mathbf{C}(\{x\})$, resp. $f \in R = \mathcal{O}(\mathbf{C}^*, 0)$, such that $x\sigma_q f = f$.

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But $\theta_q(x) := \sum_{n \in \mathbf{Z}} q^{n(n-1)/2} x^n \in \mathcal{O}(\mathbf{C}^*, 0) \subset R$ satisfies $\sigma_q \theta_q = x^{-1}\theta_q$, *i.e.* $\theta_q \in \mathrm{Hom}_{\mathcal{S}_q}(\mathcal{F}_A, \mathcal{F}_B)$.

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Therefore $\mathrm{Hom}_{\mathcal{E}_q}(M_A, M_B) = 0$ while $\mathrm{Hom}_{\mathcal{S}_q}(\mathcal{F}_A, \mathcal{F}_B) \neq 0$.

Example 2, courtesy of Julien Roques

Let $n = p = 2$, $A := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $B := \begin{pmatrix} x^{-1} & 1 \\ 1 & 0 \end{pmatrix}$.

Then one can prove that $\text{Hom}_{\mathcal{E}_q}(M_A, M_B) = 0$.

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However, setting:

$$c(x) := \sum_{n \geq 0} \frac{q^{n(n+3)/2}}{(1-q^2)\cdots(1-q^{2n})} x^{-n} \text{ and } d(x) := c(-x),$$

we get an isomorphism $\begin{pmatrix} \sigma_q c & -\sigma_q d \\ c & d \end{pmatrix} : \mathcal{F}_A \to \mathcal{F}_B$ in $\mathcal{S}_q$.

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Therefore $\mathrm{Hom}_{\mathcal{E}_q}(M_A, M_B) = 0$ while $\mathcal{F}_A \simeq \mathcal{F}_B$

According to Ismaïl-Zhang (2006), functions a, b, c, d can be seen as q-analogues of the Airy function.

Plan

To every $L := {\sigma_q}^n + a_1 {\sigma_q}^{n-1} + \cdots + a_n \in \mathcal{D}_{K,q}$ is associated a *Newton polygon*, the convex hull of $\{(i,j) \mid j \geq v_x(a_i)\}$. The finite part of its boundary is made of k vectors $(r_i, d_i) \in \mathbf{N}^* \times \mathbf{Z}$ with slopes $\mu_i := d_i/r_i \in \mathbf{Q}$ decreasing from right to left: $\mu_1 > \cdots > \mu_k$. Slope μ_i has multiplicity r_i.

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The Newton poygon is a formal invariant

Given a qdm M, all q-difference operators L such that $M \underset{Kf}{\sim} \mathcal{D}_{K,q}/\mathcal{D}_{K,q}L$ (formal isomorphism) have the same Newton polygon. We write $S(M) := \{\mu_1, \ldots, \mu_k\}$.

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We say that M is

- *fuchsian* if $S(M) = \{0\}$,
- *pure isoclinic of slope* μ if $S(M) = \{\mu\}$,
- *pure* if $M = P_1 \oplus \cdots \oplus P_k$, each P_i being pure isoclinic (the slopes "don't mix").

In the last case, if $S(P_i) = \{\mu_i\}$, then $S(M) = \{\mu_1, \ldots, \mu_k\}$ and the multiplicities are $r_i = \text{rk } P_i$.

1. Baranovsky-Ginzburg 1996: to a *formal* fuchsian qdm is associated a *flat* vector bundle (*i.e.* one that can be written $\mathcal{F}_A$ with $A \in \mathrm{GL}_n(\mathbf{C})$); this is an equivalence of categories.
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In both cases, one obtains an *analytic* object (a HVB) from a *formal* one (a qdm over $\hat{K}$) and the construction is rather involved.

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The 1999 result has as corollary the equivalence of pure isoclinic qdms of given *integral* slope μ with "pure" HVBs of slope $\lambda := -\mu$, where we (provisionally) define a "pure" HVB as:

$$\text{line bundle} \otimes \text{flat bundle}$$

Plan

The canonical slope filtration

Every qdm M over K admits a unique tower of submodules:

$$M_0 = 0 \subset \cdots \subset M_k = M$$

such that the successive quotients $P_i := M_i/M_{i-1}$ are pure isoclinic of slope μ_i and $\mu_1 < \cdots < \mu_k$. One then has $S(M) = \{\mu_1, \ldots, \mu_k\}$ and the ranks are $r_i = \text{rk } P_i$.

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The functor $M \rightsquigarrow \text{gr } M := P_1 \oplus \cdots \oplus P_k$ is exact, $\mathbf{C}$-linear, faithful (and $\otimes$-compatible).
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Applying the functor $M \rightsquigarrow \mathcal{F}_M$ yields a tower of sub HVBs:

$$0 = \mathcal{F}_0 \subset \cdots \subset \mathcal{F}_k = \mathcal{F}_M$$

such that each $\mathcal{F}_i = \mathcal{F}_{M_i}$ and the successive subquotients $\mathcal{G}_i := \mathcal{F}_i/\mathcal{F}_{i-1}$ are the associated HVBs $\mathcal{G}_i = \mathcal{F}_{P_i}$.

Case of integral slopes $S(M) \subset \mathbf{Z}$:
Then each P_i can be put into the form $(K^{r_i}, \Phi_{x^{\mu_i} C_i})$ where $C_i \in \mathrm{GL}_{r_i}(\mathbf{C}_i)$.
It follows that the subquotients $\mathcal{G}_i = \mathcal{F}_{x^{\mu_i}} \otimes \mathcal{F}_{C_i}$ are "pure" of slopes $\lambda_i = -\mu_i$ and the slopes *decrease*: $\lambda_1 > \cdots > \lambda_k$.

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Case of arbitrary slopes $S(M) \subset r^{-1}\mathbf{Z}$, $r \in \mathbf{N}^*$:
Ramification in the form $x' := x^{1/r}$, $K' := K[x']$, $q' := q^{1/r}$ multiplies the slopes by r so $M' := K' \otimes_K M$ has integral slopes $\mu'_i := r\mu_i \in \mathbf{Z}$.

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The corresponding HVB $\mathcal{F}_{M'}$ over $\mathbf{E}_{q'}$ is the pullback $\rho^* \mathcal{F}_M$ by the isogeny $\rho : \mathbf{E}_{q'} \to \mathbf{E}_q$. It has filtration by the $\rho^* \mathcal{F}_i$ with successive quotients $\rho^* \mathcal{G}_i$, which are "pure" of integral slopes $\lambda_i' = r\lambda_i$.

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Therefore, in the general case (arbitrary slopes), the quotients $\mathcal{G}_i$ in the filtration of $\mathcal{F}_M$ are the HVBs which become "pure" (*i.e.* of the form line bundle $\otimes$ flat bundle) after an isogeny making their slope integral.

We are led to characterize HVBs which become "pure" after isogeny.

Those are exactly the semi-stable HVBs.
(Recall that the HVB $\mathcal{G}$ is *semi-stable* if for every subbundle $\mathcal{G}'$ the slopes satisfy $\lambda(\mathcal{G}') \leq \lambda(\mathcal{G})$.)

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For an HVB with integral slope, "pure" is equivalent to semi-stable.

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If $\rho : E' \to E$ is an isogeny of elliptic curves and if $\mathcal{G}$ is an HVB on E, then $\mathcal{G}$ is semi-stable iff $\rho^*\mathcal{G}$ is.

(Direct implication is by Galois descent and actually much more general; the converse is easy.)

Plan

How HVBs (= holomorphic vector bundles) come in

The q-world (local analytic view)

The functor $M \rightsquigarrow \mathcal{F}_M$

Slopes and some positive results

The canonical slope filtration

"Anti-HN" filtrations

The filtration on $\mathcal{F}_M$ deduced from the slope filtration on M has successive quotients that are semi-stable with decreasing slopes $\lambda_1 > \cdots > \lambda_k$.
We write $\underline{\mathcal{F}}_M$ the HVB $\mathcal{F}_M$ equipped with that filtration.

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Main result (so far): *local* classification at 0

The functor $M \rightsquigarrow \underline{\mathcal{F}}_M$ from $\mathcal{E}_q$ to $\underline{\mathcal{S}}_q$ is exact, $\mathbf{C}$-linear, *fully* faithful (and $\otimes$-compatible).
Its essential image is $\underline{\mathcal{V}}_q$.

For *global* classification, we should take q-difference modules over $\mathbf{C}(x)$ (*i.e.* $A \in \mathrm{GL}_n(\mathbf{C}(x))$, $F \in \mathrm{Mat}_{p,n}(\mathbf{C}(x))$, etc). Since $\mathbf{C}(x)$ embeds into $K = \mathbf{C}(\{x\})$, all our constructions remain meaningful, but we should decorate the notations: $\mathcal{F}_M^{(0)}$, $\underline{\mathcal{F}}_M^{(0)}$, $\mathcal{E}_q{}^{(0)}$, etc. Then we should complete them by their analogs at infinity: $\mathcal{F}_M^{(\infty)}$, $\underline{\mathcal{F}}_M^{(\infty)}$, $\mathcal{E}_q{}^{(\infty)}$, etc.

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The patching has previously been defined using *Birkhoff's connection matrix*, seen as a *meromorphic isomorphism* $\mathcal{F}_M^{(0)} \to \mathcal{F}_M^{(\infty)}$. Following an idea of Kontsevich-Soibelman, we rather use as patching data the inclusions

$$\mathcal{F}_M^{(0)}, \mathcal{F}_M^{(\infty)} \subset \mathcal{F}_M^{(glob)} := \mathcal{F}_M^{(0)} + \mathcal{F}_M^{(\infty)} \subset \mathcal{M}_{\mathbf{E}_q}^n.$$

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The torsion sheaves $\mathcal{F}_M^{(glob)}/\mathcal{F}_M^{(0)}$ and $\mathcal{F}_M^{(glob)}/\mathcal{F}_M^{(\infty)}$ have been precisely described in a paper by Roques-Sauloy (2019). We hope to soon obtain in this way a satisfying global result.

The q-Riemann-Hilbert correspondence according to Kontsevich-Soibelman

The category of holonomic $\mathcal{D}_q$-modules over $\mathbf{C}^*$ is equivalent to the category of coherent sheaves over $\mathbf{E}_q$ endowed with two anti-HN filtrations labeled by $\mathbf{Q} \cup \{\infty\}$.

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  (The corresponding condition for us is finite length.)

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- The label ∞ is here for torsion sheaves, which have infinite slope.
  (In our case, the highest subquotients of the anti-HN filtrations are $\mathcal{F}_M^{(glob)}/\mathcal{F}_M^{(0)}$ and $\mathcal{F}_M^{(glob)}/\mathcal{F}_M^{(\infty)}$.)

EN SOUVENIR DE MICHÈLE AUDIN

qui connaissait la différence entre bandits et makhnovistes ...

Michèle, Arnold et Makhno

Dans un texte traduit par Michèle Audin et publié par la Gazette des Mathématiciens en 1992, V.I. Arnold, pour illustrer l'inadéquation de l'enseignement mathématique de l'époque (surtout en France), reprend le récit par le mathématicien I.E. Tamm de son aventure avec les "bandits" pendant la "guerre civile".

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Michèle me répondit gentiment que c'est Arnold qui avait choisi cette terminologie et que son rôle de traductrice ne lui avait permis que l'insertion d'une N.d.T. précisant ce point. Pour Michèle, non seulement il y avait une différence, mais c'était important.